СВЯЗАНО-ДОМИНАНТНЫЕ ТРЕУГОЛЬНЫЕ ГРАФЫ, СОВЕРШЕННЫЕ СВЯЗАНО-ДОМИНАНТНЫЕ ТРЕУГОЛЬНЫЕ ГРАФЫ И СВЯЗНЫЕ ОКРЕСТНОСТНЫЕ МНОЖЕСТВА

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Вводится и характеризуется класс графов, в котором каждое связное доминирующее множество является (связным) окрестностным множеством, и класс графов, в котором все связные индуцированные подграфы имеют одинаковые минимальные окрестностные множества и минимальные мощности связных окрестностных множеств. Предполагая, что $\mathrm{P} \neq \mathrm{NP}$, мы доказываем, что проблема минимальности связных окрестностных множеств не может быть приближена в пределах логарифмического множителя за полиномиальное время в их общем подклассе, классе симплициальных расщепленных графов.

Ключевые слова: симплициальный расщепленный граф; число связной доминантности; число связной окрестностности.

CONNECTED-DOMINATION TRIANGLE GRAPHS, PERFECT CONNECTED-NEIGHBOURHOOD GRAPHS AND CONNECTED NEIGHBOURHOOD SETS

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We introduce and characterize the class of graphs in which every connected dominating set is a (connected) neighbourhood set and the class of graphs whose all connected induced subgraphs have equal minimum neighbourhood set and minimum connected neighbourhood set cardinalities. Assuming $\mathrm{P} \neq \mathrm{NP}$, we also prove that the minimum connected neighbourhood set problem cannot be approximated within a logarithmic factor in polynomial time in their common subclass, the class of simplicial split graphs.

Keywords: simplicial split graph; connected domination number; connected neighbourhood number.

INTRODUCTION AND PRELIMINARY NOTIONS

We follow [7] for the graph-theoretical definitions and notation unless explicitly stated otherwise. The graphs considered are finite, undirected, without loops and multiple edges. Let $V(G)$ and $E(G)$ denote the sets of the vertices and the edges of a graph G, respectively, and let $|G|=|V(G)|$ denote the number of its vertices (the *order* of G). For a vertex v, the set of all the vertices adjacent to it is called its *neighbourhood* and is denoted by $N(v)$; a *closed neighbourhood* set is a set defined as $N[v]=N(v)\cup\{v\}$. For a vertex set $S\subseteq V(G)$, define its *neighbourhood* $N(S)$ and *closed neighbourhood* $N[S]$ as $N(S)=\bigcup_{v\in S}N(v)\setminus S$ and $N[S]=N(S)\cup S$.

A set of pairwise nonadjacent vertices of a graph is called an *independent set*, and a set of its mutually adjacent vertices is called a *clique*. A clique is called *simplicial* if it encompasses the whole neighbourhood of one of its vertices (such a vertex is called *simplicial*). A set of vertices $D\subseteq V(G)$ is called a *dominating set* if $N[D] = V(G)$. A dominating set D in a graph G is called a *neighbourhood set* [1] if for every edge e of G there exists a vertex of D which closed neighbourhood contains e. A dominating set (neighbourhood set) of a graph is called a *connected dominating set* [2] (respectively, a *connected neighbourhood set* [3]) if it induces a connected subgraph of the considered graph. By $nb_c(G)$ we shall denote the *connected neighbourhood number* of a graph, the maximum cardinality of its connected neighbourhood sets. Note that connected dominating sets may exist only in connected graphs, therefore the parameters $\gamma_c(G)$ and $nb_c(G)$ are undefined for a

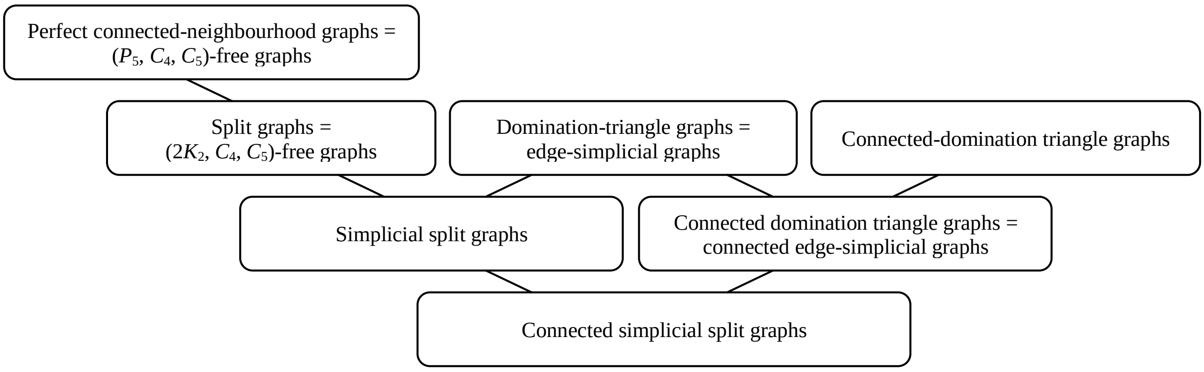

Fig. 1. The relations between the considered graph classes

disconnected graph G, while for every connected graph G, the obvious relations $\gamma(G)\le\gamma_c(G)$ and $nb(G)\le nb_c(G)$ hold.

We introduce the class of *connected-domination triangle graphs*, where every connected dominating set is a neighbourhood set (and thus a connected neighbourhood set). The characterization obtained for this class implies a simple polynomial algorithm for its recognition. Moreover, we introduce the class of *perfect connected-neighbourhood graphs*, where every connected induced subgraph has equal minimum connected dominating set and minimum connected neighbourhood set cardinalities, and characterize this class in terms of forbidden induced subraphs. The characterization implies that this class contains the widely known class of split graphs [5, 6]. (A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set.) The relations between the mentioned graph classes are shown in fig. 1 (some of them follow from the further details), where, as usual, a graph G is called F-free for a collection of graphs F if G does not contain any of the graphs in F as induced subgraphs. Assuming $\mathrm{P} \neq \mathrm{NP}$, we show that it is hard to approximate the minimum connected neighbourhood set cardinality with up to a logarithmic factor in the class of simplicial split graphs, the latter being a proper subclass of the classes of connected-domination triangle, perfect connected-neighbourhood and split graphs. (We define a *simplicial split graph* as a graph whose vertex set can be partitioned into a simplicial clique and an independent set.) To our knowledge, this is the first result on the complexity of the corresponding problem in the class of all graphs.

CONNECTED-DOMINATION TRIANGLE GRAPHS

A connected graph G is called *domination triangle* if for every its minimal dominating set D and every its edge $uv \in E(G-D)$ there exists a vertex $w \in D$ that is adjacent to both u and v (i.e. the vertex set $\{u, v, w\}$ induces a triangle in the graph G). In other words, every minimal dominating set of a domination triangle graph is also its neighbourhood set. Domiation triangle graphs were introduced and characterized in [4].

Call a connected graph G *connected-domination triangle* (a *CDT graph* for short) if for every its minimal connected dominating set D and every its edge $uv \in E(G-D)$ there exists a vertex $w \in D$ that is adjacent to both u and v (i.e. the vertex set $\{u, v, w\}$ induces a triangle in the graph G). In other words, every minimal connected dominating set of a CDT graph is also its (connected) neighbourhood set.

We introduce the following characterization of the class of CDT graphs. Call a vertex v of a graph G *private for the edge* $e \in E(G)$ if $N[v] \subseteq PN[e]$.

Theorem 1. For a connected graph G, the following claims are equivalent:

1) G is a CDT graph;

2) for every edge $e \in E(G)$ without private vertices, the graph $G - PN[e]$ is disconnected.

Proof. $1) \Rightarrow 2)$. Assume there is an edge uv without private vertices in the CDT graph G such that the graph $G - PN[uv]$ is connected. Then for every vertex $x \in PN[uv]$ the relation $N[x] \setminus PN[uv] \neq \varnothing$ holds. Therefore $x \in N[D]$, where $D = V(G) \setminus PN[uv]$, i.e. the set D is a dominating set in the graph G. On the other hand, the graph $G(D) = G - PN[uv]$ is connected, so D is a connected dominating set of the graph G. But D contains no vertex adjacent both to u and v, which leads to a contradiction with the fact that G is a CDT graph.

$2) \Rightarrow 1)$. Let the connected graph G be not a CDT graph. Then there exist its connected dominating set $D \subseteq V(G)$ and an edge $uv \in E(G-D)$ such that every vertex in D is not adjacent to at least one of the vertices u and v. In particular, this means that $PN[uv] \cap D = \varnothing$. If we now assume that the edge uv has a private vertex x, i.e. $N[x] \subseteq PN[uv]$, then $N[x] \cap D = \varnothing$, a contradiction with the fact that D is a dominating set. Thus, the edge uv has no private vertices and the induced subgraph $H = G - PN[uv]$ is disconnected. As $D \subseteq V(G) \setminus PN[uv] = V(H)$, the connected induced subgraph $G(D)$ is also a subgraph of the graph H. It folows that all the vertices of the set D belong to a single connected component of the graph H. But then every vertex of any other connected component of H is outside of $N[D]$ in the graph G, which is impossible for the dominating set D. This finishes the proof of the theorem.

Corollary 1. For every CDT graph G, the following equality holds: $\gamma_c(G) = nb_c(G)$.

Proof. Every connected neighbourhood set of an arbitrary connected graph G is a dominating set, so $\gamma_c(G) \le nb_c(G)$. For a CDT graph G the opposite is also true: every connected dominating set of such a graph is a (connected) neighbourhood set, therefore $\gamma_c(G) \ge nb_c(G)$. This finishes the proof of the corollary.

Theorem 1 implies a straightforward polynomial recognition algorithm for the recognition of the class of CDT graphs.

PERFECT CONNECTED-NEIGHBOURHOOD GRAPHS

Similar to the notion of perfect connected-domination graphs of [9] (which are shown to be precisely all the (P_5, C_5)-free graphs there), we define and characterize the following class of graphs.

Call a graph G *perfect connected-neighbourhood* (a *PCN graph* for short) if for every its connected induced subgraph H, the equality $nb(H) = nb_c(H)$ holds.

Theorem 2. A graph G is a PCN graph if and only if it is a (P_5, C_4, C_5)-free graph (fig. 2).

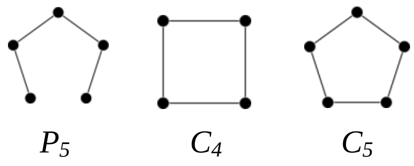

Fig. 2. Minimal forbidden induced subgraphs for the class of PCN graphs

The proof of this theorem is similar to the proof of the corresponding characterization in [9] and is omitted for brevity.

Corollary 2. Every split graph G is a PCN graph. In particular, if G is connected, then $nb(G) = nb_c(G)$.

Proof. It is easy to check that every split graph does not contain induced paths and cycles of length more than 3. This also follows from the characterization of the class of split graphs [5, 6], according to which a graph is split if and only if it is $(2K_2, C_4, C_5)$-free (because the graph $2K_2$ is an induced subgraph of a simple path P_5). This proves the corollary.

HARDNESS OF APPROXIMATION FOR THE CONNECTED NEIGHBOURHOOD NUMBER

We adopt the terminology of [8] for the computational complexity-theoretic notions.

It is easy to show the validity of the following relation (we omit the proof).

Theorem 3. The equality $\gamma_c(G) = \gamma(G)$ holds for every connected split graph G.

Together with corollaries 1 and 2, this implies that for every connected simplicial split graph G, we have $\gamma(G) = \gamma_c(G) = nb_c(G) = nb(G)$. As every such graph is a domination triangle (or edge-simplicial) graph [4], we arrive to the following result on the inapproximability of these parameters in connected simplicial split graphs.

Theorem 4. It is NP-hard to approximate the domination number $\gamma(G)$, the connected domination number $\gamma_c(G)$, the neighbourhood number $nb(G)$ and the connected neighbourhood number $nb_c(G)$ within the factor of $\kappa \ln n$ for arbitrary connected simplicial split graphs G of order n, where κ is a fixed constant.

The theorem follows from the proof of the more special result in [4] where the inapproximability of domination and neighbourhood numbers is considered in domination triangle graphs. It can be seen that the proof there remains valid in the class of connected simplicial split graphs, and applying theorem 3 this proves theorem 4.

Theorem 4 establishes the first result on the hardness for the connected neighbourhood set problem in the class of all graphs.

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LITERATURE

1. Sampathkumar E., Neeralagi P. S. The neighbourhood number of a graph // Indian J. Pure Appl. Math. 1985. Vol. 16. P. 126–132.
2. Sampathkumar E., Walikar H. B. The connected domination number of a graph // J. Math. Phys. Sci. 1979. Vol. 13. P. 607–613.
3. Sampathkumar E., Neeralagi P. S. Independent, perfect and connected neighbourhood numbers of a graph // J. Combin. Inf. Syst. Sci. 1994. Vol. 19. P. 139–145.
4. Kartynnik Y. A., Orlovich Y. L. Domination triangle graphs and upper bound graphs // Dokl. NAN Belarusi. 2014. Vol. 58, № 1. P. 16–25. Картынник Ю. А., Орлович Ю. Л. Доминантно-треугольные графы и графы верхних границ // Докл. НАН Беларуси. 2014. Т. 58, № 1. С. 16–25.
5. Földes S., Hammer P. L. Split graphs // Congress. Numer. 1978. № XIX. P. 311–315.
6. Tyshkevich R. I., Chernyak A. A. Canonical partition of a graph defined by the degrees of its vertices // Isv. Akad. Nauk BSSR, Ser. Fiz.-Mat. Nauk. 1979. Vol. 5. P. 14–26. Тышкевич Р. И., Черняк А. А. Каноническое разложение графа, определяемого степенями его вершин // Изв. АН БССР. Сер.: физ.-мат. наук. 1979. Т. 5. С. 14–26.
7. Lectures on Graph Theory / O. Melnikov [et al.]. Mannheim : B.I. Wissenschaftsverlag, 1994.
8. Garey M. R., Johnson D. S. Computers and Intractability: A Guide to the Theory of NP-Completeness. N. Y. : W. H. Freeman and co., 1979.
9. Zverovich I. E. Perfect connected-dominant graphs // Discuss. Math. Graph Theory. 2003. Vol. 23. P. 159–162.