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LA-SALLE'S INVARIANCE PRINCIPLE AND METHOD OF SEMI DEFINITE FUNCTIONS*

В статье «Принцип инвариантности Ля-Салля и метод знакопостоянных функций Ляпунова» исследованы притяжение и устойчивость инвариантных множеств динамических систем методом функций Ляпунова. Показано, что известный принцип инвариантности Ля-Салля [1] может гарантировать свойство устойчивости лишь в случае, когда используемая вспомогательная функция является знакопостоянной или знакоопределенной. Дается сравнение двух методов исследования динамических систем.

La-Salle's invariance principle, originally proved for systems of ordinary differential equations [1], has received further development for abstract dynamical systems [2-5] and can be regarded as a method of localization of positive limit sets with the help of continuous or semi-continuous functions (called Lyapunov functions). These functions have a derivative along the trajectories of the considered system with is non positive. The positive limit set can be found in the greatest invariant subset contained in the set of points where the derivative of the Lyapunov functional vanishes. The solution of a problem of localization essentially will use topological and dynamical properties of positive semi-trajectories and positive limit sets.

From this theorem of localization follows (with the same auxiliary Lyapunov function) the theorem of the Lyapunov's second method about stability or asymptotic stability and, as far as we are concerned with this question, the applicable results known up to the present time make use of definite or semi-definite Lyapunov's functions. So, we have an open problem: which minimal properties should have a Lyapunov's function, in terms of sign, in the study of the stability. The answer to this problem for dynamical systems on locally compact metric space makes the purpose of the present work. Below, we show that under the conditions of the application of the La-Salle's invariance principle, the Lyapunov's function is necessarily semi-definite positive in presence of an asymptotically stable equilibrium.

1. Preliminaries. Let *(X, d)* be a metric space; recall that the continuous mapping $\pi: X\times R\to X$ defines a dynamical system if:

- $\pi(x, 0)=x$ for every $x\in X$;
- $\pi(\pi(x, t_1), t_2)=\pi(x, t_1+t_2)$ for every $x\in X$ and every $t_1, t_2\in\Re$.

In the sequel, we will say that the triple $(X, \Re, \pi)$ is a *dynamical system* [6] and we always assume that *X* is locally compact. Clearly this notion of dynamical system generalizes the notion of solution of an autonomous ordinary differential equation.

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1.1. Some notations. Given a dynamical system $(X, \Re, \pi)$ and (x, t) in $X\times\Re$, we will write xt in place of $\pi(x, t)$. Given $x\in X$ we will denote by

- $\gamma(x)=\{xt : t\in\Re\}$, the orbit of x;
- $\gamma^+(x)=\{xt : t\in\Re^+\}$ and $\gamma^-(x)=\{xt : t\in\Re^-\}$ the semi-orbits issued from x.

The set of ω-limit points (resp. α-limit points) of x is denoted by $L^+(x)$ (resp. $L^-(x)$). A point y belongs to $L^+(x)$ (resp. $L^-(x)$) if end only if there exists a sequence of scalars $(t_n)_{n\geq 0}$ such that $\lim\limits_{n\to+\infty} t_n = +\infty$ (resp. $-\infty$) and $y=\lim\limits_{n\to+\infty} xt_n$.

1.2. Stable and attractive sets. A subset S of X is said positively invariant (resp. negative invariant) if $\gamma^+(x)\subset S$ (resp. $\gamma^-(x)\subset S$) for every $x\in S$. The following definition generalizes the notion of stability and attraction of an equilibrium.

Definition 1. Let M be a compact subset of X, we will say that M is

- *stable* if every neighborhood of M contains a positively invariant neighborhood of M or, equivalently, if

$$\forall\ \varepsilon>0,\ \exists\alpha>0,\ \gamma^+(B(M,\alpha))\subset B(M,\varepsilon)$$

where $B(M, r)$ - $\{x\in X : d(M, x)<r\}$ is an open neighborhood of M and, if S is a subset of X, $\gamma^+(S) = \{xt : x\in S \text{ and } t\geq 0\}$;

- *attractive* if the set $A^+(M)=\{x\in X : L^+(x)\neq\varnothing \text{ and } L^+(x)\subset M\}$ is a neighborhood of M;
- *asymptotically stable* if its stable and attractive;
- *globally asymptotically stable* if its asymptotically stable and $A_+(M)=X$.

The set $A_+(M)$ is called the basin of attraction of M.

Assume now that the compact set M is included in a positively invariant set Y; if, in the above definition, we restrict the dynamical system to $(Y, \Re, \pi)$, we will speak of stability, attraction, etc. with regard to Y. Thus, we will say that M is stable with regard to Y if for every neighborhood V of M, there exists a neighborhood W of M such that $\gamma^+(W\cap Y)\subset V$.

2. Localization of the attractive sets. Let $(X, \Re, \pi)$ be a dynamical system, Ω a positively invariant subset of X and $V\colon \Omega\to\Re$ a continuous function, we denote by $\dot V(x)$ the number (possibly infinite) defined by

$$\dot V(x) = \liminf_{t\to 0,\, t\geq 0} \frac{1}{t}(V(xt) - V(x)).$$

The following lemma is about the limit sets and is proved in [6].

Lemma 1. Let $(X, \Re, \pi)$ be a dynamical system, K a subset of X and V: $X\to\Re$ a function such that $\dot V(x)\leq 0$ for all $x\in K$. If for some x $\overline{\gamma^+(x)}$ (resp. $\overline{\gamma^-(x)}$) is included in K, then $V(y)=V(z)$ for all y, $z\in L^+(x)$ (resp. in $L_-(x)$).

We state below the La-Salle's invariance principle in the framework of dynamical systems, notice that this result, stated by La-Salle in [1] has been reformulated by Saperstone [7] in the framework of semidynamical systems. Recall that a dynamical system is said Lagrange stable if $\overline{\gamma^+(x)}$ is compact for all x.

Principle Invariance. Let $(X, \Re, \pi)$ be a dynamical system which is Lagrange stable and Ω a subset of X, positively invariant. Assume that there exists a function $V\colon\Omega\to\Re$ such that $\dot V(x)\leq 0$ for every $x\in\Omega$. We put $E=\{x\in\Omega : \dot V(x)=0\}$ and we call Y the maximal positively invariant set included in E. Then for every $x\in\Omega$, $L^+(x)\subset Y$ and V is constant on $L_+(x)$.

Since, for every x, the closure of the positive semi-orbit $\gamma^+(x)$ is compact, we can write also $\lim\limits_{t\to+\infty} d(xt, Y) = 0$, so we can say, equivalently, that set Y is attractive.

As for as we are concerned with the sign of function V, we have the following result.

Lemma 2. Let $(X, \mathfrak{R}, \pi)$ be a dynamical system which is Lagrange stable and Ω a subset of X, positively invariant. Assume that there exists a function $V\colon \Omega \to \mathfrak{R}$ such that $\dot{V}(x) \le 0$ for every $x \in \Omega$. Then for every $x \in \Omega$ $V(x) \ge \inf\limits_Y V$.

Proof. Since $\dot{V}(x) \le 0$ the function $t \to V(xt)$ is non increasing so $V(x) \ge V(xt)$ for every $t \ge 0$, from which we can deduce that $V(x) \ge \inf\limits_Y V$ since $L^+(x) \subset Y$.

In the particular case where Y is compact, $\inf\limits_Y V > -\infty$ and so there exists a constant c such that $W(x) = V(x) + c$ is positive for every $x \in \Omega$ and $\dot{W}(x) = \dot{V}(x) \le 0$.

When Y is not compact, $\inf\limits_Y V$ can be equal to $-\infty$. As a matter of fact, consider the following example given in $\mathfrak{R}^3$

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x, \\ \dot{z} = -z \end{cases}$$

and $V(x, y, z) = z^2 - (x^2 + y^2)$ we have $\dot{V}(x, y, z) = -z^2 \le 0$, $E = Y = \{(x, y, z) \in \mathfrak{R}^3 : z = 0\}$ and $\inf\limits_Y V = -\infty$, so there does not exists a constant c such that $V(x) + c \ge 0$.

Notice also that, if V is constant on every limit set $L^+(x)$, this is not longer true on Y, consider the following example given in $\mathfrak{R}^2$:

$$\begin{cases} \dot{x} = -x\varphi(x, y) - y, \\ \dot{y} = x - y\varphi(x, y) \end{cases}$$

where

$$\varphi(x, y) = \begin{cases} \exp\left(\dfrac{1}{1 - (x^2 + y^2)}\right) & \text{if } x^2 + y^2 > 1, \\ 0 & \text{if } x^2 + y^2 \le 1. \end{cases}$$

Let

$$V(x, y) = \frac{1}{2}(x^2 + y^2),$$

we have

$$\dot{V}(x, y) = -(x^2 + y^2)\varphi(x, y),$$

and $E = Y = \{(x, y) \in \mathfrak{R}^2 : x^2 + y^2 \le 1\}$, it can be easily seen that V is not constant on E.

Finally, we emphasize on the fact that the La-Salle's principle does not allow, in general, to conclude to the stability of a set. The following example is well known in the literature and is due to R.E. Vinograd

$$\begin{cases} \dot{x} = \dfrac{x^2(y - x) + y^5}{(x^2 + y^2)(1 + (x^2 + y^2))^2}, \\ \dot{y} = \dfrac{y^2(y - 2x)}{(x^2 + y^2)(1 + (x^2 + y^2))^2}. \end{cases}$$

For this system, it is well known that the origin is an unstable and attractive equilibrium (see [8] for more details), if we add to these two equations the equation $\dot{z} = -z$, we obtain a system defined in $\mathfrak{R}^3$ for which the origin is still an unstable and attractive equilibrium. For this last system the function $V\colon (x, y, z) \to \frac{1}{2} z^2$ is such that $\dot{V} = -z^2 \le 0$.

In conclusion, if we do not assume that V is definite positive we cannot conclude to the stability of a set by means of the La-Salle's principle.

3. The method of semi-definite positive functions. The following result was first published in 1978 [9], see also [10].

Theorem 1. Let $(X, \mathfrak{R}, \pi)$ be a dynamical system and M a compact positively invariant subset of X. Suppose that there exists a continuous function V defined on a neighborhood U of M such that:

- $V(x)\geq 0$ for all $x\in U$ and $V(x)=0$ if $x\in M$;
- $\dot{V}(x)\leq 0$ for all $x\in U$;
- M is asymptotically stable with regard to the set $Y_0=\{x\in U : V(x)=0\}$.

Then M is stable.

The following theorem is about the asymptotic stability.

Theorem 2. Let $(X, \mathfrak{R}, \pi)$ be a dynamical system, M a compact positively invariant subset of X and V a function defined in a neighborhood U of M. Assume that

- $V(x)\geq 0$ for all $x\in U$ and $V(x)=0$ if $x\in M$;
- $\dot{V}(x)$ is non positive;
- M is asymptotically stable with regard to Y the maximal positive invariant subset included in the set $\{x\in X : \dot{V}(x)=0\}$ (notice that $Y_0\subset Y$).

Then M is asymptotically stable.

Notice that if the conditions of **Principle Invariance** are fulfilled and if $Y=Y_0$ then M is asymptotically stable. Conversely suppose that function V satisfies all the conditions of **Principle Invariance** and that M is asymptotically stable. Take x in Y, since Y is positively invariant and $Y\subset\{y\in X : \dot{V}(x)=0\}$, xt belongs to Y and $V(x)=V(xt)$ for all $t\geq 0$. Now since M is compact and У continuous and zero on M, for every $\varepsilon>0$, there exists $\alpha>0$ such that $V(y)<\varepsilon$ if $y\in B(M, \alpha)$, but $xt\in B(M, \alpha)$ provided t is large enough, for such a t we here $V(x)=V(xt)<\varepsilon$ and so $V(x)=0$ since $V(x)<\varepsilon$ for all $\varepsilon>0$.

Finally Theorem 2 can be reformulated as follows:

Theorem 3. Let $(X, \mathfrak{R}, \pi)$ be a dynamical system and M a positively invariant compact subset of X Suppose that there exists a continuous function V defined on a neighborhood U of M such that:

- $V(x)\geq 0$ for all $x\in U$ and $V(x)=0$ if $x\in M$;

  $\dot{V}(x)\leq 0$ for all $x\in U$;
- M is asymptotically stable with regard to the set $Y_0=\{x\in U : V(x)=0\}$ and Y_0 is the maximal positively invariant set contained in $E=\{x\in\Omega : \dot{V}(x)=0\}$ (so $Y=Y_0$).

Then M is asymptotically stable.

As far as we are concerned with global asymptotic stability, we have the following result.

Theorem 4. Let $(X, \mathfrak{R}, \pi)$ be a dynamical system, M a compact positively invariant subset of X and V a function defined in a neighborhood U of M. Assume that

- $V(x)\geq 0$ for all $x\in U$ and $V(x)=0$ if $x\in M$;
- $\dot{V}(x)$ is non positive;
- M is globally asymptotically stable with regard to Y, the maximal positive invariant subset included in the set $\{x\in X : \dot{V}(x)=0\}$;
- the dynamical system $(X, \mathfrak{R}, \pi)$ is Lagrange stable.

Then $U\subset A^+(M)$, and so if $U=X$, M is globally asymptotically stable.

Consider in the space $\mathfrak{R}^2$ the differential equations

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(y + x^3)(y + a(x))^2 - 3x^2 y \end{cases} \qquad (1)$$

where a is smooth function such that $a(x)=0 \Leftrightarrow x=0$.

Take $V(x, y) = \frac{1}{2}(y + x^3)^2$ then an easy computation gives us

$$\dot{V}(x, y) = -(y+x^3)(y+a(x))^2,$$

the sets Y_0 and E being

$$Y_0 = \{(x, y) \in \Re^2 : y = -x^3\}, \; E = Y_0 \cup \{(x, y) \in \Re^2 : y = -a(x)\}.$$

On Fo we have $\dot{x} = -x^3$, so the origin is asymptotically stable with regard to Y_0, from Principle Invariance it following that the origin is stable.

If $a \equiv 0$. it is clear that $Y = Y_0 \cup \{(x, 0) : x \in \Re\}$ and so 0 is stable but not asymptotically stable.

If $a \neq 0$, in a neighborhood of 0, then $Y = Y_0$ and 0 is asymptotically stable.

Now we claim that the dynamical system (1) is Lagrange stable if function a satisfies the following property: $\forall A>0, \exists B>0, |x| > A \Rightarrow |a(x) - x^3| < B$.

As a matter of fact, the expression $|x^3(t) + y(t)|$ is bounded when t varies on $\Re^+$ and since

$$(y(t)+x^3(t))(y(t)+ax(t))^2=(y(t)+x^3(t))(y(t)+x^3(t)+(a(t)-x^3(t)))^2,$$

the expression $(y(t)+x^3(t))(y(t)+ax(t))^2$ is also bounded as t varies $\Re^+$ so for $|x(t)|$ and $|y(t)|$ large enough we have $y(t)\dot{y}(t) \leq 0$ which proves that $y(f)$ is bounded and so, $x(t)$ is bounded too.

4. Localization of the asymptotically stable sets. Let $(X, \Re, \pi)$ be a dynamical system, Ω an open set of X and $V: \Omega \to \Re$ a continuous function, we adopt the some notations as in lemma 2.

Theorem 5. Let $(X, \Re, \pi)$ be a dynamical system which is Lagrange stable and Ω a subset of X, positively invariant. Assume that there exists a function $V: \Omega \to \Re$ such that $\dot{V}(x) \leq 0$ for every $x \in \Omega$. Let M and Y be a positively invariant compact sets ($M \subset Y$). Suppose that

- $V(x)=0$ for every $x \in Y$;
- M is asymptotically stable with regard to Y.

Then we have:

- $V(x) \geq 0$ for all $x \in \Omega \backslash Y$;
- M is asymptotically stable.

Proof. From Lemma 2, we know that $V(x) \geq \inf_Y V(x) = 0$ we have $V(xt)=0$ for all $t \geq 0$ because $\dot{V}(x) \leq 0$ and so $\gamma^+(x) \subset Y$. So if $x \in \Omega \backslash Y$, $V(x) \geq 0$.

The second point follows from Theorem 2.

Remark. If we add to the conditions of this theorem the hypothesis that the dynamical system $(X, \Re, \pi)$ is Lagrange stable, we can conclude t l$\Omega \subset A^+(M)$n d so M is globally asymptotically stable if $\Omega = X$ (see Theorem 2).

Corollary. Assume that $\dot{V}(x) \leq 0$ for all $x \in \Omega$ and suppose that

- $V(x)=0$ for every $x \in Y$;
- Y is compact.

Then we have:

- $V(x) \geq 0$ for all $x \in \Omega \backslash Y$;
- Y is asymptotically stable and, if the dynamical system is Lagrange stable, $\Omega \subset A^+(M)$.

Proof. With the choice $M=Y$, M is obviously asymptotically stable with regard to Y and all the assumptions of Theorem 5 are satisfies. So we can conclude as this theorem.

Conclusion. The invariance principle gives us a condition about the attraction of an invariant set (or an equilibrium) and does not permit us to conclude to the stability, unless the function V is definite positive. As for as we are concerned with the sign of V, we can see that this sign is almost always non negative. On the other hand, the use of semi-definite positive function allows us to address the problem of the stability (or the asymptotic stability) of an equilibrium.

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$t\geq 0$ $X\subset R^n$

$$\dot{x} = Ax + b(x)u, \ x \in X, \quad (1)$$