On the application of a viscoelastic model with Rabotnov's fractional exponential function for assessment of the stress-strain state of the periodontal ligament

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Abstract – In the present paper, the mathematical modeling of the viscoelastic periodontal membrane is carried out. Internal surface of the periodontal ligament is adjacent to the outer surface of the tooth root, the geometric shape of which is described by the equation of an elliptic hyperboloid. The external surface of the periodontal membrane is shifted along the normal to the outer surface of the root and fixed on the dental alveolus bone. Relationships between displacements and deformations of the periodontal ligament are formulated with due account for incompressibility of the periodontal tissue. Viscoelastic properties of the periodontium are described by the relaxation kernel with Rabotnov's fractional exponential function. A system of equations of motion in terms of the translational displacements and rotation angles of the tooth root is obtained. Particular cases of the equations of motion corresponding to the translational motion in the vertical and horizontal directions are formulated.

Keywords – elliptical hyperboloid, equations of motion, Rabotnov's fractional exponential function, periodontal ligament, single-root tooth, translational displacements, fractional calculus viscoelastic model.

I. Introduction

THE periodontal ligament is a thin membrane which keeps the root of the tooth within the alveolus bone, amortizes and distributes the occlusal load on the tooth by means of the collagen fibers. Under normal conditions, the contact between the tooth root and bone tissue is absent. The load acting on the tooth crown is transmitted to the alveolar bone via the periodontal ligament. Short term (initial) and long-term (orthodontic) tooth displacements, generally considered as regulated by the strains and stresses of the periodontal ligament because teeth are assumed to be almost completely rigid, are connected with almost the same rigid alveolar bone [1]–[5]. These circumstances explain the urgency of developing a mathematical model of the periodontal ligament, which allows one to determine the stress-strain state of periodontal tissue under the short-term and long-term loads.

Viscoelastic equations of the state of the periodontal ligament allow one to describe adequately the function of tissues of the supporting apparatus of the tooth without using simplified or too complex mathematical models [6], [7]. In particular, a viscoelastic model enables one to avoid discrepancies in the values of physiological and calculated stresses in the periodontal tissues, as well as to explain the dependence of the physiological response of periodontal tissue to the action of the load with time and to combine the motion of a nonstationary viscous liquid phase with instantaneous, like a rigid body, behavior [8], [9].

The known existing viscoelastic models are based on the use of the Maxwell single element [10], the Kelvin-Voigt model (spring and shock absorber in parallel connection) [11], [12], or nonlinear springs with three parameters [13], [14]. Attempt to execute the simulation of a periodontal ligament via a linear viscoelastic law has been undertaken in [15] and [16]. Nevertheless, it is shown in [17] that the nonlinear simulation of properties of periodontal tissue provides a more accurate and reliable calculation of stresses and strains in a wide range of tooth displacements. Some researchers have demonstrated the viscoelastic behavior of the periodontal ligament of human and primate, but have not offered a quantitative description [18]–[21]. An improved approach to the study of the mechanical behavior of periodontium features based on the quasi-linear viscoelastic phenomenological model was proposed in [22]. However, these results have been questioned, since the nonlinear behavior of the periodontal ligament may not be well described by a quasi-linear viscoelastic theory, which is used usually in biomechanics of tissue [23].

The most important results related to the finite element calculation of the viscoelastic models of the "root of the tooth–periodontal ligament–alveolar bone" could be found in [24]–[27].

A review of the results concerning the viscoelastic models of the periodontal ligament shows that an accurate information about the relationships between the viscoelastic response and

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periodontium structure are lacking, as well as a unified approach to the description of the properties of periodontal tissue is absent. Great bulk research on the properties and behavior of periodontium still treats it as a linearly elastic material [7]. A universal model of the periodontal ligament allowing one to describe the phenomenological behavior of periodontium under a system of forces acting for a long time (orthodontics) or discontinuous load (occlusal load) is not formulated. To generalize viscoelastic models corresponding to different types of loading the periodontal ligament, a viscoelastic model with fractional operators involving relaxed and nonrelaxed moduli, the relaxation time and the fractional parameter (the order of a fractional derivative or fractional operator) could be used [28]. Moreover, such models have been successfully applied for solving the problems of solid mechanics [29]–[33] and biomechanics [34].

The aim of this work is the formulation of the equations of motion of the periodontal ligament using a viscoelastic model with relaxed and nonrelaxed moduli, relaxation time and the fractional parameter, which allows one to determine the translational displacements and rotation angles of the periodontium points under the action of external loading.

II. Mathematical Model of the Periodontal Ligament

The outer surface of the tooth root and the adjacent inner surface of the periodontal ligament (we assume that the root of the tooth is an absolutely rigid body) are described by an elliptic hyperboloid:

$$F(x, y, z) = y - \frac{h}{\sqrt{1+p^2}-p} \times \left(\sqrt{\left(1-e^2\right)\left(\frac{x}{b}\right)^2 + \left(\frac{z}{b}\right)^2 + p^2} - p \right) = 0, \tag{1}$$

where h is the height of the tooth root, $e=\sqrt{1-(b/a)^2}$ is the ellipse eccentricity in the cross section of a tooth in an alveolar crest, a and b are the axles of ellipse in the cross section of the tooth root, and p is a parameter of rounding of the tooth root.

The internal surface of the periodontal ligament adjacent to the dental alveoli bone is shifted along the normal to the surface of the tooth root on the value δ. Its equation is as follows:

$$F_1(x, y, z) = y + n_y\delta - \frac{h}{\sqrt{1+p^2}-p} \times \left(\sqrt{\left(1-e^2\right)\left(\frac{1}{b}\left(x+n_x\delta\right)\right)^2 + \left(\frac{1}{b}\left(z+n_z\delta\right)\right)^2 + p^2} - p \right) = 0, \tag{2}$$

where n_x, n_y, and n_z are components of the unit normal vector to the surface (1). Considering (1), these components are defined as follows:

$$n_x = -\frac{1}{\Delta}\frac{h\left(1-e^2\right)x}{A}, \quad n_y = \frac{1}{\Delta}, \quad n_z = -\frac{1}{\Delta}\frac{hz}{A}, \tag{3}$$

$$A = b\left(\sqrt{1+p^2}-p\right)\sqrt{\left(1-e^2\right)x^2+z^2+(bp)^2},$$

$$\Delta = \sqrt{1+\frac{h^2\left(\left(1-e^2\right)x^2+z^2\right)}{b^2\left(\sqrt{1+p^2}-p\right)\left((bp)^2+\left(1-e^2\right)x^2+z^2\right)}}.$$

Under the action of a concentrated force on a tooth, the points of the periodontal ligament contiguous to surface of the tooth root (1) begin to experience some displacements, which are equal to those of the root. The external surface of the periodontal ligament (2) is fixed. There are no significant differences between the model calculations considering the fixing of the outer surface of the periodontal ligament in the alveolar bone or it rigid fixing. Therefore, when calculating the initial movement of the teeth in the periodontal ligament, both the teeth and the alveolar bone could be defined as solids [35].

Further we will consider a periodontal incompressible material with Poisson's ratio equal to 0.49. This means that the periodontal tissue begins to flow around the surface of the root of the tooth when the root is displaced to the wall of the dental alveolus [36]. Therefore, the components of the strain tensor in the coordinate system associated with the normal, generatrix and guide to the external surface of the tooth root could be represented as follows [36, 37]:

$$\varepsilon_{nn} = -\frac{u_n}{\delta},\ \varepsilon_{tt} = \varepsilon_{\theta\theta} = 0,\ \gamma_{n\theta} = -\frac{u_\theta}{\delta},\ \gamma_{nt} = -\frac{u_t}{\delta},\ \gamma_{t\theta} = 0, \tag{4}$$

where u_n, u_t and u_θ are the displacements of the periodontium points along the normal, generatrix and guide to the tooth surface, respectively, and δ is a width of the periodontal ligament in the normal direction. The normal, generatrix and guide to the root surface of the tooth, as well as its geometrical dimensions are shown in Fig. 1.

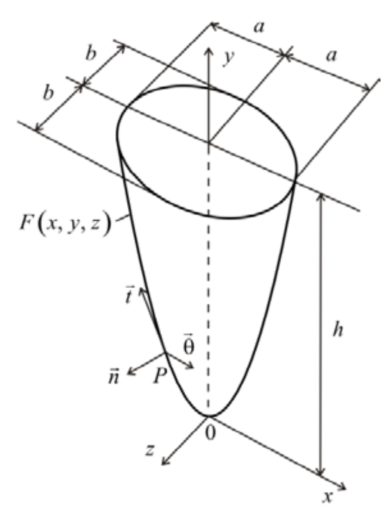

Fig. 1 root of the tooth in the elliptic hyperboloid form: $\vec{n}$ is the normal, t is the generatrix, and θ is the guide to the surface of the hyperboloid at the point P

Let us express the components of the strain tensor in the coordinate system (x, y, z) in terms of the components of the strain tensor in the coordinate system (n, t, θ) [37]:

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} = T_2 \cdot T_1 \cdot \begin{pmatrix} \varepsilon_{nn} & \varepsilon_{tn} & \varepsilon_{\theta n} \\ \varepsilon_{tn} & 0 & 0 \\ \varepsilon_{\theta n} & 0 & 0 \end{pmatrix} \cdot T_1^{\mathrm{T}} \cdot T_2^{\mathrm{T}}, \tag{5}$$

$$\varepsilon_{tn} = \frac{1}{2}\gamma_{tn}, \varepsilon_{\theta n} = \frac{1}{2}\gamma_{\theta n},$$

$$T_1 = \begin{pmatrix} \sin(\alpha) & \cos(\alpha) & 0 \\ -\cos(\alpha) & \sin(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} H & 0 & -G \\ 0 & 1 & 0 \\ G & 0 & H \end{pmatrix},$$

$$H = \frac{x\left(1-e^2\right)}{\sqrt{x^2\left(1-e^2\right)^2 + z^2}}, G = \frac{z}{\sqrt{x^2+z^2}},$$

where T_1 is the rotation matrix relative to the guide $\vec{\theta}$ on the angle α, T_2 is the rotation matrix relative to the z-axis on the angle φ, and T_1^{T} and T_2^{T} are the matrixes transpose of the matrixes T_1 and T_2, respectively. In the matrix T_1, the angle α between the generatrix to the root surface and the xz-plane is given by

$$\mathrm{tg}(\alpha) = \frac{h\sqrt{\left(1-e^2\right)^2 x^2 + z^2}}{b\left(\sqrt{1+p^2} - p\right)\sqrt{(bp)^2 + \left(1-e^2\right)x^2 + z^2}}.$$

Transform the displacement vector (u_n, u_t, u_θ) of the point on the external surface of the tooth root (the internal surface of the periodontal ligament) from the coordinate system (n, t, θ) to the coordinate system (x, y, z):

$$\begin{pmatrix} u_n \\ u_t \\ u_\theta \end{pmatrix} = T_1^T \cdot T_2^T \cdot \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}. \tag{6}$$

Substituting sequentially (4) and (6) in (5), we obtain

$$\begin{aligned} &\varepsilon_{11} = -\frac{H\left(G^2+H^2\right)u_x \sin\alpha}{\delta}, \quad \varepsilon_{22} = \frac{u_y \cos\alpha}{\delta}, \\ &\varepsilon_{33} = -\frac{G\left(G^2+H^2\right)u_z \sin\alpha}{\delta}, \\ &\varepsilon_{12} = \frac{\left(G^2+H^2\right)u_x \cos\alpha - Hu_y \sin\alpha}{2\delta}, \\ &\varepsilon_{23} = \frac{\left(G^2+H^2\right)u_z \cos\alpha - Gu_y \sin\alpha}{2\delta}, \\ &\varepsilon_{13} = -\frac{\left(G^2+H^2\right)\left(u_z H + Gu_x\right)\sin\alpha}{2\delta}. \end{aligned} \tag{7}$$

Any displacements of the tooth root could be described by a combination of the translational displacements u_{0x}, u_{0y}, and u_{0z} along the coordinate axes and the angles of rotation θ_x, θ_y, and θ_z relative to the root apex of the same axes. Since the thickness of periodontium is very small, the rotation angles are also very small. Therefore, we can use the following linearized formula [37]:

$$\begin{aligned} u_x &= u_{0x} + z\theta_y - y\theta_z, \\ u_y &= u_{0y} - z\theta_x + x\theta_z, \\ u_z &= u_{0z} + y\theta_x - x\theta_y. \end{aligned} \tag{8}$$

To find the translational displacements and the rotation angles, the conditions of the dynamic equilibrium of the tooth root are utilized:

$$\begin{aligned} &\iint_F (\vec{n} \cdot \sigma) dF + M\frac{d^2\vec{u}_0}{dt^2} - \vec{P} = 0, \\ &\iint_F \vec{r} \times (\vec{n} \cdot \sigma) dF + J\frac{d^2\vec{\theta}}{dt^2} - \vec{m} = 0, \end{aligned} \tag{9}$$

where $\vec{m} = \left(m_x, m_y, m_z\right)$ is the principal moment of external forces, $\vec{f} = \left(f_x, f_y, f_z\right)$ is the principal vector of external forces, $\vec{r}$ is the radius-vector, $\vec{n} = \left(n_x, n_y, n_z\right)$ is the unit normal vector to the surface (1), σ is the stress tensor, M is the mass of the tooth root, J is the axial moment of inertia of the tooth root, $\vec{u}_0 = \left(u_{0x}, u_{0y}, u_{0z}\right)$ is the vector of translational displacements of the tooth root along the coordinate axes, and $\vec{\theta} = \left(\theta_x, \theta_y, \theta_z\right)$ is the vector of rotation angles of the tooth root with respect to the coordinate axes.

The relationships between the components of the stress tensor and the strain tensor taking the viscoelastic properties of the periodontal ligament into account are represented in the following form:

$$\sigma_{ij} = \frac{E_\infty}{(2\nu-1)(1+\nu)}\Big[(2\nu-1)\varepsilon_{ij} - \nu_\varepsilon \int_0^t \ni_\gamma(-\tau/\tau_\varepsilon)\varepsilon_{ij}(t-\tau)\,d\tau + \nu\delta_{ij}\left(\sum_{k=1}^{3}\varepsilon_{kk} - \nu_\varepsilon\int_0^t \ni_\gamma(-\tau/\tau_\varepsilon)\sum_{k=1}^{3}\varepsilon_{kk}(t-\tau)d\tau\right)\Big], \tag{10}$$

where τ_ε is the relaxation time, $\nu_\varepsilon = \frac{E_\infty - E_0}{E_\infty}$, E_0 and E_∞ are the relaxed (prolonged modulus of elasticity, or the rubbery modulus) and nonrelaxed (instantaneous modulus of elasticity, or the glassy modulus) magnitudes of the elastic modulus, respectively [28], and $\ni_\gamma(-\tau/\tau_\varepsilon)$ is the Rabotnov fractional exponential function, which describes the relaxation of volume and shear stresses [33]. It was introduced in 1948 by Rabotnov [38] in the form

$$\ni_\gamma(-t/\tau_\varepsilon) = \frac{t^{\gamma-1}}{\tau_\varepsilon^\gamma}\sum_{n=0}^{\infty}(-1)^n \frac{(t/\tau_\varepsilon)^{\gamma n}}{\Gamma(\gamma(n+1))},$$

where $0 < \gamma < 1$ is a fractional parameter.

Substitute (3), (5), (9) and (10) in (5). After transformations, we obtain a system of homogeneous algebraic equations with respect to the translational displacements and the rotation angles of the tooth root of the following form:

$$\begin{gathered}
c_x u_{0x} - \nu_\varepsilon\int_0^t \ni_\gamma(-\tau/\tau_\varepsilon)c_x u_{0x}(t-\tau)d\tau + c_{\theta xy}\theta_z \\
-\nu_\varepsilon\int_0^t \ni_\gamma(-\tau/\tau_\varepsilon)c_{\theta xy}\theta_z(t-\tau)d\tau + M\frac{d^2u_{0x}}{dt^2} = f_x, \\
c_y u_{0y} - \nu_\varepsilon\int_0^t \ni_\gamma(-\tau/\tau_\varepsilon)c_y u_{0y}(t-\tau)d\tau + M\frac{d^2u_{0y}}{dt^2} = f_y, \\
c_z u_{0z} - \nu_\varepsilon\int_0^t \ni_\gamma(-\tau/\tau_\varepsilon)c_z u_{0z}(t-\tau)d\tau + c_{\theta yz}\theta_x \\
-\nu_\varepsilon\int_0^t \ni_\gamma(-\tau/\tau_\varepsilon)c_{\theta yz}\theta_x(t-\tau)d\tau + M\frac{d^2u_{0z}}{dt^2} = f_x, \\
c_{\theta z} u_{0z} - \nu_\varepsilon\int_0^t \ni_\gamma(-\tau/\tau_\varepsilon)c_{\theta z} u_{0z}(t-\tau)d\tau + \mu_x\theta_x \\
-\nu_\varepsilon\int_0^t \ni_\gamma(-\tau/\tau_\varepsilon)\mu_z\theta_x(t-\tau)d\tau + J_x\frac{d^2\theta_x}{dt^2} \\
= y_f f_z - z_f f_y, \\
\mu_y\theta_y - \nu_\varepsilon\int_0^t \ni_\gamma(-\tau/\tau_\varepsilon)\mu_y\theta_y(t-\tau)d\tau + J_y\frac{d^2\theta_y}{dt^2} \\
= z_f f_x - x_f f_z, \\
c_{\theta x} u_{0x} - \nu_\varepsilon\int_0^t \ni_\gamma(-\tau/\tau_\varepsilon)c_{\theta x} u_{0x}(t-\tau)d\tau + \mu_z\theta_z \\
-\nu_\varepsilon\int_0^t \ni_\gamma(-\tau/\tau_\varepsilon)\mu_z\theta_z(t-\tau)d\tau + J_z\frac{d^2\theta_z}{dt^2} \\
= x_f f_y - y_f f_x,
\end{gathered} \tag{11}$$

where c_x, c_y, and c_z are the stiffness coefficients of the periodontal ligament at the tooth root translation along the coordinate axes, $c_{\theta xy}$ and $c_{\theta yz}$ are the static moments of stiffness, $c_{\theta x}$ and $c_{\theta z}$ are the stiffness coefficients of the periodontal ligament at the tooth root rotations relative to the x-axis and z-axis, respectively, under the force acting along this coordinate axis, μ_x, μ_y, and μ_z are the stiffness coefficients of the periodontal ligament at the tooth root rotations relative to the axes x, y and z, respectively, and x_f, y_f and z_f are the coordinates of the point where the load is applied.

We note that the stiffness of the periodontal ligament and the moments of stiffness depend on the geometrical shape of the tooth root, Poisson's ratio and the relaxed and nonrelaxed elastic moduli of periodontal tissue and are time-independent. Therefore, the stiffness and the moments of stiffness could be eliminated from the integrals in (11).

III. Particular Cases of Movement of the Tooth Root

To find the material constants and the relaxation time, the experimental data on the stress-strain state of the periodontal ligament could be used and, in particular, the time-dependence of the periodontal points displacements. Typically, such data are obtained for the translational movement of the tooth root in the vertical and horizontal directions under the action of load, which takes on discrete values with time, or which is changed with a predetermined frequency.

During the motion of the tooth root along the y-axis, the corresponding extrusion (or intrusion), the translational displacement along the x- and z-axes, as well as the angles of rotation are equal to zero, i.e., $u_{0x} = u_{0z} = 0$, and $\theta_x = \theta_y = \theta_z = 0$. Load is acting only in the y-axis direction. In this case, from (11) we obtain

$$c_y\left(u_{0y} - \nu_\varepsilon\int_0^t \ni_\gamma(-\tau/\tau_\varepsilon)u_{0y}(t-\tau)d\tau\right) + M\frac{d^2u_{0y}}{dt^2} = f_y, \tag{12}$$

where

$$c_y = \iint_F E_\infty\Big[2b(\nu-1)A_1B^2\cos\alpha + h(2\nu-1)\big((1-e^2)Hx + Gz\big)B\sin\alpha\Big]\frac{dF}{C},$$

$$A_1 = \sqrt{p^2+1} - p, \; B = \sqrt{b^2p^2 + (1-e^2)x^2 + z^2},$$

$$C = 2b(1+\nu)(2\nu-1)A_1B^2\delta.$$

Under the action of a periodically varying load, particularly under the vertical component of the mastication load, relationship (12) can be represented as

$$c_y\left(u_{0y}-\nu_\varepsilon\int_0^t \Im_\gamma\left(-\tau/\tau_\varepsilon\right)u_{0y}\left(t-\tau\right)d\tau\right)$$
$$+M\frac{d^2u_{0y}}{dt^2}=F\sin\left(\omega t\right), \tag{13}$$

Note that (12) and (13) are similar to the equations of motion of a viscoelastic oscillator considered in [39] and [40].

Equations of motion for the translational displacement of the tooth root in a horizontal plane, in particular, in the x-axis direction, we obtain from (11) that $u_{0y}=u_{0z}=0$ and $\theta_x=\theta_y=\theta_z=0$. The load is acting along the x-axis, and the line of action of the force passes through the center of resistance of the tooth root with the coordinates $(0,\,y_1,\,0)$. As a result, we have

$$c_xu_{0x}-\nu_\varepsilon\int_0^t \Im_\gamma\left(-\tau/\tau_\varepsilon\right)c_xu_{0x}\left(t-\tau\right)d\tau+M\frac{d^2u_{0x}}{dt^2}=f_x, \tag{14}$$
$$c_{\theta x}u_{0x}-\nu_\varepsilon\int_0^t \Im_\gamma\left(-\tau/\tau_\varepsilon\right)c_{\theta x}u_{0x}\left(t-\tau\right)d\tau=-y_1f_x,$$
$$c_x=\iint_F E_\infty\left(G^2+H^2\right)\Big\{b(2\nu-1)A_1B^2\cos\alpha$$
$$+h\Big[\left(2(1-e^2)H(\nu-1)x+G(2\nu-1)z\right]B\sin\alpha\Big\}\frac{dF}{C},$$
$$c_{\theta x}=-\iint_F E_\infty\left(G^2+H^2\right)$$
$$\times\Big\{(2\nu-1)\Big[A_1B^2by+\left(1-e^2\right)hx^2B\Big]\cos\alpha$$
$$+\Big[2bH\nu A_1xB^2+hy\left(2(1-e^2)H(\nu-1)x\right.$$
$$\left.+G(2\nu-1)z\right)B\Big]\sin\alpha\Big\}\frac{dF}{C}.$$

To formulate the system of equations describing the translational motion of the tooth root in the x-axis, in (11) it is necessary to vanish to zero the displacements u_{0x} and u_{0y} and all angles of rotation. In this case, only the z-component of load is acting on a tooth, and its line of action passes through the center of resistance with the coordinates $(0,\,y_2,\,0)$.

IV. Conclusion

To generalize viscoelastic models corresponding to different types of modeling the stress-strain state of the periodontal ligament under the action of concentrated forces and moments, the equations of motion of the tooth root involving the fractional exponential function are suggested. The advantage of this model is the use of the fractional parameter to describe the various pathological processes and age-related changes in the periodontium. Fractional parameter allows one to take the different behavior of the periodontal tissue during the action of different short-term and long-term loads into account. To find the material constants, the experimental data on the intrusion or extrusion of the tooth [41] - [44] can be used together with the solution of (12), or the results of experiments on the cyclic loading of the tooth [45, 46] together with the solution of (13), as well as the experimental data on the root translational displacement in a horizontal plane [47] together with the solution of (14).

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