CONNECTED-DOMINATION TRIANGLE GRAPHS AND CONNECTED NEIGHBOURHOOD NUMBERS

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In [1], Sampathkumar and Neeralagi have introduced the notion of a *neighbourhood set* of vertices: in a graph G, a set $S \subseteq V(G)$ of its vertices is called a neighbourhood set if

$$G = \bigcup_{v \in S} G(N[v]),$$

where $N[v]$ is the closed neighbourhood of a vertex v and $G(X)$ denotes the subgraph of G induced by a vertex set X. In other words, a neighbourhood set S in G is a special kind of dominating set with an additional constraint that for every edge e of G, there exists a vertex $v \in S$ adjacent to both endvertices of e. The minimum size of the neighbourhood sets in a graph G is called its *neighbourhood number* $nb(G)$.

In [2], the same authors have proposed a related notion of a *connected neighbourhood set* of vertices. A neighbourhood set of vertices in a graph G is called a connected neighbourhood set if it induces a connected subgraph of G. The minimum size of connected neighbourhood sets in a graph G is called its *connected neighbourhood number* $nb_c(G)$.

In [3], we have shown that in a certain class of graphs (hence called *domination triangle* and also known as *edge-simplicial*) characterized by every domination set being a neighbourhood set, it is NP-hard to approximate $nb(G)$ (which is equal to the domination number $\gamma(G)$ in every graph of that class) within a factor of $c \ln |V(G)|$ for some constant $c > 0$. This remains true in the class of *simplicial split graphs*, i.e. split graphs [4] having such a partition $V(G) = I \cup C$ of the vertex set that I is an independent set and C is a *simplicial clique* (there exists a vertex $s \in C$ such that $N[s] = C$).

Following this idea, we introduce the class of *connected-domination triangle graphs* characterized by every its *connected dominating set* (a dominating set of vertices inducing a connected subgraph) being a connected neighbourhood set. Let $\gamma_c(G)$ denote the connected domination number, i.e. the minimum size of the domination sets of the graph G that induce connected subgraphs. Then, clearly, for every connected-domination triangle graph G, $nb_c(G) = \gamma_c(G)$.

For an edge $uv \in E(G)$ of a graph G, call $N[u] \cap N[v]$ the *private neighbourhood* $PN[uv]$ of uv. Call a vertex v *private to the edge* e if $N[v] \subseteq PN[e]$. We characterize the class of connected-domination triangle graphs with the following theorem, which implies a trivial polynomial recognition algorithm.

Theorem 1. *A graph G is connected domination triangle* iff *for every its edge $e \in E(G)$ having no private vertices, the vertex set $V(G) \setminus PN[e]$ induces a disconnected subgraph.*

Corollary 1. *Every simplicial split graph is connected-domination triangle.*

Corollary 2. *For every simplicial split graph G,*

$$nb_c(G) = \gamma_c(G).$$

The following result helps us link the approximation hardness for the parameter nb_c with that of the parameter γ.

Theorem 2. *For a simplicial split graph G,*

$$\gamma_c(G) = \gamma(G).$$

Corollary 3. *It is* NP-*hard to approximate $nb_c(G)$ within a factor of $c \ln |V(G)|$ for some constant $c > 0$ in the class of simplicial split graphs.*

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References

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