ON RANDOM GRAPHS IN RANDOM ENVIRONMENT

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Abstract

We consider a configuration graph with N vertices whose degrees are independent identically distributed according to power-law distribution under the condition that the sum of vertex degrees is equal to n. A random graph dynamics as $N, n \to \infty$ to take place in a random environment when parameter of vertex degree distribution following uniform distribution on the finite fixed interval. The limit distributions of the maximum vertex degree and the number of vertices with a given degree were obtained.

1 Introduction

The study of random graphs has been gaining interest in connection with the wide use of these models for the description of different complex networks (see e. g. [3]). One of the ways for constructed such models based on configuration graphs introduced in [2]. Configuration random graphs are being a good implementation of the social, telecommunication networks and Internet topology. While considering real networks it has been noted that they could be adequate representing by random graphs with the vertex degrees being independent identically distributed random variables following the power-law distribution [4]. In [7] it was shown that the distribution of a random variable ξ, being equal to an arbitrary vertex degree could be defined as follows:

$$\mathbf{P}\{\xi = k\} = k^{-\tau} - (k+1)^{-\tau}, \tag{1}$$

where $k = 1, 2, \ldots; \tau > 0$. Moreover in [4] it was found that for present-day complex telecommunication networks the typical values of the distribution (1) parameter τ belongs to the interval (1, 2). Research in the last years showed that configuration power-law random graphs could be used also for modeling forest fires as well as banking system defaults, but in these cases usually $\tau > 2$ [6]. Let N be a number of vertices in the graph and random variables $\xi_1, \ldots, \xi_N$ are equal to the degrees of vertices with the numbers $1, \ldots, N$. These variables are independent and following the distribution (1). The vertex degree is the number of its semiedges, i. e. edges for which adjacent vertices are not yet determined. All of semiedges are numbered in an arbitrary order. The graph is constructed by joining all of the semiedges pairwise equiprobably to form edges. Those models admit multiple edges and loops. The sum of vertex degrees in any graph has to be even, so if the sum $\xi_1 + \ldots + \xi_N$ is odd we add one extra vertex with degree one. In [7] it was note that addition of this vertex together with its semiedge does not influence the graph behaviour as $N \to \infty$. That is why further we will consider only vertex degrees $\xi_1, \ldots, \xi_N$. An interesting fact (see e. g. [1]) that parameter τ of the distribution (1) can be depended on N and even can be random.

We consider the subset of random graphs under the condition that sum of vertex degrees is equal to n. It means that $\xi_1+\ldots+\xi_N = n$ and $\xi_1, \ldots, \xi_N$ are not independent. Such conditional graphs can be useful for modeling of networks for which we can estimate the number of links. They are useful also for studying networks without conditions on the number of edges by averaging the results of conditional graphs with respect to the distribution of the sum of degrees. We assume that as $N \to \infty$ a dynamics of our graph to take place in a random environment when τ is a random variable following uniform distribution on the interval $[a, b], 0 < a < b < \infty$. Then from (1) we find

$$p_1 = \mathbf{P}\{\xi = 1\} = 1 - \frac{1}{(b-a)\ln 2}\left(\frac{1}{2^a} - \frac{1}{2^b}\right),$$

$$p_k = \mathbf{P}\{\xi = k\} = \frac{1}{(b-a)\ln k}\left(\frac{1}{k^a} - \frac{1}{k^b}\right) - \frac{1}{(b-a)\ln(k+1)}\left(\frac{1}{(k+1)^a} - \frac{1}{(k+1)^b}\right),$$

where $k = 2, 3, \ldots$

Denote by $\xi_{(N)}$ and μ_r the maximum vertex degree and the number of vertices with degree r respectively. We obtained the limit distributions of $\xi_{(N)}$ and μ_r as $N, n \to \infty$. The technique of obtaining these results is based on so called generalized allocation scheme supported by V. F. Kolchin [5].

2 Proof Strategy

Let $\eta_1, \ldots, \eta_N$ be auxiliary independent identically distributed random variables such that

$$p_k(\lambda) = \mathbf{P}\{\eta_i = k\} = \lambda^k p_k / B(\lambda), \tag{2}$$

where $k = 1, 2, \ldots;\ i = 1, \ldots, N;\ 0 < \lambda < 1$ and

$$B(\lambda) = \sum_{k=1}^{\infty} \lambda^k p_k.$$

It is readily seen that for our subset of graphs

$$\mathbf{P}\{\xi_1 = k_1, \ldots, \xi_N = k_N\} = \mathbf{P}\{\eta_1 = k_1, \ldots, \eta_N = k_N \,|\eta_1 + \ldots + \eta_N = n\}. \tag{3}$$

This equation means that for random variables $\xi_1, \ldots, \xi_N$ and $\eta_1, \ldots, \eta_N$ the generalized allocation scheme is valid and we can apply the known properties of this scheme to the study of conditional random graphs.

Let $\eta_i^{(r)}, \nu_i^{(r)}, i = 1, \ldots, N$, be two sets of random variables such that

$$\mathbf{P}\{\eta_i^{(r)} = k\} = \mathbf{P}\{\eta_i = k | \eta_i \le r\}, \quad \mathbf{P}\{\nu_i^{(r)} = k\} = \mathbf{P}\{\eta_i = k | \eta_i \ne r\}.$$

It is shown in [5] that from (3) it is not hard to get:

$$\mathbf{P}\{\xi_{(N)} \le r\} = (1 - \mathbf{P}\{\eta_1 > r\})^N \frac{\mathbf{P}\{\eta_1^{(r)} + \ldots + \eta_N^{(r)} = n\}}{\mathbf{P}\{\eta_1 + \ldots + \eta_N = n\}} \tag{4}$$

and

$$\mathbf{P}\{\mu_r = k\} = \binom{N}{k} p_r^k(\lambda)(1 - p_r(\lambda))^{N-k} \frac{\mathbf{P}\{\nu_1^{(r)} + \ldots + \nu_{N-k}^{(r)} = n - kr\}}{\mathbf{P}\{\eta_1 + \ldots + \eta_N = n\}}. \tag{5}$$

From (4) and (5) we see that to obtain the limit distributions of $\xi_{(N)}$ and μ_r it suffices to consider the asymptotic behaviour of the sums of independent random variables, binomial $(1 - \mathbf{P}\{\eta_1 > r\})^N$ and binomial probabilities. By this way we proved the main results of this paper (see the next section).

3 Results

Let parameter $\lambda = \lambda(N, n)$ of the distribution (2) be determined by the relation

$$m = \mathbf{E}\eta_1 = n/N$$

and let also $\sigma^2 = \mathbf{D}\eta_1$. We have the following results.

Theorem 1. *Let $N, n \to \infty$ in such a way that $n/N \to 1, (n-N)^3/N^2 \to \infty$ and sequence $r = r(N, n)$ are minimal natural numbers such that $N\lambda^r p_{r+1}/p_1 \to \gamma$, where γ is a non-negative constant. Then $\mathbf{P}\{\xi_{(N)} = r\} \to e^{-\gamma}$, $\mathbf{P}\{\xi_{(N)} = r+1\} \to 1 - e^{-\gamma}$.*

Theorem 2. *Let $N, n \to \infty$ in such a way that $1 < C_1 \le n/N \le C_2 < \infty$ and $r = r(N, n)$ are chosen such that*

$$\frac{aN\lambda^{r+1}}{(b-a)B(\lambda)r^{a+1}\ln r} \to \gamma,$$

where γ is a positive constant. Then for any fixed $k = 0, \pm 1, \pm 2, \ldots$

$$\mathbf{P}\{\xi_{(N)} \le r + k\} = \exp\{-\gamma\lambda^k(1-\lambda)^{-1}\}(1 + o(1)).$$

Theorem 3. *Let $N, n \to \infty$ in such a way that $n/N \to \infty, a \le 1$ and $N(1-\lambda)^{2+\delta} \to \infty$ for some $\delta > 0$. Then*

$$\mathbf{P}\{|\ln\lambda|\xi_{(N)} - u \le z\} \to e^{-e^{-z}},$$

where $-\infty < z < \infty$ and $u = u(N, n)$ are chosen so that

$$\frac{N|\ln\lambda|^a}{e^u u^{a+1}\ln(u/|\ln\lambda|)} \to \frac{b-a}{a}.$$

Theorem 4. *Let $N, n \to \infty$ in such a way that $n/N \to 1, n - N \to \infty$. Then for $r > 2$*

$$\mathbf{P}\{\mu_r = k\} = \frac{(Np_r(\lambda))^k}{k!} e^{-Np_r(\lambda)}(1 + o(1))$$

uniformly in the integer k such that $(k - Np_r(\lambda))/\sqrt{Np_r(\lambda)}$ lies in any fixed finite interval.

Theorem 5. *Let $N, n \to \infty$ and one of the following conditions hold:*

$$1. \quad 1 < C_1 \leq n/N \leq C_2 < \infty;$$

$$2. \quad a \leq 1, \quad n/N \to \infty, \quad N(1 - \lambda)^{2+\delta} \to \infty,$$

where δ is a some positive constant. Then for any fixed natural r

$$\mathbf{P}\{\mu_r = k\} = (\sigma_{rr}\sqrt{2\pi N}))^{-1} e^{-u_r^2/2}(1 + o(1))$$

uniformly in the integer k such that $u_r = (k - Np_r(\lambda))/(\sigma_{rr}\sqrt{N})$ lies in any fixed finite interval, where

$$\sigma_{rr}^2 = p_r(\lambda)\left(1 - p_r(\lambda) - \frac{(m - r)^2}{\sigma^2} p_r(\lambda)\right).$$

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