MULTIVARIATE LINEAR REGRESSION WITH HETEROGENEOUS STRUCTURE AND ASYMMETRIC DISTRIBUTIONS OF ERRORS

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Abstract

Multivariate econometric models with heterogeneous structure arise in economic and financial processes influenced by exogenous shocks. If structural heterogeneity is driven by presence of several classes of states in modeled complex systems, multivariate regime-switching econometric models is a choice. An assumption of normally distributed errors, which is traditionally held for such models, is often violated on real data. Therefore it is actual to develop multivariate regime-switching econometric models in presence of non-gaussian errors. In this paper, a multivariate regression model with switching regimes and asymmetrically distributed errors is proposed. A maximum likelihood approach is used to estimate the parameters of the model.

1 The model

Let us introduce an independent-switching multivariate linear regression model with errors distributed according to a class SNI [1] of asymmetric distributions, hereafter the IS-MLR-SNI model. The relation between endogenous and exogenous variables in the IS-MLR-SNI is expressed as follows:

$$x_t = B_{d(t)} z_t + \eta_{d(t),t}, \quad t = 1, \dots, T, \tag{1}$$

where for a period of time t: $x_t = (x_{t1}, ..., x_{tN})' \in \mathbf{X}$, $\mathbf{X} \subset \Re^N$ $(N \geq 1)$ — vector of endogenous variables, $z_t = (z_{t1}, ..., z_{tM})' \in \mathbf{Z}$, $\mathbf{Z} \subset \Re^M$ $(M \geq 1)$ — vector of exogenous variables, $d(t) \in S(L) = \{1, ..., L\}$ — a state of a system modeled, $B_{d(t)}$ — regression coefficients matrix with a dimension $N \times M$, $\eta_{d(t),t} \in \Re^N$ — a random vector of heterogeneous errors.

For the model (1) the following assumptions are used.

1. Assumptions about observation errors:

a) observation errors have zero means and are mutually uncorrelated:

$$\mathbf{E}\{\eta_{d(t),t}\} = 0_N, \ \mathbf{E}\{\eta_{d(t),t}(\eta_{d(\tau),\tau})'\} = 0_{M\times N}, \quad t \neq \tau, \ (t, \tau = 1, \dots, T); \tag{2}$$

b) observation errors have asymmetric distribution from a class SNI:

$$\eta_{d(t),t} \sim SNI_N(b\Delta_{d(t)}, \Sigma_{d(t)}, \lambda_{d(t)}, \nu), \quad t = 1, \dots, T, \tag{3}$$

where $SNI_N(\mu, \Sigma, \lambda, \nu)$ — a class of multivariate asymmetric distributions [1] including skewed normal distribution and skewed t-distribution. The distributions from the

SNI class have the following parameters: $\mu \in \Re^N$ – location parameter; Σ – scale parameter, a covariance matrix with a dimension of $N \times N$; $\lambda \in \Re^N$ – skewness parameter; $H(u|\ \nu)$ – mixing distribution with a parameter $\nu \in \Re^{m_\nu}(m_\nu \geq 1)$; $\Sigma_{d(t)}$ $\lambda_{d(t)}$ – covariance matrix and skewness parameter for a state $d(t) \in S(L)$; $b\Delta_{d(t)}$ – parameter ensuring the condition $\mathbf{E}\{\eta_{d(t),t}\} = 0_N$, $b = -K_1\sqrt{2/\pi}$, $K_1 = \mathbf{E}\{U^{-1/2}|\nu\}$ – expectation of $U^{-1/2}$, where the random variable U is distributed according to $H(u|\nu)$, $\Delta_l = \Sigma_l^{1/2}\delta_l$, $\delta_l = \lambda_l/\sqrt{1+\lambda'_l\lambda_l}$, $l \in S(L)$.

2. Assumptions about the regime-switching model: the sequence of states following discrete time and space process with the distribution

$$P\left\{d_t = l\right\} = \pi_l > 0\,(l \in S(L)), \ \sum\nolimits_{l=1}^{L} \pi_l = 1, \tag{4}$$

where parameters $\{\pi_l\}$ $(l \in S(L))$ correspond to prior probabilities of states.

3. Condition of structural parametric heterogeneity:

$$B_k \neq B_l,\ k \neq l,\ k,l \in S(L). \tag{5}$$

4. Assumption about exogenous variables.

A vector of exogenous variables z_t is fixed for all realizations $\{z_t\},\ t = 1,\dots,T$.

With assumption (3), the model IS-MLR-SNI may be represented in the form of the mixture of distributions with the following density function:

$$p(x_t|\Theta; z_t) = \sum\nolimits_{l=1}^{L} \pi_l sni_N\left(x_t|B_l z_t + b\Delta_l, \Sigma_l, \lambda_l, \nu\right), \quad t = 1,\dots,T, \tag{6}$$

where $sni_N\left(x_t|B_l z_t + b\Delta_l, \Sigma_l, \lambda_l, \nu\right)$ — distribution density function for a random vector $x_t \in \Re^N$ against parameters Θ and fixed vector $z_t \in \Re^M$.

For model (1) in assumptions (2)–(5), let $\Theta = (\pi_1,\dots,\pi_{L-1},\theta'_1,\dots,\theta'_L)' \in \Re^m$ be the stacked vector of all independent parameters, where $\theta_l = (b'_l, S'_l, \lambda'_l)' \in \Re^K$, $b_l = vec(B_l)$ denotes the vector of all elements of matrix B_l, S_l denotes the vector with the elements of upper triangular matrix of Σ_l, $\nu \in \Re^{m_\nu}$. Then the overall number of parameters equals $m = L - 1 + LK + m_\nu$, where $K = NM + N(N+1)/2 + N$.

2 Parameter estimation

To estimate the parameters of the model, we use an approach based on maximizing the likelihood function for the parameters Θ given a sample of regression observations $\{x_t, z_t\},\ t = 1,\dots,T$. For derivation of the parameter estimates introduce the parameterization:

$$\Delta_l = \Sigma_l^{1/2}\delta_l, \qquad \Gamma_l = \Sigma_l^{1/2}(I_N - \delta_l\delta'_l)\Sigma_l^{1/2} = \Sigma_l - \Delta_l\Delta'_l, \quad l \in S(L) \tag{7}$$

where $\delta_l = \lambda_l/\sqrt{1+\lambda'_l\lambda_l}$, $\lambda_l \in \Re^N$ – skewness parameter for class l.

Let $X = (x'_1,\dots,x'_T)' \in \Re^{TN}$ – stacked vector of all endogenous variables' realizations from the sample, $Z = (z'_1,\dots,z'_T)' \in \Re^{TM}$ – stacked vector of all exogenous variables' realizations; $v = (v_1,\dots,v_T)'$, $u = (u_1,\dots,u_T)'$ – vectors of all realizations of random variables V_t, U_t accordingly, where random variable V_t has the truncated univariate normal distribution with mean b and variance u_t^{-1} on the interval $(0, \infty)$ and depends on realization u_t of random variable U_t distributed according to the mixing distribution $H(u|\nu)$, and $\alpha_t = (\alpha_{t1}, \dots, \alpha_{tL})'$ denotes the state indicator that has the multinomial distribution $M(1;\ \pi_1, \dots, \pi_L)$.

Define the following expectations:

$$\begin{gathered}\rho_{tl} = \mathbf{E}_\Theta \left\{\alpha_{tl} | x_t, z_t\right\}, \beta_{tl} = \mathbf{E}_\Theta \left\{\alpha_{tl} U_t |\ x_t, z_t\right\}, \\ \xi_{tl} = \mathbf{E}_\Theta \left\{\alpha_{tl} U_t V_t |\ x_t, z_t\right\}, \omega_{tl} = \mathbf{E}_\Theta \left\{\alpha_{tl} U_t V_t^2 |\ x_t, z_t\right\}, t = 1, \dots, T,\ l \in S(L),\end{gathered} \tag{8}$$

where α_{ti}, V_t, U_t – random variables, and the expectations (8) derived against fixed vector of parameters Θ and regression observations x_t, z_t with the following formula [1]:

$$\begin{gathered}\rho_{tl} = \frac{\pi_l sni_N \left(x_t | B_l z_t + b\Delta_l, \Sigma_l, \lambda_l, \nu\right)}{\sum_{j=1}^{L} \pi_j sni_N \left(x_t | B_j z_t + b\Delta_j, \Sigma_j, \lambda_j, \nu\right)}, \quad t = 1, \dots, T,\ l \in S(L), \\ \beta_{tl} = \rho_{tl} \beta(x_t, z_t, \theta_l),\ \xi_{tl} = \rho_{tl} \xi(x_t, z_t, \theta_l), \\ \omega_{tl} = \rho_{tl} \omega(x_t, z_t, \theta_l), \quad t = 1, \dots, T,\ l \in S(L),\end{gathered} \tag{9}$$

where $\beta(\cdot),\ \xi(\cdot),\ \omega(\cdot)$ are defined for basic distributions from SNI class as in [1].

Theorem 1. *Let $\tilde{\rho}_{tl}, \tilde{\beta}_{tl}, \tilde{\xi}_{tl}, \tilde{\omega}_{tl}$ be conditional expectations (8) derived against fixed vector of parameters $\tilde{\Theta}$ and regression observations sample $\{x_t, z_t\},\ t = 1, \dots, T$. Then the maximum likelihood estimates of the parameters $\{\pi_l, B_l, \Delta_l, \Gamma_l\}$, $l \in S(L)$ have the following representation:*

$$\hat{\pi}_l = 1/T \ \sum\nolimits_t^T \tilde{\rho}_{tl}, \tag{10}$$

$$\hat{B}_l = \sum\nolimits_{t=1}^{T} \left(\tilde{\beta}_{tl} x_t z_t' - \tilde{\xi}_{tl} \tilde{\Delta}_l z'_t\right) / \left(\sum\nolimits_{t=1}^{T} \tilde{\beta}_{tl} z_t z'_t\right)^{-1}, \tag{11}$$

$$\hat{\Delta}_l = \left[\sum\nolimits_{t=1}^{T} \tilde{\xi}_{tl} \left(x_t - \hat{B}_l z_t\right)\right] / \sum\nolimits_{t=1}^{T} \tilde{\omega}_{tl}, \tag{12}$$

$$\begin{gathered}\hat{\Gamma}_l = \left(\sum\nolimits_{t=1}^{T} \tilde{\rho}_{tl}\right)^{-1} \sum\nolimits_{t=1}^{T} \left\{\tilde{\beta}_{tl} \left(x_t - \hat{B}_l z_t\right) \left(x_t - \hat{B}_l z_t\right)' + \tilde{\omega}_{tl} \hat{\Delta}_l \left(\hat{\Delta}_l\right)' - \right. \\ \left. - \tilde{\xi}_{tl} \left[\left(x_t - \hat{B}_l z_t\right) \left(\hat{\Delta}_l\right)' + \hat{\Delta}_l \left(x_t - \hat{B}_l z_t\right)'\right]\right\}, \quad l \in S(L).\end{gathered} \tag{13}$$

To prove the theorem we follow the corresponding results from [1] considering density (6) from model (1) based on assumptions (2)–(5).

To recover the initial parameters λ_l, Σ_l the following formulae are used:

$$\begin{gathered}\lambda_l = \left(\Gamma_l + \Delta_l \Delta'_l\right)^{-1/2} \Delta_l / \left[1 - \Delta'_l (\Gamma_l + \Delta_l \Delta'_l)^{-1} \Delta_l\right]^{1/2}, \\ \Sigma_l = \Gamma_l + \Delta_l \Delta'_l, \quad l \in S(L).\end{gathered} \tag{14}$$

3 Objectives of the study

Regime-switching models are widely used in such applications as macroeconomics (real business cycles modeling), microeconomics (company credit risk modeling), financial markets (modeling and analysis of cyclical changes on financial markets) [2]. The problem of cyclical changes analysis with a help of the models mentioned may be considered in a context of more general problem of structural breaks analysis [3]. Structural breaks may be partial or full, that is parameters $\{B_l, \Sigma_l, \lambda_l, \pi_l\}$, $l \in S(L)$ of the IS-MLR-SNI model may partially or fully distinguish across the states. The changes in the parameters may take place in any period of time $t = 1, \ldots, T$. A vector of states $d = (d_1, \ldots, d_T)'$ is unobserved.

To estimate structural breaks, a classification based approach is proposed. Therefore for the IS-MLR-SNI model (1) on the assumptions (2)–(5) we have the following problems to solve: 1) estimation of the parameters Θ of the model and the vector of states $d = (d_1, \ldots, d_T)'$ on unclassified sample of regression observations $\{x_t, z_t\}$, $t = 1, \ldots, T$; 2) classification of new observations $\{x_\tau, z_\tau\}$, $\tau = T+1, \ldots, T+h$ $(h \geq 1)$ with the model estimated on the train data sample of size T. Problems 1 and 2 are solved with cluster and discriminant analysis algorithms respectively. For cluster analysis we use Expectation-Maximization (EM) algorithm. Earlier these problems were solved for multivariate regression models with switching regimes and normally distributed errors [4]. In [5] algorithms for analysis of multivariate regression observations with markov-switching regimes were presented.

In this study, for the solution of the problems an EM-type algorithm has been developed for the model (1) on the assumptions (2)–(5). An experimental study of the proposed algorithm is conducted on the simulated data.

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