ON THE PROBABILITY DISTRIBUTION PROCESSES SOME MODELS OF INTEREST RATES

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Abstract

Presents the probability density and their properties for some stochastic models of short-term interest rates of yield, the authors previously proposed constructively without probabilistic analysis of their properties.

1 Introduction

There are many different models of short-term interest rates of the class of diffusion processes. Most of them are well documented by the authors, which offered them, or those who use them for their studies. However, there is a set of models tend to be fairly complex, probabilistic description of the properties which are absent in the literature. It is they who are the subject of our consideration. The main problem that we are interested is getting analytical expressions for the stationary probability densities and its main moments. Some models, such as models Vasiek (1977), Cox - Ingersoll - Ross (CIR) (1985), Duffie - Kan (1996), Ahn - Gao (1999), are well documented in the literature, therefore are not described here and not mentioned in the list of references. All considered models belong to the class of diffusion models, that generate processes $X(t)$, described by the equation

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t), t > t_0, X(t_0) = X_0,$$

where a specific determination of drift $\mu(x)$ and volatility $\sigma(x)$ defines one or another particular model.

2 The Ait-Sahalia model [1]

Ait-Sahalia has tested the based models of short interest rates (including described here) by fitting them to the actually time series of rates. It was found that an acceptable level of goodness-of-fit all these rates were rejected because the drift and volatility properties. As a result he proposed the following functions drift and diffusion

$$\mu(r) = \alpha_0 + \alpha_1 r + \alpha_2 r^2 + \alpha_{-1}\frac{1}{r}, \sigma^2(r) = \beta_0 + \beta_1 r + \beta_2 r^2.$$

In this model, the non-linear functions of drift and diffusion allow a wide variety of forms. To $\sigma^2(r) > 0$ for any r, it is necessary that the diffusion function parameters ensure the fulfilment of inequalities

$$\beta_0 > 0, \beta_2 > 0, \gamma^2 \equiv 4\beta_0\beta_2 - \beta_1^2 \geq 0.$$

Relevant in this function a probability density is given by expression

$$f(x) = Nx^B(\beta_0 + \beta_1 x + \beta_2 x^2)^{C-1} e^{Ax + Garctg(E+Fx)}, x > 0,$$

where N is normalization constant,

$$A = \frac{2\alpha_2}{\beta_2} < 0, B = \frac{2\alpha_{-1}}{\beta_0} > 0, C = \frac{\alpha_1}{\beta_2} - \frac{\alpha_2\beta_1}{\beta_2^2} - \frac{\alpha_{-1}}{\beta_0},$$

$$G = 2\left(2\alpha_0 + \frac{\alpha_2\beta_1^2}{\beta_2^2} - \frac{\alpha_1\beta_1}{\beta_2} - \frac{2\alpha_2\beta_0}{\beta_2} - \frac{\alpha_{-1}\beta_1}{\beta_0}\right)\Big/\gamma,$$

$$E = \beta_1/\gamma,\ F = \beta_2/\gamma.$$

Since the density $f(x)$ at $x \to 0$ has order $O(x^B), B > 0$, and at $x \to \infty$ its order is $O(x^{B+C}e^{Ax}), A < 0$, then for every finite m the moments $E[X^m]$ are exist, but their analytical expressions can not be obtained, and they can be calculated only numerically.

3 The CKLS model [2]

In Chan - Karolyi - Longstaff - Sanders (CKLS) model it is assumed that $\mu(x) = k(\theta - x), \sigma^2(x) = \sigma^2 x^3$. It turns out that a random process corresponding to this model has a stationary density

$$f(x) = \frac{n}{x^3} e^{-c((\frac{\theta}{x})^2 - 2\frac{theta}{x})}, x > 0,$$

where $c = \frac{k}{\theta\sigma^2}$, n is normalization constant. Note that such random process has only the first stationary moment $E[X] = \theta$.

4 The unrestricted model I [3]

In “unrestricted model I”

$$dr = (\alpha_1 + \alpha_2 r + \alpha_3 r^2)dt + \sqrt{\alpha_4 + \alpha_5 r + \alpha_6 r^3}dW$$

are embedded some known models, that is, at a certain setting parameters $\{\alpha\}$ can get any of these known models. Table of according in this case has the form

Stationary probability density “unrestricted I” process has the form

$$f(x) = \frac{c(w)}{\sigma^2(x)} e^{\int_w^x \frac{2\mu(u)}{\sigma^2(u)}du} = \frac{c(w)}{\alpha_4 + \alpha_5 x + \alpha_6 x^3} e^{\int_w^x \frac{2(\alpha_1 + \alpha_2 u + \alpha_3 u^2)}{\alpha_4 + \alpha_5 u + \alpha_6 u^3}du},$$

where $c(w)$ is normalization constant, w is a fixed number from the set of possible values of a random process, the specific value of which does not play some role.

To get the explicit form of expression for $f(x)$ is possible, but it will be in general case quite cumbersome, and we restrict ourselves to the case when the values of the

Restrictions of parameters	Model	Equation of processes
$\alpha_3 = \alpha_5 = \alpha_6 = 0$	Vasicek	$dr = k(\theta - r)dt + \sigma dW$
$\alpha_3 = \alpha_4 = \alpha_6 = 0$	CIR	$dr = k(\theta - r)dt + \sigma\sqrt{r}dW$
$\alpha_3 = \alpha_6 = 0$	Duffie - Kan	$dr = k(\theta - r)dt + \sqrt{\alpha + \beta r}dW$
$\alpha_1 = \alpha_4 = \alpha_5 = 0$	Ahn - Gao	$dr = k(\theta - r)rdt + \sigma r^{1.5}dW$
$\alpha_3 = \alpha_4 = \alpha_5 = 0$	CKLS	$dr = k(\theta - r)dt + \sigma r^{1.5}dW$

parameters $\{\alpha\}$ provide performance properties of the probability density $f(x)$. First, we note that the volatility of the real process needs to be a real function, so $\sigma^2(r) = \alpha_4+\alpha_5 r+\alpha_6 r^3 \geq 0$ for all values of r. At the same analytic properties of the probability density depends on the type of the roots of equation $\alpha_4 + \alpha_5 r + \alpha_6 r^3 = 0$,$\alpha_6 > 0$. The sign of the discriminant $\Delta = (\frac{\alpha_5}{3\alpha_6})^3 + (\frac{\alpha_4}{2\alpha_6})^2$ specifies the number of real and complex roots of the equation. When $\Delta > 0$, there is one real and two complex conjugate roots. When $\Delta < 0$, there are three different real roots. When $\Delta = 0$, real roots are multiples.

Let $\Delta > 0$ and the real root is $r = r_0$, then we can write

$$\alpha_4 + \alpha_5 r + \alpha_6 r^3 = \alpha_6(r - r_0)(r^2 + pr + q),$$

where r_0, p and q are relatively sophisticated analytical expression and because of that are not listed here. However, if $\alpha_4 = 0$, then $r_0 = 0, p = 0, q = \frac{\alpha_5}{\alpha_6}$. In this case, the probability density is given by

$$f(x) = \frac{c(w}{\alpha_6 x(x^2 + \frac{\alpha_5}{\alpha_6})} e^{\int_w^x \frac{2(\alpha_1+\alpha_2 u+\alpha_3 u^2)}{\alpha_6 u(u^2+\frac{\alpha_5}{\alpha_6})}du} =$$

$$n x^{\frac{2\alpha_1}{\alpha_5}-1}(\alpha_6 x^2 + \alpha_5)^{\frac{\alpha_3}{\alpha_6}-\frac{\alpha_1}{\alpha_5}-1} e^{\frac{2\alpha_2}{\sqrt{\alpha_5\alpha_6}} arctg[x\sqrt{\frac{\alpha_6}{\alpha_5}}]},$$

where n is the normalization constant. For the existence of the probability density its parameters must satisfy the inequalities: $\frac{\alpha_1}{\alpha_5} > 1, \frac{\alpha_3}{\alpha_6} < 1$. In order to at the same time there exist stationary moments it is necessary for the expectation $\frac{\alpha_3}{\alpha_6} < 0,5$, for variance $\frac{\alpha_3}{\alpha_6} < 0$, for the third moment $\frac{\alpha_3}{\alpha_6} < -0,5$ and for the fourth moment $\frac{\alpha_3}{\alpha_6} < -1$.

If $\Delta < 0$, denote the roots of the equation $r_0 > r_1 > r_2$ so

$$\alpha_4 + \alpha_5 r + \alpha_6 r^3 = \alpha_6(r - r_0)(r - r_1)(r - r_2).$$

Then the probability density is expressed in the form

$$f(x) = n \prod_{i=0}^{2} (x - r_i)^{-1+2(\alpha_1+\alpha_2 r_i+\alpha_3 r_i^2)/\alpha_6 \prod_{j\neq i}(r_i-r_j)}. \tag{9}$$

In this case must be performed the inequalities

$$2(\alpha_1 + \alpha_2 r_0 + \alpha_3 r_0^2) > \alpha_6(r_0 - r_1)(r_0 - r_2),\ \alpha_3/\alpha_6 < 1.$$

For the existence of the m-th moment other than that necessary to perform the conditions $\frac{m}{2}+\frac{\alpha_3}{\alpha_6} < 1$. Unfortunately, the analytical expression of the normalization constant n and moments $E[r^m]$ very cumbersome, they include hypergeometric functions. Under these assumptions the process with such density has a bottom line equal to the largest root, i.e. $r(t) \geq r_0$.

Model	γ	$E[X]$	$Var[X]$	Skewness	Kurtosis
Vasicek	0	θ	$\frac{\sigma^2}{2k}$	0	3
CIR	0.5	$\frac{q}{c}=\theta$	$\frac{q}{c^2}=\frac{\sigma^2\theta}{2k}$	$2\sqrt{q}$	$3+\frac{6}{q}$
Brennan - Schwartz	1.0	$\frac{q}{c}=\theta$	$\frac{\theta^2}{c-1}$	$\frac{4\sqrt{c-1}}{c-2}$	$\frac{3(c-1)(c+6)}{(c-2)(c-3)}$
CKLS	1.5	$\frac{q}{c}=\theta$	not exist	not exist	not exist

5 The unrestricted model II [2]

In the "unrestricted model II" process of short rate follows the equation

$$dr = k(\theta - r)dt + \sigma r^{\gamma} dW,\ \gamma > 0. \tag{1}$$

Therefore $\mu(x) = k(\theta - x), \sigma^2(x) = \sigma^2 x^{2\gamma}$ and the stationary density $f(x)$ has form

$$f(x) = \frac{n}{x^{2\gamma}} e^{\frac{1}{x^{2\gamma}}\left(\frac{qx}{1-2\gamma} - \frac{cx^2}{2-2\gamma}\right)},\ x > 0, \tag{2}$$

where $q = \frac{2k\theta}{\sigma^2}, c = \frac{2k}{\sigma^2}, n$ is the normalization constant. Values of parameter γ, allowing the convergence of the integral of $f(x)$ on the interval $(0, \infty)$, determined by the inequality $\gamma > 0.5$. At the same time, there are two critical points: $\gamma = 0.5$ (in this case, the model is transformed into a short-term rate model CIR) and $\gamma = 1$, when the probability density is reduced to form that corresponds to process of the Brennan - Schwartz model [4]

$$f(x) = \frac{q^{1+c}}{x^{2+c}\Gamma(1+c)} e^{-\frac{q}{x}},\ x > 0.$$

When $\gamma = 1.5$, model "unrestrictions II" is known as the model CKLS. Vasicek model is also a model embedded in the model "unrestrictions II" at $\gamma = 0$. For existence of moments of order m, it is necessary the fulfilment of inequality $2\gamma > m + 1$. Unfortunately, the expression for the probability density in general case does not allow to calculate moments in analytical form, although for referred particular cases they simply calculated. For the model CIR

$$E[X^m] = \Gamma(m+q)/c^m\Gamma(q);$$

for Brennan - Schwartz model

$$E[X^m] = q^m\Gamma(1+c-m)/\Gamma(1+c),$$

the moments of order m exist if the inequality $m < 1 + c$ is fulfilled. So that

Even before the appearance of the model "unrestrictions II" there were used models, which then turned out to be special cases of this model. This is the model of the CIR (1980) [5], which is obtained from the equation (1), if we assume that $\gamma = 1.5$ and $k = 0$. Another particular version is the CEV model, i.e. model of constant elasticity of variance that was proposed J. Cox and S. Ross (1976) [6], as in equation (1) made

$\theta = 0$. Properties of the processes generated by these models can be understood by considering the limiting transition $k \to 0$ in the first model or $\theta \to 0$ in the second. When k and θ still finite the stationary regimes in the models exist and the probability density of processes for these models is expressed in the form (2). However, in the limiting case $k = 0$ or $\theta = 0$ stationary regimes of processes no longer exist, and the probability density can not be expressed in the form (2), and can be obtained as solutions of partial differential equations

$$\frac{\partial f(x,t|y,s)}{\partial t} - \frac{1}{2}\frac{\partial^2[\sigma^2 x^3 f(x,t|y,s)]}{\partial x^2} = 0$$

for model CIR (1980) and

$$\frac{\partial f(x,t|y,s)}{\partial t} + \beta\frac{\partial[x f(x,t|y,s)]}{\partial x} - \frac{\sigma^2}{2}\frac{\partial^2[x^{2\gamma} f(x,t|y,s)]}{\partial x^2} = 0$$

for model CEV at the boundary condition for both equations

$$\lim_{t\to s} f(x,t|y,s) = \delta(x-y).$$

Unfortunately, these equations can not be solved analytically, but we can say that for $k = 0$ or $\theta = 0$ the process generated by the equation (1) becomes unsteady for the CIR model (1980) with the constant expectation and increasing with time variance, and for model CEV changing with time both the expectation and the variance.

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