MIXED POWER VARIATIONS WITH STATISTICAL APPLICATIONS

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Abstract

We obtain results on both weak and almost sure asymptotic behavior of power variations of a linear combination of independent Wiener process and fractional Brownian motion. These results are used to construct strongly consistent parameter estimators in mixed models.

1 Introduction

These results are common with G. Shevchenko and M. Dozzi.

A fractional Brownian motion (fBm) is frequently used to model short- and long-range dependence. By definition, an fBm with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $\{B_t^H, t \geq 0\}$ with the covariance function

$$\mathsf{E}[B_t^H B_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

For $H > 1/2$, an fBm has a property of long-range dependence; for $H < 1/2$, it is short-range dependent and, in fact, is counterpersistent, i.e. its increments are negatively correlated. For $H = 1/2$, an fBm is a standard Wiener process.

Two important properties of an fBm are the stationarity of increments and self-similarity. However, these properties restrict applications of an fBm. So, let us consider some generalizations. A simplest approach is to consider a linear combination

$$X_t = \sum_{k=1}^{N} a_k B_t^{H_k}, t \geq 0, \tag{1}$$

of independent fBms B^{H_k} with different Hurst parameters $H_1 < H_2 < \dots < H_N$.

We consider a particular version of the process (1) with $N = 2$ and one of the Hurst parameters equal to 1/2. In other words, we consider a process

$$M_t^H = aB_t^H + bW_t, t \geq 0 \tag{2}$$

where a and b are some non-zero coefficients. Such process is frequently called a mixed fractional Brownian motion. Its applications were considered in many papers, see [2, 7, 10, 11].

There are only few papers concerned with parameter estimation in the mixed model, but they address questions different from the one we are interested in. In particular, [6, 9] address the estimation of drift parameter in a model with mixed fractional

Brownian motion. In [1], the authors construct several estimators based on discrete variation, so their research is quite close to ours, but they also work in the low-frequency setting, which is essentially different from the high-frequency setting we consider. In both settings, the first order difference of the observed series is a stationary sequence. However, in the low-frequency setting the covariance does not depend on the number of observations, while in the high-frequency one, the covariance structure is very different. As it was mentioned above, for $H > 1/2$, in a small scale the mixed fractional Brownian motion behaves like Wiener process. Thus, the increments of Wiener process become more and more dominating as the partition becomes finer, which makes estimation of the Hurst parameter much harder in the case where $H > 1/2$.

As it was already mentioned, our main aim is the estimation of the parameters of the process (2) based on its single observation on a uniform partition of a fixed interval. To this end, we use power variations of this process.

In the future we plan to consider more advanced techniques as those developed in [1, 4, 5, 8] to construct more efficient estimators.

2 Asymptotic behavior of mixed power variations

Let $W = \{W_t, t \geq 0\}$ be a standard Wiener process and $B^H = \{B^H_t, t \geq 0\}$ be an independent of W fBm with Hurst parameter $H \in (0,1)$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$.

For a function $X : [0,1] \to \mathbb{R}$ and integers $n \geq 1$, $i = 0, 1, \dots, n-1$ we denote $\Delta_i^n X = X_{(i+1)/n} - X_{i/n}$. In this section we will study the asymptotic behavior as $n \to \infty$ of the following mixed power variations

$$\sum_{i=0}^{n-1} (\Delta_i^n W)^p \left(\Delta_i^n B^H\right)^r,$$

where $p \geq 0$, $r \geq 0$ are fixed integer numbers. Since $\Delta_i^n W$ and $\Delta_i^n B^H$ are centered Gaussian with variances $n^{-1/2}$ and n^{-H} respectively, we get that

$$\mathsf{E}\left[(\Delta_i^n W)^p \left(\Delta_i^n B^H\right)^r\right] = n^{-rH-p/2}\mu_p\mu_r,$$

where for an integer $m \geq 1$

$$\mu_m = \mathsf{E}\left[N(0,1)^m\right] = (m-1)!!\mathbf{1}_{m \text{ is even}}$$

is the mth moment of the standard Gaussian law; $(m-1)!! = (m-1)(m-3)\dots$ is the double factorial.

In view of this, we will study centered sums of the form

$$S_n^{H,p,r} = \sum_{i=0}^{n-1} \left(n^{rH+p/2} (\Delta_i^n W)^p \left(\Delta_i^n B^H\right)^r - \mu_p\mu_r\right).$$

We start with studying the almost sure behavior of $S_n^{H,p,r}$. For brevity, the phrase “almost surely” will be omitted throughout the article.

Proposition 1. *Let $\varepsilon > 0$ be arbitrary.*

If $r = 0$, then $S_n^{H,p,r} = o(n^{1/2+\varepsilon})$, $n \to \infty$.

If p and $r \geq 2$ are even, then

- *for $H \in (0, 3/4]$ $\;S_n^{H,p,r} = o(n^{1/2+\varepsilon})$, $n \to \infty$.*
- *for $H \in (3/4, 1)$ $\;S_n^{H,p,r} = o(n^{2H-1+\varepsilon})$, $n \to \infty$.*

If p is odd and $r \geq 1$ is arbitrary, then for any $H \in (0, 1)$ $\;S_n^{H,p,r} = o(n^{1/2+\varepsilon})$, $n \to \infty$.

If p is even and r is odd, then

- *for $H \in (0, 1/2]$ $\;S_n^{H,p,r} = o(n^{1/2+\varepsilon})$, $n \to \infty$.*
- *for $H \in (1/2, 1)$ $\;S_n^{H,p,r} = o(n^{H+\varepsilon})$, $n \to \infty$.*

In particular, for any $H \in (0, 1)$ the following version of the ergodic theorem takes place: $S_n^{H,p,r} \to 0$, $n \to \infty$.

The following theorem summarizes the weak limit behaviour of $S_n^{H,p,r}$. We remark that some (but not all) of the results can be obtained either from the limit theorems for stationary Gaussian sequences of vectors, see e.g. [3] or from the limit theorems for arrays of Gaussian vectors, see [4]. However, we believe that our approach (using one-dimensional limit theorems) is more accessible and leads quicker to the desired results.

Denote

$$\rho_H(m) = \mathsf{E}\left[B_1^H(B_{m+1}^H - B_m^H)\right] = \frac{1}{2}\left(|m+1|^{2H} + |m-1|^{2H} - 2m^{2H}\right)$$

the covariance of the so-called fractional Gaussian noise $\{B_{k+1}^H - B_k^H\}$. It is easy to see that $\rho_H(m) \sim H(2H-1)m^{2H-2}$, $m \to \infty$.

Theorem 1. *If p and r are even, $r \geq 2$, then*

- *for $H \in (0, 3/4)$*

$$n^{-1/2} S_n^{H,p,r} \Rightarrow N(0, \sigma_{H,r}^2 \mu_p^2 + \sigma_{p,r}^2),\ n \to \infty, \tag{3}$$

  where

$$\sigma_{H,r}^2 = \sum_{l=1}^{r/2} \frac{(l!)^2}{(2l)!((r-2l)!!)^2} \sum_{m=-\infty}^{\infty} \rho_H(m)^{2l}, \quad \sigma_{p,r}^2 = \mu_{2r}\left(\mu_{2p} - \mu_p^2\right);$$

- *for $H = 3/4$*

$$\frac{S_n^{3/4,p,r}}{\sqrt{n \log n}} \Rightarrow N(0, \sigma_{3/4,r}^2 \mu_p^2 + \sigma_{p,r}^2),\ n \to \infty, \tag{4}$$

  where $\sigma_{3/4,r} = 3r(r-1)/4$;

- *for* $H \in (3/4, 1)$

$$n^{1-2H} S_n^{H,p,r} \Rightarrow \zeta_{H,p,r}, \ n \to \infty, \tag{5}$$

where $\zeta_{H,p,r}$ *is a special "Rosenblatt" random variable.*

If p *is odd and* $r \geq 1$ *is arbitrary, then for any* $H \in (0, 1)$

$$n^{-1/2} S_n^{H,p,r} \Rightarrow N(0, \mu_{2p}\mu_{2r}). \tag{6}$$

If p *is even and* r *is odd, then*

- *for* $H \in (0, 1/2]$

$$n^{-1/2} S_n^{H,p,r} \Rightarrow N(0, \sigma_{H,r}^2 \mu_p^2 + \sigma_{p,r}^2), \ n \to \infty, \tag{7}$$

where $\sigma_{H,1} = 0$,

$$\sigma_{H,r}^2 = \sum_{l=1}^{(r-1)/2} \frac{(r!)^2}{(2l+1)!((r-2l-1)!!)^2} \sum_{m=-\infty}^{\infty} \rho_H(m)^{2l+1}, \quad r \geq 3;$$

- *for* $H \in (1/2, 1)$

$$n^{-H} S_n^{H,p,r} \Rightarrow N(0, \mu_p^2 \mu_{r+1}^2), \ n \to \infty. \tag{8}$$

Remark. For $r = 0$ we have the pure Wiener case, so for any $H \in (0, 1)$

$$n^{-1/2} S_n^{H,p,r} \Rightarrow N(0, \mu_{2p} - \mu_p^2), \quad n \to \infty.$$

3 Statistical estimation in mixed model based on quadratic variation

Now we turn to the question of parametric estimation in the mixed model

$$M_t^H = aB_t^H + bW_t, \ t \in [0, T], \tag{9}$$

where a, b are non-zero numbers, which we assume to be positive, without loss of generality. Our primary goal is to construct a strongly consistent estimator for the Hurst parameter H, given a single observation of M^H.

It is well-known (see [7]) that for $H \in (3/4, 1)$ the measure induced by M^H in $C[0, T]$ is equivalent to that of bW. Therefore, the property of almost sure convergence in this case is independent of H. Consequently, no strongly consistent estimator for $H \in (3/4, 1)$ based on a single observation of M^H exists.

In this section we denote $\Delta_i^n X = X_{T(i+1)/n} - X_{Ti/n}$ and

$$V_n^{H,p,r} = \sum_{i=0}^{n-1} (\Delta_i^n W)^p \left(\Delta_i^n B^H\right)^r.$$

Consider the quadratic variation of M^H, i.e.

$$V_n^{H,2} := \sum_{i=0}^{n-1} \left(\Delta_i^n M^H\right)^2 = a^2 V_n^{H,0,2} + 2ab V_n^{H,1,1} + b^2 V_n^{H,2,0}.$$

Note that $V_n^{H,2}$ depends only on the observed process but not on H. We use this notation to specify the distribution. Namely, we will use it to refer to the limit behavior of the quadratic variation for a specified value of the Hurst parameter H.

By Proposition 1, we have that $V_n^{H,0,2} \sim T^{2H} n^{1-2H}$, $V_n^{H,2,0} \to T$, $V_n^{H,1,1} = o(n^{1/2-H})$, $n \to \infty$. Therefore, the asymptotic behavior of $V_n^{H,2}$ depends on whether $H < 1/2$ or not. Precisely, for $H \in (0, 1/2)$,

$$V_n^{H,2} \sim a^2 T^{2H} n^{1-2H},\ n \to \infty, \tag{10}$$

so the quadratic variation behaves similarly to that of a scaled fBm.

For $H \in (1/2, 1)$,

$$V_n^{H,2} \to b^2 T,\ n \to \infty, \tag{11}$$

so the quadratic variation behaves similarly to that of a scaled Wiener process.

Let us consider the cases $H < 1/2$ and $H > 1/2$ individually in more detail.

3.1 $H \in (0, 1/2)$

We have seen above that this case is similar to the pure fBm case. Unsurprisingly, the same estimators work, which is precisely stated below.

Theorem 2. *For $H \in (0, 1/2)$, the following statistics*

$$\widehat{H}_k = \frac{1}{2}\left(1 - \frac{1}{k}\log_2 V_{2^k}^{H,2}\right)$$

and

$$\widetilde{H}_k = \frac{1}{2}\left(\log_2 \frac{V_{2^{k-1}}^{H,2}}{V_{2^k}^{H,2}} + 1\right)$$

are strongly consistent estimators of the Hurst parameter H.

Remark. At the first sight, there is no clear advantage of $\widehat{H}_k$ or $\widetilde{H}_k$. But a careful analysis shows that $\widetilde{H}_k$ is better. Indeed, it is easy to see that

$$\widehat{H}_k = H - \frac{\log_2 a + H \log_2 T}{k} + o(k^{-1}), k \to \infty, \tag{12}$$

while

$$\widetilde{H}_k = H + O(2^{k(2H-1)}) + o(2^{k(-1/2+\varepsilon)}), k \to \infty. \tag{13}$$

Now it is absolutely clear that $\widetilde{H}_k$ performs much better (unless one hits the jackpot by having $aT^H = 1$).

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