EVALUATION OF SEQUENTIAL TEST CHARACTERISTICS FOR TIME SERIES WITH A TREND

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Abstract

The problem of sequential testing of simple hypotheses for time series with a trend is considered. Analytical expressions and asymptotic expansions of error probabilities and expected numbers of observations are obtained. The result is illustrated numerically.
Keywords: sequential test, time series, trend, error probability, expected number of observations

1 Introduction

The sequential approach to test parametric hypotheses was proposed by Wald (see [6]) and is applied in many practical problems of statistical data analysis. The problem of sequential test characteristics (error probabilities and expected number of observations) evaluation is well studied for the case of identical distribution of observations (see [1] – [6]). In this paper, the model of non-identical distribution is considered.

Let $x_1, x_2, ...$ be observations of time series with a trend:

$$x_t = \theta^T \psi(t) + \xi_t,\ t = 1, 2, 3, ..., \tag{1}$$

where $\psi(t) = (\psi_1(t), \psi_2(t), ..., \psi_m(t))^T$, $t \geq 1$, are the vectors of basic functions of trend, $\theta = (\theta_1, \theta_2, ..., \theta_m)^T \in \mathbb{R}^m$ is an unknown vector of coefficients, and $\{\xi_t, t \geq 1\}$ is the sequence of independent identically distributed random variables, $\xi_t \sim N(0, \sigma^2)$.

Consider two simple hypotheses:

$$H_0 : \theta = \theta^0, H_1 : \theta = \theta^1, \tag{2}$$

where $\theta^0, \theta^1 \in \mathbb{R}^m$ are known vectors.

Denote the accumulated log-likelihood ratio statistic:

$$\Lambda_n = \Lambda_n(x_1, x_2, ..., x_n) = \sum_{t=1}^{n} \lambda_t, \tag{3}$$

where $\lambda_t = ln\left(\frac{p_t(x_t, \theta^1)}{p_t(x_t, \theta^0)}\right)$ is the log-likelihood ratio calculated on the observation x_t, and $p_t(x, \theta)$ is the probability density function of x_t provided the parameter value is θ.

To test these hypotheses, after n observations one makes the decision:

$$d = \mathbf{1}_{[C_+,+\infty)}(\Lambda_n) + 2 \cdot \mathbf{1}_{(C_-,C_+)}(\Lambda_n). \tag{4}$$

The thresholds C_- and C_+ are the parameters of the test. Decisions $d = 0$ and $d = 1$ mean stopping of the observation process and acceptance of H_0 or H_1 correspondently. According to Wald (see [6]) we use $C_+ = ln\left(\frac{1-\beta_0}{\alpha_0}\right)$ and $C_- = ln\left(\frac{\beta_0}{1-\alpha_0}\right)$, where α_0, β_0 are the given values for probability errors of types I and II respectively.

2 Main results

Introduce the notation: $E^{(k)}(\cdot), D^{(k)}(\cdot)$ are conditional expected value and variance provided hypothesis H_k is true $(k = 0, 1)$; for $n \geq 1$,

$$\sigma_n^2 = \frac{(\theta^0-\theta^1)^T\psi(n)\psi^T(n)(\theta^0-\theta^1)}{\sigma^2}, \mu_n^{(k)} = \frac{(-1)^{k+1}\sigma_n^2}{2}, s_n^2 = \sum_{t=1}^{n}\sigma_t^2,\ m_n^{(k)} = \frac{(-1)^{k+1}s_n^2}{2},$$

$$A_n = \{a_{ij}\}_{n\times n}, \quad a_{ij} = \begin{cases} 1, & i \geq j, \\ 0, & otherwise; \end{cases} \qquad X_n = (\lambda_1, \lambda_2, ..., \lambda_n)^T,$$

$$T_n = (\Lambda_1, \Lambda_2, ..., \Lambda_n)^T = A_n X_n, \quad \mu_{T_n}^{(k)} = A_n E^{(k)}(X_n), \Sigma_{T_n} = A_n Cov(X_n, X_n) A_n^T;$$

$\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution $N(0,1)$. Put $N = \inf\{n \in \mathbb{N} : \Lambda_n \notin (C_-, C_+)\}, \Gamma = (\theta^0-\theta^1)(\theta^0-\theta^1)^T$ and $H_n = \sum_{i=1}^n \psi(i)\psi^T(i)$. Let α, β be the factual values of the error type I and II probabilities for test (3), (4).

Theorem 1. *If the trace of the matrix ΓH_n tends to $+\infty$ when $n \to +\infty$, then the test terminates finitely with probability 1.*

Proof. The proof is derived from the fact that $P_k(N > n) \leq P_k(\Lambda_n \in (C_-, C_+))$. □

Corollary 1. *If $tr\{\Gamma H_n\}$ is bounded, then there exists a positive constant L so that $s_n^2 \to L$ when $n \to +\infty$. In this case, we have:*

$$\lim_{n\to+\infty} P_k(\Lambda_n \in (C_-, C_+)) = \Phi\left(\frac{2C_+ - (-1)^{k+1}L}{2\sqrt{L}}\right) - \Phi\left(\frac{2C_- - (-1)^{k+1}L}{2\sqrt{L}}\right) > 0.$$

Theorem 2. *Under the Theorem 1 condition following expressions are valid for the characteristics of test* (2)*:*

$$E^{(k)}(N) = 1 + \sum_{i=1}^{+\infty}\int_{C_-}^{C_+} ds_i \int_{C_-}^{C_+} ds_{i-1} ... \int_{C_-}^{C_+} n_i(s, \mu_{T_i}^{(i)}, \Sigma_{T_i}) ds_1,\ k = 0, 1;$$

$$\alpha = \int_{C_+}^{+\infty} n_1(s_1, \mu_1^{(0)}, \sigma_1^2) ds_1 + \sum_{i=2}^{+\infty}\int_{C_+}^{+\infty} ds_i \int_{C_-}^{C_+} ds_{i-1} ... \int_{C_-}^{C_+} n_i(s, \mu_{T_i}^{(0)}, \Sigma_{T_i}) ds_1,$$

$$\beta = \int_{-\infty}^{C_-} n_1(s_1, \mu_1^{(1)}, \sigma_1^2) ds_1 + \sum_{i=2}^{+\infty}\int_{-\infty}^{C_-} ds_i \int_{C_-}^{C_+} ds_{i-1} ... \int_{C_-}^{C_+} n_i(s, \mu_{T_i}^{(1)}, \Sigma_{T_i}) ds_1.$$

Proof. The results above are proved directly by using the properties of multivariate normal distributions. □

Corollary 2. *Under the Theorem 1 condition, the following inequalities hold:*

$$E^{(k)}(N) \le 1 + \sum_{i=1}^{+\infty} \int_{iC_-}^{iC_+} n_1(x, \bar{m}_i^{(k)}, \bar{s}_i^2)dx,\ k = 0, 1;$$

$$\alpha \le 1 - \Phi\left(\frac{C_+ - \mu_1^{(0)}}{\sigma_1}\right) + \sum_{i=2}^{+\infty} \int_{C_+}^{+\infty} \int_{C_-}^{C_+} n_1(x, m_{i-1}^{(0)}, s_{i-1}^2) n_1(y, x + \mu_i^{(0)}, \sigma_i^2) dxdy,$$

$$\beta \le \Phi\left(\frac{C_- - \mu_1^{(1)}}{\sigma_1}\right) + \sum_{i=2}^{+\infty} \int_{-\infty}^{C_-} \int_{C_-}^{C_+} n_1(x, m_{i-1}^{(1)}, s_{i-1}^2) n_1(y, x + \mu_i^{(1)}, \sigma_i^2) dxdy,$$

where $\bar{m}_i^{(k)} = \dfrac{(-1)^{k+1}}{2} \sum\limits_{j=1}^{i} (i+1-j)\sigma_j^2,\ \bar{s}_i^2 = \sum\limits_{j=1}^{i} (i+1-j)^2 \sigma_j^2.$

To construct asymptotic expansions, split the state space of Λ_n into $K+2$ cells:

$$A_0 = (-\infty, C_-),\ A_i = [C_{i-1}, C_i),\ i = \overline{1, K},\ A_{K+1} = [C_+, +\infty)$$

$$C_- = C_0 < C_1 < C_2 < ... < C_K = C_+,\ C_i = C_- + ih,\ h = \frac{C_+ - C_-}{K}, i = \overline{1, K}.$$

Denote $f_{C_-}^{C_+}(x) = \left(\left[\dfrac{x - C_-}{h}\right] + 1\right) \cdot \mathbf{1}_{(C_-, C_+)}(x) + (K+1) \cdot \mathbf{1}_{[C_+, +\infty)}(x).$
For the random sequence Λ_n, let us introduce the discrete random sequence Z_n with the finite state space $V = \{0, 1, ..., K+1\}$. Put $Z_1 = f_{C_-}^{C_+}(\Lambda_1)$ and for $n \ge 2$:

$$Z_n = \begin{cases} 0, & \text{if } Z_{n-1} = 0, \\ K+1, & \text{if } Z_{n-1} = K+1, \\ f_{C_-}^{C_+}(\Lambda_n), & \text{otherwise.} \end{cases}$$

In this case, Z_n is an inhomogeneous Markov chain with a finite state space $\{0, ..., K+1\}$, in which 0 and $K+1$ are absorbing states. In order to simplify the notation, let us renumerate the states space of Z_n: $V = \{\{0\}, \{K+1\}, \{1\}, ..., \{K\}\}$.

Introduce the notation:

$$P^{(n)}(\theta^i) = \left(\begin{array}{c|c} I_2 & \mathbf{O}_{2\times K} \\ \hline R_n(\theta^i) & Q_n(\theta^i) \end{array}\right), \quad i = 0, 1; P^{(n)}(\theta^i) = \{p_{kl}^{(n)}(\theta^i)\}_{(K+2)\times(K+2)},$$

$$p_{kl}^{(n)}(\theta^i) = \frac{\int_{A_k} n_1(y, m_{n-1}^{(i)}, s_{n-1}^2) \int_{A_l} n_1(x, y + \mu_n^{(i)}, \sigma_n^2) dxdy}{\int_{A_k} n_1(y, m_{n-1}^{(i)}, s_{n-1}^2) dy},$$

$$S(\theta^i) = I_K + \sum_{k=1}^{+\infty} \prod_{j=1}^{k+1} Q_j(\theta^i), \quad B(\theta^i) = R_2(\theta^i) + \sum_{k=2}^{+\infty} \prod_{j=1}^{k} Q_j(\theta^i) R_{k+1}(\theta^i);$$

$B_{(j)}(\cdot)$ is the j^{th}-column of matrix $B(\cdot), \pi(\theta^i)$ is the probability distribution of Z_1, $\mathbf{1}_K$ is the vector of size K with all components equal to 1, $t(\theta^i) = E(N|\theta^i), i = \overline{0, 1}.$

Theorem 3. *If* $\inf_n tr(\Gamma\psi(n)\psi^T(n)) \geq C, C = const > 0$, *then the characteristics of the test* (2) *satisfy the following expansions:*

$$t(\theta^i) = 1 + (\pi(\theta^i))' S(\theta^i) \cdot \mathbf{1}_K + O(h),\ i = \overline{0,1};$$

$$\alpha = (\pi(\theta^0))' B_{(2)}(\theta^0) + \pi_{K+1}(\theta^0) + O(h), \quad \beta = (\pi(\theta^1))' B_{(1)}(\theta^1) + \pi_0(\theta^1) + O(h).$$

Proof. The approximations are derived from properties of inhomogeneous Markov chains. □

3 Numerical results

The probability model (1) was considered and the hypotheses (2) were tested by (3), (4) with the following values of parameters:

$$m = 4, \sigma = 2, \psi(t) = (1, t/10, t^2/100, t^3/1000)^T, \theta^0 = (1, 2, 3, 0.9)^T, \theta^1 = (1, 1, 1, 1)^T.$$

The infinite sum was limited to 1000 summands. The thresholds C_-, C_+ were calculated according to Wald. Denote the sample estimate of a characteristic γ with Monte-Carlo method by $\hat{\gamma}$. The number of runs used in this method was 100 000. The results of Corollary 2 are given in Table 1, where $t_i = E(N|\theta^i), i = 0, 1$.

α_0	β_0	$\alpha \leq$	$\beta \leq$	$\hat{\alpha}$	$\hat{\beta}$	$E^{(0)}(N) \leq$	$E^{(1)}(N) \leq$	$\hat{t_0}$	$\hat{t_1}$
0.1	0.1	0.0545	0.0545	0.0477	0.0480	12.674	12.674	9.428	9.434
0.05	0.05	0.0230	0.0230	0.0207	0.0216	13.685	13.685	10.275	10.266
0.01	0.01	0.0037	0.0037	0.0034	0.0036	15.359	15.359	11.532	11.538

Table 1. Upper bounds and Monte-Carlo estimates

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