ON THE INEQUALITY IN OPEN MULTISERVER QUEUEING NETWORKS

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Abstract

The paper is devoted to the analysis of queueing systems in the context of the network and communications theory. We investigate the inequality in an open multiserver queueing network and its applications to the theorems in heavy traffic conditions (fluid approximation, functional limit theorem, and law of the iterated logarithm) for a queue of jobs in an open multiserver queueing network.

1 Statement of the problem and the network model

This paper is devoted to the analysis of queueing systems in the context of the network and communications theory. We investigate the inequality in an open multiserver queueing network and its applications to the theorems in heavy traffic conditions for a queue of jobs in an open multiserver queueing network. Familiar researches made by [1], [2], and others, were used in this paper.

Consider a network of j stations, indexed by $j = 1, 2, \ldots, J$, and the station j has c_j servers, indexed by $(j, 1), \ldots, (j, c_j)$. A description of the primitive data and construction of processes of interest are the focus of this section. No probability space will be mentioned in this section, and of course, one can always think that all the variables and processes are defined on the same probability space.

First, $\{u_j(e), e \geq 1\}, j = 1, 2, \ldots, J$, are J sequences of exogenous interarrival times, where $u_j(e) \geq 0$ is the interarrival time between the $(e-1)$-st job and the e-st job that arrive at the station j exogenously (from the outside of the network). Define $U_j(0) = 0, \quad U_j(n) = \sum_{e=1}^{n} u_j(e),\ n \geq 1$ and $A_j(t) = \sup\{n \geq 0 : U_j(n) \leq t\}$, where $A_j = \{A_j(t), t \geq 0\}$ is called an exogenous arrival process of the station j, i.e., $A_j(t)$ counts the number of jobs that arrived at the station j from the outside of the network.

Second, $\{v_{jk_j}(e), e \geq 1\},\ j = 1, 2, \ldots, J,\ k_j = 1, 2, \ldots, c_j$, are $c_1 + \ldots + c_J$ sequences of service times, where $v_{jk_j}(e) \geq 0$ is the service time for the e-th customer served by the server k_j of the station j. Define $V_{jk_j}(0) = 0, \quad V_{jk_j}(n) = \sum_{e=1}^{n} v_{jk_j}(e),\ n \geq 1$ and $x_{jk_j}(t) = \sup\{n \geq 0 : V_{jk_j}(n) \leq t\}$, where $x_{jk_j} = \{x_{jk_j}(t), t \geq 0\}$ is called a service process for the server k_j at the station j, i.e., $x_{jk_j}(t)$ counts the number of services completed by the server k_j at the station j during the server's busy time. We define $\mu_{jk_j} = \left(M\left[v_{jk_j}(e)\right]\right)^{-1} > 0,\ \sigma_{jk_j} = D\left(v_{jk_j}(e)\right) > 0$ and $\lambda_j = (M\left[u_j(e)\right])^{-1} > 0$, $a_j = D\left(u_j(e)\right) > 0,\ j = 1, 2, ..., k$; with all of these terms assumed finite.

Also, let $\tilde{\tau}_j(t)$ be the total number of jobs routed to the jth station of the network in the interval $[0,t]$, $\tau_j(t)$ be the total number of jobs after service departure from the jth station of the network in the interval $[0,t]$, $\tilde{\tau}_{jk_j}(t)$ be the total number of jobs routed to the k_j server of the jth station of the network in the interval $[0,t]$, let $\tau_{jk_j}(t)$ be the total number of jobs after service departure from the k_j server of the jth station of the network in the interval $[0,t]$, and $\tau_{ijk_i}(t)$ be the total number of jobs after service departure from the k_i server of the ith station of the network and routed to the k_j server of the jth station of the network in the interval $[0,t]$. Let p_{ij} be a probability of the job after service at the ith station of the network routed to the jth station of the network. Denote $p^t_{ijk_i} = \dfrac{\tau_{ijk_i}(t)}{\tau_{ik_i}(t)}$ as part of the total number of jobs which, after service at the k_i server of the ith station of the network, are routed to the jth station of the network in the interval $[0,t]$, $i,j = 1,2,\ldots,J$, $k_i = 1,\ldots,c_i$ and $t > 0$.

The processes of primary interest are the queue length process $Q = (Q_j)$ with $Q_j = \{Q_j(t), t \geq 0\}$, where $Q_j(t)$ indicates the number of jobs at the station j at time t. Now we introduce the following processes $Q_{jk_j} = \{Q_{jk_j}(t),\ t \geq 0\}$, where $Q_{jk_j}(t)$ indicates the number of customers waiting to be served by the server k_j of the station j at time t; clearly, we have $Q_j(t) = \sum\limits_{k_i=1}^{c_j} Q_{jk_i}(t)$, $j = 1,2,\ldots,J$.

The dynamics of the queueing system (to be specified) depends on the service discipline at each service station. To be more precise, "first come, first served" (FCFS) service discipline is assumed for all J stations. When a customer arrives at a station and finds more than one server available, it will join one of the servers with the smallest index. We assume that the service station is work-conserving; namely, not all servers at a station can be idle when there are customers waiting for service at that station. In particular, we assume that a station must serve at its full capacity when the number of jobs waiting is equal to or exceeds the number of servers at that station. Suppose that the queue of jobs in each station of the open queueing network is unlimited. All random variables are defined on one common probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

2 The main results

First, denote $\beta_j = \sum\limits_{i=1}^{J} \sum\limits_{k_i=1}^{c_i} \mu_{ik_i} \cdot p_{ij} + \lambda_j - \sum\limits_{k_j=1}^{c_j} \mu_{jk_j} > 0$, $\hat{\sigma}^2_j = \sum\limits_{i=1}^{J} \sum\limits_{k_i=1}^{c_i} \mu^3_{ik_i} \cdot \sigma_{ik_i} \cdot p^2_{ij} + \lambda^3_j \cdot a_j + \sum\limits_{k_j=1}^{c_j} \mu^3_{jk_j} \cdot \sigma_{jk_j} > 0$, $j = 1,2,\ldots,J$.

We assume that the following conditions are fulfilled:

$$\sum_{i=1}^{J} \sum_{k_i=1}^{c_i} \mu_{ik_i} \cdot p_{ij} + \lambda_j > \sum_{k_j=1}^{c_j} \mu_{jk_j},\ j = 1,2,\ldots,J. \tag{1}$$

Theorem 1. *If $Q_j(0)=0$, then*

$$|Q_j(t)-\hat{x}_j(t)| \le \sum_{i=1}^{k} w_i(t) + \sum_{i=1}^{k} \gamma_i(t), \quad where \ \ \hat{x}_j(t) =$$

$\sum\limits_{i=1}^{J}\sum\limits_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ij} + A_j(t) - \sum\limits_{k_j=1}^{c_j} x_{jk_j}(t), \ \ w(t) = \sum\limits_{j=1}^{J}\sum\limits_{i=1}^{J}\sum\limits_{k_i=1}^{c_i} x_{ik_i}(t)\cdot |p^t_{ijk_i} - p_{ij}|,$

$\gamma(t) = \sum\limits_{i=1}^{J}\sum\limits_{k_i=1}^{c_i} \sup\limits_{0\le s\le t}(x_{ik_i}(s) - \tau_{ik_i}(s)), \ j=1,2,\ldots,J.$

Proof. By definition of the queue of customers at the stations of the network, we get that, for $j=1,2,\ldots,J, \ k_j = 1,2,\ldots,c_j$

$$Q_j(t) = \tilde{\tau}_j(t) - \tau_j(t) = \sum_{k_i=1}^{c_j} Q_{ik_i}(t) = \sum_{k_i=1}^{c_j} \tilde{\tau}_{ik_i}(t) - \sum_{k_i=1}^{c_j} \tau_{ik_i}(t)$$

$$= \sum_{k_i=1}^{c_j} \tilde{\tau}_{ik_i}(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t) + \sum_{k_i=1}^{c_j} x_{ik_i}(t) - \sum_{k_i=1}^{c_j} \tau_{ik_i}(t)$$

$$\le \sum_{k_i=1}^{c_j} \tilde{\tau}_{ik_i}(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t) + \sum_{k_i=1}^{c_j} \sup_{0\le s\le t}(x_{ik_i}(s) - \tau_{ik_i}(s))$$

$$= \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} \tau_{ijk_i}(t) + A_j(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t) + \sum_{k_i=1}^{c_j} \sup_{0\le s\le t}(x_{ik_i}(s) - \tau_{ik_i}(s))$$

$$\le \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} \tau_{ik_i}(t)\cdot \frac{\tau_{ijk_i}(t)}{\tau_{ik_i}(t)} + A_j(t) - \sum_{k_j=1}^{c_j} x_{jk_j}(t) + \sum_{k_i=1}^{c_j} \sup_{0\le s\le t}(x_{ik_i}(s) - \tau_{ik_i}(s))$$

$$\le \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p^t_{ijk_i} + A_j(t) - \sum_{k_j=1}^{c_j} x_{jk_j}(t) + \sup_{0\le s\le t}(x_{jk_j}(s) - \tau_{jk_j}(s))$$

$$= \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot (p^t_{ijk_i} - p_{ij} + p_{ij}) + A_j(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t)$$

$$\le \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ij} + A_j(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t) + \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot |p^t_{ijk_i} - p_{ij}|$$

$$+ \sum_{k_i=1}^{c_j} \sup_{0\le s\le t}(x_{ik_i}(s) - \tau_{ik_i}(s)) = \hat{x}_j(t) + w(t) + \gamma(t), j = 1,2,\ldots,J \text{ and } t>0.$$

Hence it follows that

$$Q_j(t) \le \hat{x}_j(t) + w(t) + \gamma(t), \ j=1,2,\ldots,J \text{ and } t>0. \tag{2}$$

Also, note that

$$
\begin{aligned}
&Q_j(t) \geq \tilde{\tau}_j(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t) = \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} \tau_{ik_i}(t)\cdot p_{ijk_i}^t + A_j(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t) \\
&= \sum_{i=1}^{J}\sum_{k_i=1}^{c_i}(x_{ik_i}(t) + \tau_{ik_i}(t) - x_{ik_i}(t))\cdot p_{ijk_i}^t + A_j(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t) \\
&= \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ijk_i}^t + \sum_{i=1}^{J}\sum_{k_i=1}^{c_i}(\tau_{ik_i}(t) - x_{ik_i}(t))\cdot p_{ijk_i}^t + A_j(t) \\
&- \sum_{k_i=1}^{c_j} x_{ik_i}(t) = \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ijk_i}^t - \sum_{i=1}^{J}\sum_{k_i=1}^{c_i}(x_{ik_i}(t) - \tau_{ik_i}(t))\cdot p_{ijk_i}^t \\
&+ A_j(t) - \sum_{k_j=1}^{c_j} x_{jk_j}(t) \geq \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ijk_i}^t + A_j(t) - \sum_{k_j=1}^{c_j} x_{jk_j}(t) \\
&- \sum_{i=1}^{J}\sum_{k_i=1}^{c_i}(x_{ik_i}(t) - \tau_{ik_i}(t)) \geq \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ijk_i}^t + A_j(t) - \sum_{k_j=1}^{c_j} x_{jk_j}(t) \\
&- \sup_{0\leq s\leq t}\sum_{i=1}^{J}\sum_{k_i=1}^{c_i}(x_{ik_i}(s) - \tau_{ik_i}(s)) \geq \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ijk_i}^t + A_j(t)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{k_j=1}^{c_j} x_{jk_j}(t) - \sum_{i=1}^{J}\sum_{k_i=1}^{c_i}\sup_{0\leq s\leq t}(x_{ik_i}(s) - \tau_{ik_i}(s)) \\
&= \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot(p_{ijk_i}^t - p_{ij} + p_{ij}) + A_j(t) - \sum_{k_j=1}^{c_j} x_{jk_j}(t) \\
&- \sum_{i=1}^{J}\sum_{k_i=1}^{c_i}\sup_{0\leq s\leq t}(x_{ik_i}(s) - \tau_{ik_i}(s)) = \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ij} + A_j(t) \\
&- \sum_{k_i=1}^{c_j} x_{ik_i}(t) + \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot(p_{ijk_i}^t - p_{ij}) \\
&- \sum_{i=1}^{J}\sum_{k_i=1}^{c_i}\sup_{0\leq s\leq t}(x_{ik_i}(s) - \tau_{ik_i}(s)) \geq \hat{x}_j(t) - \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot|p_{ijk_i}^t - p_{ij}| \\
&- \sum_{i=1}^{J}\sum_{k_i=1}^{c_i}\sup_{0\leq s\leq t}(x_{ik_i}(s) - \tau_{ik_i}(s)) = \hat{x}_j(t) - w(t) - \gamma(t),\ j = 1, 2, \ldots, J \text{ and } t > 0.
\end{aligned} \tag{3}
$$

Hence it follows that

$$Q_j(t) \geq \hat{x}_j(t) - \sum_{i=1}^{k} w_i(t) - \sum_{i=1}^{k} \gamma_i(t), \tag{4}$$

$j = 1, 2, \ldots, J$ and $t > 0$.

By combining (2) and (4), we can prove an inequality. □

3 Application of the inequality

Note that the inequality is the key inequality to prove several laws (fluid approximation, functional limit theorem, and law of the iterated logarithm) for a queue of jobs in open multiserver queueing networks in heavy traffic conditions. At first we present a theorem on the fluid approximation for a queue of jobs in open multiserver queueing networks under heavy traffic conditions.

Theorem 2. *Under conditions (1) the weak convergence holds:*

$$t^{-1}(Q_j(t))_{j=1}^J \Rightarrow (\beta_j)_{j=1}^J, \ 0 \le t \le 1.$$

Next, we present a theorem on the functional limit for a queue of jobs in open multiserver queueing networks in heavy traffic conditions.

Theorem 3. *Under conditions (1) the following CLT holds:*

$$n^{-1/2}\left(\frac{Q_j(nt) - nt\beta_j}{\hat{\sigma}_j}\right)_{j=1}^J \Rightarrow (W_j(t))_{j=1}^J, \ 0 \le t \le 1,$$

for independent standard Wiener processes $W_j(t)$, $j = 1, \ldots, J$.

One of the results of the paper is the following theorem on the law of the iterated logarithm for a queue of jobs in an open multiserver queueing network.

Theorem 4. *Under conditions (1) the following law of the iterated logarithm holds:*

$$P\left(\overline{\lim_{t\to\infty}} \frac{Q_j(t) - t\beta_j}{\hat{\sigma}_j\sqrt{2t\ln\ln t}} = 1\right) = 1, \ j = 1, \ldots, J.$$

The proof of these theorems is similar to that in [3].

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