ON THE LIMIT DISTRIBUTION OF THE MAXIMUM VERTEX DEGREE IN A CONDITIONAL CONFIGURATION GRAPH

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Abstract

Configuration graphs where vertex degrees are independent identically distributed random variables are often used for modeling complex networks, such as the Internet, social media and others. We consider a random graph consisting of N vertices. The random variables $\eta_1, \dots, \eta_N$ are equal to the degrees of vertices with the numbers $1, \dots, N$. The probability $\mathbf{P}\{\eta_i = k\}$, $i = 1, \dots, N$, is equivalent to $h(k)/k^\tau$ as $k \to \infty$, where $h(x)$ is a slowly varying function integrable in any finite interval, $\tau > 1$. We obtain the limit distribution of the maximum vertex degree under the condition that the sum of degrees is equal to n and $N, n \to \infty$.

1 Introduction

Much attention has been paid to studying the asymptotic behaviour and the structure of random graphs which simulate various complex networks, such as the Internet or telecommunication networks (see e.g. [4], [5]). One of the most commonly used random graphs is the configuration model with the degree of vertices distributed identically and independently. The notion of the configuration graph was introduced in [1] for the first time. The process of graph construction consists of two stages. First, each numbered vertex of such a graph is assigned a certain degree in accordance with a given distribution. The vertex degree is the number of stubs that are numbered in an arbitrary order. Stubs are vertex edges for which adjacent nodes are not yet determined (semiedges). The graph is constructed at the second stage by joining each stub to another equiprobably to form edges. It is clear that we need to use the auxiliary vertex for the sum of degrees to be even. This vertex has the degree 0 if the sum of all other vertices is even, else the degree is 1.

A fundamental trait of many real networks is that the number of nodes with the degree k is near proportional to $k^{-\tau}$, $k \to \infty$, $\tau > 1$. There are many papers where the results describing the limit behaviour of different random graph characteristics were obtained. In [10] the configuration graph was considered where vertex degrees η have the distribution

$$\mathbf{P}\{\eta \geq k\} = h(k)k^{-\tau+1}, \quad k = 1, 2, \dots, \tag{1}$$

where $h(k)$ is a slowly varying function. The authors of this paper are convinced (without proof) that the function $h(k)$ does not influence limit results and that to study the configuration graph one can replace $h(k)$ with the constant 1. Various characteristics of such graphs were studied, for example in [7] the limit theorem for the sum of vertex degrees was obtained. In our work we will show that the role of the slowly varying function $h(k)$ is more complicated.

We consider a random graph where random variables $\eta_1, \dots, \eta_N$ equal to the degrees of vertices with the numbers $1, \dots, N$ have the distributions

$$p_k = \mathbf{P}\{\eta_1 = k\} = \frac{h(k)}{k^\tau \Sigma(1,\tau)}, \tag{2}$$

where $k = 1, 2, \dots,$ $\tau > 1$, $h(x)$ is a slowly varying function integrable in any finite interval and

$$\Sigma(x, y) = \sum_{k=1}^{\infty} x^k \frac{h(k)}{k^y}. \tag{3}$$

Further we consider the subset of random graphs under the condition that $\eta_1 + \ldots + \eta_N = n$. Analysis of conditional random graphs was first carried out in [9]. It is not difficult to see that the addition of function $h(x)$ to the distribution (2) allows to consider this model as a generalization of the random graphs considered in [7]- [9]. For such random graphs, in [2], [3] the limit distributions of the maximum vertex degree were obtained as $n, N \to \infty$ and $1 < n/N \le C < \Sigma(1, \tau - 1)/\Sigma(1, \tau)$, where C is a positive constant and $\Sigma(x, y)$ is determined by the relation (3). Now we obtain the limit distributions of the maximum vertex degree as $n, N \to \infty$ and $n/N \nearrow \Sigma(1, \tau - 1)/\Sigma(1, \tau)$. Note that if $\tau < 2$, then $n/N \to \infty$.

2 The main result

We denote by $\xi_1, \dots, \xi_N$ auxiliary independent identically distributed random variables such that

$$p_k(\lambda) = \mathbf{P}\{\xi_i = k\} = \frac{\lambda^k p_k \Sigma(1,\tau))}{\Sigma(\lambda,\tau)}, \quad i = 1, 2, \dots, N, \quad k = 1, 2, \dots, \quad 0 < \lambda < 1.$$

From this we obtain

$$m = \mathbf{E}\xi_1 = \frac{\Sigma(\lambda, \tau - 1)}{\Sigma(\lambda, \tau)}.$$

Let $\lambda = \lambda(N, n)$ be determined by the relation

$$\frac{\Sigma(\lambda, \tau - 1)}{\Sigma(\lambda, \tau)} = \frac{n}{N}.$$

We introduce the conditions:

(A1) $\quad \tau > 4;$
(A2) $\quad 3 < \tau \le 4, \quad (1-\lambda)^{\tau-4-\epsilon}/\sqrt{\mathrm{N}} \to 0;$
(A3) $\quad 5/2 < \tau \le 3, \quad \mathrm{N}(1-\lambda)^{11-3\tau+\epsilon} \ge \mathrm{C}_3 > 0;$
(A4) $\quad \tau = 5/2, \quad \mathrm{N}(-\ln(1-\lambda))^2(1-\lambda)^{7/2+\epsilon} \ge \mathrm{C}_4 > 0;$
(A5) $\quad 1 < \tau < 5/2, \quad \mathrm{N}(1-\lambda)^{6-\tau+\epsilon} \ge \mathrm{C}_5 > 0,$

where ϵ is some sufficiently small positive constant.

We denote by $\eta_{(N)}$ the maximum vertex degree.

Theorem. *Let $N, n \to \infty$, $n/N \nearrow \Sigma(1,\tau-1)/\Sigma(1,\tau)$, parameters τ, N, n are determined by one of the conditions (A1)–(A5), and $r = r(N,n)$ satisfies*

$$\frac{N\lambda^{r+1}h(r+1)}{(r+1)^{\tau}\Sigma(\lambda,\tau)(1-\lambda)} \to \gamma,$$

where γ is a positive constant. Then for any fixed $k = 0, \pm 1, \ldots$

$$\mathbf{P}\{\eta_{(N)} \le r\} = e^{-\gamma}(1+o(1)).$$

3 Proof of the theorem

The technique for obtaining these theorems is based on the generalized allocation scheme suggested by V.F.Kolchin [6]. It is readily seen that for our subset of graphs

$$\mathbf{P}\{\eta_1 = k_1, \ldots, \eta_N = k_N\} = \mathbf{P}\{\xi_1 = k_1, \ldots, \xi_N = k_N | \xi_1 + \ldots + \xi_N = n\}.$$

Therefore the conditions of the generalized allocation scheme are valid.

Let $\xi_1^{(r)}, \ldots, \xi_N^{(r)}$ and $\tilde{\xi}_1^{(r)}, \ldots, \tilde{\xi}_N^{(r)}$ be two sets of independent identically distributed random variables such that

$$\mathbf{P}\{\xi_1^{(r)} = k\} = \mathbf{P}\{\xi_1 = k | \xi_1 \le r\}.$$

We also put $\zeta_N = \xi_1 + \ldots + \xi_N$, $\zeta_N^{(r)} = \xi_1^{(r)} + \ldots + \xi_N^{(r)}$, $P_r = \mathbf{P}\{\xi_1 > r\}$. It is shown in [6] that

$$\mathbf{P}\{\eta_{(N)} \le r\} = (1-P_r)^N \frac{\mathbf{P}\{\zeta_N^{(r)} = n\}}{\mathbf{P}\{\zeta_N = n\}}. \tag{4}$$

From (4) we see that to obtain the limit distributions of $\eta_{(N)}$ it suffices to consider the asymptotic behaviour of the sums of auxiliary independent identically distributed random variables ζ_N, $\zeta_N^{(r)}$. To solve these problems one has to find both integral and local convergence of the distributions of these sums to limit laws under the conditions of array schemes, which is the main difficulty.

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