SOME FRACTIONAL EXTENSIONS OF THE POISSON PROCESS

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In recent years some fractional generalisations of the Poisson process appeared. The space-fractional Poisson process $N^\alpha(t)$, $t > 0$, recently studied, share with the homogeneous Poisson process $N(t), t > 0$, the property of independence of increments. The space-fractional Poisson process $N^\alpha(t)$ is a time-changed Poisson process

$$N^\alpha(t) = N(S^\alpha(t)), \quad 1 < \alpha < 0, \tag{1}$$

where $S^\alpha(t)$ is a stable subordinator. The probability generating function of N^α reads

$$\mathbb{E}u^{N^\alpha(t)} = e^{-t(\lambda(1-u))^\alpha}, \quad t > 0, \lambda > 0, |u| \leq 1. \tag{2}$$

From (2) the probability distribution $p_k(t)$, $k \geq 0$ of $N^\alpha(t)$ can be extracted together with the moments and satisfies the difference-differential equation

$$\frac{dp_k}{dt} = -\lambda^\alpha (I - B)^\alpha p_k, \tag{3}$$

where B is the shift operator.
For the hitting tims T_k of levels k, namely

$$T_k^\alpha := \inf\,(s : N^\alpha(s) = k), \quad k \geq 1, \tag{4}$$

we are able to show that

$$P\{T_k^\alpha < \infty\} = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} \frac{1}{k!} < 1, \quad \forall\, k \geq 1, \tag{5}$$

and study its behavior with respect to k and α.
For the n-times iterated subordinator

$$N^\alpha(S^{\gamma_1}(S^{\gamma_2}(\ldots S^{\gamma_n}(t)))) \stackrel{d}{=} N^{\alpha \prod_{j=1}^n \gamma_j}(t), \quad \gamma_j \in (0,1) \tag{6}$$

we study the limiting behavior. When $\prod_{j=1}^\infty \gamma_j = 0$ the limiting process of (6) is a degenerate r.v. with values 0 and ∞. If $0 < \prod_{j=1}^n \gamma_j < 1$, instead, we still have a space-fractional Poisson process. Also the space-time fractional Poisson process $N^{\alpha,\nu}(t), t > 0$ is studied and we show that it has not a renewal structure. Its p.g.f. has the form

$$G_{\alpha,\nu}(t) = E_{\nu,1}(-t^\nu(\lambda(1-u))^\alpha), \quad |u| \leq 1, \nu, \alpha \in (0,1) \tag{7}$$

and for $\nu = 1$ coincides with the space-fractional Poisson while for $\alpha = 1$ gives the time-fractional Poisson process. A large class of generalized Poisson processes is obtained by considering processes with p.g.f.

$$\mathbb{E}u^{N^f(t)} = e^{-tf(\lambda(1-u))}, \tag{8}$$

where f is a Bernstein function, that is a function with integral representation

$$f(x) = \int_0^\infty (1 - e^{-xs})\nu(ds), \tag{9}$$

ν being the so-called Lévy measure on $(0, \infty)$. For $f = x^\alpha$, we have the space-fractional Poisson process while for different forms of f we have a large class of generalized Poisson processes, sharing the property of independence of increments and the characteristic that jumps have arbitrary size. Furthermore $N^f(t)$ is a time-changed Poisson process

$$N^f(t) = N(S^f(t)), \tag{10}$$

where $S^f(t)$ is a general subordinator related to the Bernstein function f. A special attention is devoted to the cases $f(x) = (x + \lambda)^\alpha - \lambda^\alpha$ and $f(x) = \ln(1 + x)$.

References

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