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MODELING PARAMETERS OF THE LOWER AND UPPER BOUNDS AND PARAMETERS OF THE OBJECTIVE FUNCTION FOR GENERALIZED NETWORK FLOW PROGRAMMING PROBLEMS

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Abstract. In this paper, we propose the mathematical models of the inverse optimization for modeling parameters of the lower and upper bounds and parameters of the objective function for one generalized inhomogeneous linear network flow programming problem under l_1 norm. We have considered the numerical example of modeling of the parameters of the lower and upper bounds, where new parameters are adjusted as little as possible so, that the infeasible solution becomes the feasible solution for the new values parameters.

STATEMENT OF PROBLEM

Let $G = (I, U)$ be a finite oriented connected multigraph without loops with set of nodes I and set of multiarcs U, $|U| \gg |I|$. Let $K(|K| < \infty)$ be a set of different types of flow transported in G. We assume that $K = \{1, \cdots, |K|\}$. Let us denote a connected network corresponding to a certain type of flow $k \in K$: $G^k = (I^k, U^k)$, $I^k \subseteq I$, $U^k = \{(i, j)^k : (i, j) \in \widehat{U}^k\}$, $\widehat{U}^k \subseteq U$ – a set of multiarcs of G carrying the flow of type $k \in K$. Also, we define for each multiarc $(i, j) \in U$ the set $K(i, j) = \{k \in K : (i, j)^k \in U^k\}$ of types of flow transported through the multiarc (i, j). Consider the following mathematical model:

$$f(x) = \sum_{(i,j)\in U} \sum_{k\in K(i,j)} c^k_{ij} x^k_{ij} + \sum_{k\in K} \sum_{i\in I^*_k} c^k_i x^k_i \longrightarrow \min, \tag{1}$$

$$\sum_{j\in I^+_i(U^k)} x^k_{ij} - \sum_{j\in I^-_i(U^k)} \mu^k_{ji} x^k_{ji} = \begin{cases} a^k_i + x^k_i, & i \in I^*_k, \\ a^k_i, & i \in I^k \backslash I^*_k, \end{cases} \quad k \in K, \tag{2}$$

$$\sum_{k\in K} \sum_{(i,j)^k\in U^k} \lambda^{k,p}_{ij} x^k_{ij} + \sum_{k\in K} \sum_{i\in I^*_k} \lambda^{k,p}_i x^k_i = \beta_p, \quad \text{for} \quad p = \overline{1, q}, \tag{3}$$

$$\sum_{k\in K_0(i,j)} x^k_{ij} \le d^0_{ij},\ x^k_{ij} \ge 0,\ k \in K_0(i, j),\ (i, j) \in U_0, \tag{4}$$

$$0 \le x^k_{ij} \le d^k_{ij},\ k \in K_1(i, j),\ (i, j) \in U, \tag{5}$$

$$x^k_{ij} \ge 0,\ k \in K(i, j) \setminus K_1(i, j),\ (i, j) \in U \setminus U_0, \tag{6}$$

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$$b^k_{*i} \le x^k_i \le b^{*k}_i,\ i \in I^*_k,\ k \in K, \tag{7}$$

where $I^+_i(U^k) = \{j \in I^k : (i,j)^k \in U^k\}$, $I^-_i(U^k) = \{j \in I^k : (j,i)^k \in U^k\}$, $K_0(i,j) \subseteq K(i,j)$, $K_1(i,j) \subseteq K(i,j)$, $K_0(i,j) \bigcap K_1(i,j) = \emptyset$, x^k_{ij} – the flow along the arc $(i,j)^k$, a^k_i, μ^k_{ij}, $\lambda^{k,p}_{ij}$, $\lambda^{k,p}_i$, β_p, d^0_{ij}, d^k_{ij}, b^k_{*i}, b^{*k}_i – known parameters. The nodes $i^k \in I^*_k$ (further i) are named the nodes with variable intensities x^k_i, $I^*_k \subseteq I^k$, $k \in K$.

Flows of different types are interrelated and dependent, since certain types flows using the common power multi-arcs. For known values of the parameters in [1] are developed a constructive theory of constructing optimal solutions extremal problem (1) – (7), which is based on the principles of decomposition constraints [2, 3]. Application the principles of decomposition to the construction of solutions of the inhomogeneous problem (1) – (7) is also aimed at the effective solution of the set of homogeneous network tasks using advances in technology of construction their numerical solutions.

INVERSE OPTIMIZATION PROBLEM: MODELING PARAMETERS OF THE LOWER AND UPPER BOUNDS

Let $x = (x^k_{ij},\ (i,j) \in U,\ k \in K(i,j);\ x^k_i,\ i \in I^*_k,\ k \in K)$ is a infeasible solution of the extremal inhomogeneous problem (1) – (7) (are not met some restrictions of the problem). It is necessary to adjust the values of the lower and upper bounds of the constraints of the problem (1) – (7) so that the infeasible solution x becomes the feasible solution of the problem (1) – (7) for new values of that parameters, where new parameters computed as follows:

$$\begin{aligned}
&\widetilde{a}^k_i = a^k_i + t^k_i - \psi^k_i,\ t^k_i \ge 0,\ \psi^k_i \ge 0,\ i \in I^k,\ k \in K;\\
&\widetilde{\beta}_p = \beta_p + \varphi_p - \delta_p,\ \varphi_p \ge 0,\ \delta_p \ge 0,\ p = \overline{1,q};\\
&\widetilde{d}^0_{ij} = d^0_{ij} + u_{ij} - h_{ij},\ u_{ij} \ge 0,\ h_{ij} \ge 0,\ (i,j) \in U_0;\\
&\widetilde{d}^k_{ij} = d^k_{ij} + m^k_{ij} - n^k_{ij},\ m^k_{ij} \ge 0,\ n^k_{ij} \ge 0,\ k \in K_1(i,j),\ (i,j) \in U;\\
&\widetilde{b}^k_{*i} = b^k_{*i} + k^k_i - s^k_i,\ k^k_i \ge 0,\ s^k_i \ge 0, \qquad \widetilde{b}^{*k}_i = b^{*k}_i + u^k_i - h^k_i,\ u^k_i \ge 0,\ h^k_i \ge 0,\ i \in I^*_k,\ k \in K.
\end{aligned} \tag{8}$$

Values increase and decrease of each parameter for the lower and upper bounds of the constraints to the problem (1) – (7) can not be positive numbers simultaneously.

Applying the principle of inverse optimization [4, 5, 6] we change the parameters for the upper and lower bounds of the constraints to the problem (1) – (7) in accordance with the following norm l_1 a sum of vectors:

$$\begin{aligned}
&\sum_{k\in K}\sum_{i\in I^k}|t^k_i - \psi^k_i| + \sum_{p=1}^{q}|\varphi_p - \delta_p| + \sum_{(i,j)\in U_0}|u_{ij} - h_{ij}| + \\
&+ \sum_{(i,j)\in U}\ \sum_{k\in K_1(i,j)}|m^k_{ij} - n^k_{ij}| + \sum_{k\in K}\sum_{i\in I^*_k}(|k^k_i - s^k_i| + |u^k_i - h^k_i|) = \\
&= \sum_{k\in K}\sum_{i\in I^k}(t^k_i + \psi^k_i) + \sum_{p=1}^{q}(\varphi_p + \delta_p) + \sum_{(i,j)\in U_0}(u_{ij} + h_{ij}) + \\
&+ \sum_{(i,j)\in U}\ \sum_{k\in K_1(i,j)}(m^k_{ij} + n^k_{ij}) + \sum_{k\in K}\sum_{i\in I^*_k}[(k^k_i + s^k_i) + (u^k_i + h^k_i)].
\end{aligned}$$

We change the parameters of the upper and lower bounds of the problem (1) – (7) so that the vector $x = (x^k_{ij},\ (i,j) \in U,\ k \in K(i,j);\ x^k_i,\ i \in I^*_k,\ k \in K)$ becomes the feasible solution for the problem (1) – (7) with new modified parameters (8). To change the parameters of the lower and upper bounds for the problem (1) – (7) in accordance with a selected norm, we construct the mathematical model of inverse optimization problem. The inverse optimization problem for the problem (1) – (7) in accordance with the norm l_1 a sum of vectors has the form:

$$\sum_{k\in K}\sum_{i\in I^k}(t^k_i + \psi^k_i) + \sum_{p=1}^{q}(\varphi_p + \delta_p) + \sum_{(i,j)\in U_0}(u_{ij} + h_{ij}) +$$

$$+\sum_{(i,j)\in U}\sum_{k\in K_1(i,j)}(m^k_{ij}+n^k_{ij})+\sum_{k\in K}\sum_{i\in I^*_k}[(k^k_i+s^k_i)+(u^k_i+h^k_i)]\longrightarrow \min, \tag{9}$$

$$\sum_{j\in I^+_i(U^k)} x^k_{ij}-\sum_{j\in I^-_i(U^k)}\mu^k_{ji}x^k_{ji}=\begin{cases} a^k_i+t^k_i-\psi^k_i+x^k_i, & i\in I^*_k,\\ a^k_i+t^k_i-\psi^k_i, & i\in I^k\backslash I^*_k,\end{cases}\quad t^k_i\geq 0,\ \psi^k_i\geq 0,\ k\in K; \tag{10}$$

$$\sum_{k\in K}\sum_{(i,j)^k\in U^k}\lambda^{k,p}_{ij}x^k_{ij}+\sum_{k\in K}\sum_{i\in I^*_k}\lambda^{k,p}_i x^k_i=\beta_p+\varphi_p-\delta_p,\ \varphi_p\geq 0,\ \delta_p\geq 0,\quad \text{for}\quad p=\overline{1,q}; \tag{11}$$

$$\sum_{k\in K_0(i,j)} x^k_{ij}\leq d^0_{ij}+u_{ij}-h_{ij},\ u_{ij}\geq 0,\ h_{ij}\geq 0, x^k_{ij}\geq 0,\ k\in K_0(i,j),\ (i,j)\in U_0; \tag{12}$$

$$0\leq x^k_{ij}\leq d^k_{ij}+m^k_{ij}-n^k_{ij},\ m^k_{ij}\geq 0,\ n^k_{ij}\geq 0, k\in K_1(i,j),\ (i,j)\in U; \tag{13}$$

$$x^k_{ij}\geq 0,\ k\in K(i,j)\backslash K_1(i,j),\ (i,j)\in U\backslash U_0; \tag{14}$$

$$b^k_{*i}+k^k_i-s^k_i\leq x^k_i\leq b^{*k}_i+u^k_i-h^k_i,\ k^k_i\geq 0,\ s^k_i\geq 0,\ u^k_i\geq 0,\ h^k_i\geq 0,\ i\in I^*_k,\ k\in K, \tag{15}$$

The unknowns of the inverse optimization problem (9) – (15) are the values of the increase and the decrease of each parameter (8). Decrease and increase of each parameter can not both be positive. The new parameters of the lower and upper bounds adjusted as little as possible so that an infeasible solution $x=(x^k_{ij},\ (i,j)\in U,\ k\in K(i,j);\ x^k_i,\ i\in I^*_k,\ k\in K)$ in the linear optimization problem (1) – (7) becomes the feasible solution for that problem with new parameters (8).

NUMERICAL EXAMPLE OF MODELING PARAMETERS OF THE LOWER AND UPPER BOUNDS

In Figure 1 a finite connected directed multigraph $G=(I,U)$ is shown, where $|I|=4$, $|U|=6$ and the nodes with variable intensities forming the set $I^*_2=\{1\}$, $I^*_4=\{4\}$.

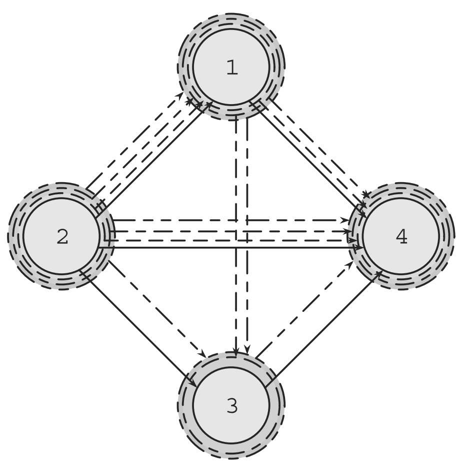

FIGURE 1. Multigraph $G=(I,U)$, $|I|=4$, $|U|=6$, $I^*_2=\{1\}$, $I^*_4=\{4\}$.

We present the restrictions and parameters of the generalized network flow programming problem of type (1) – (7) for the multigraph (see Figure 1) in the form (16) – (23).

$$x^1_{1,4}-x^1_{2,1}=0,\quad x^1_{2,1}+x^1_{2,3}+x^1_{2,4}=15,\quad x^1_{3,4}-\frac{9}{10}x^1_{2,3}=\frac{16}{5},\quad -\frac{3}{5}x^1_{1,4}-\frac{3}{10}x^1_{2,4}-\frac{3}{10}x^1_{3,4}=-\frac{63}{10}; \tag{16}$$

TABLE 1. Infeasible solution for the known parameters of the restrictions (16) – (23)

(i, j), i	(1,3)	(1,4)	(2,1)	(2,3)	(2,4)	(3,4)	1	4
x_{ij}^1		3	3	5	10	5		
x_{ij}^2		1	2		2			
x_{ij}^3	7	3	9		8			
x_{ij}^4	6		1	0	1	1		
x_1^2							6	
x_4^4								1

$$x_{1,4}^2 - \frac{3}{5}x_{2,1}^2 - x_1^2 = -\frac{1}{5}, \quad x_{2,1}^2 + x_{2,4}^2 = 4, \quad -\frac{7}{10}x_{1,4}^2 - \frac{1}{2}x_{2,4}^2 = -\frac{17}{10}; \tag{17}$$

$$x_{1,3}^3 + x_{1,4}^3 - \frac{1}{10}x_{2,1}^3 = \frac{47}{5}, \quad x_{2,1}^3 + x_{2,4}^3 = 14, \quad -\frac{1}{2}x_{1,3}^3 = -\frac{7}{2}, \quad -\frac{1}{10}x_{1,4}^3 - x_{2,4}^3 = -\frac{83}{10}; \tag{18}$$

$$x_{1,3}^4 - \frac{3}{10}x_{2,1}^4 = \frac{57}{10}, \; x_{2,1}^4 + x_{2,3}^4 + x_{2,4}^4 = 2, \; x_{3,4}^4 - \frac{7}{10}x_{1,3}^4 - \frac{4}{5}x_{2,3}^4 = -\frac{16}{5}, \; -\frac{4}{5}x_{2,4}^4 - \frac{9}{10}x_{3,4}^4 - x_4^4 = -\frac{17}{10}; \tag{19}$$

$$\begin{gathered} 3x_{1,3}^3 + 4x_{1,3}^4 + x_{1,4}^1 + 3x_{1,4}^2 + x_{1,4}^3 + 8x_{2,1}^1 + 5x_{2,1}^2 + 3x_{2,1}^3 + 4x_{2,1}^4 + x_{2,3}^1 + \\ +6x_{2,3}^4 + 9x_{2,4}^1 + x_{2,4}^2 + 2x_{2,4}^3 + 9x_{2,4}^4 + 2x_{3,4}^1 + x_{3,4}^4 = 234, \\ 8x_{1,3}^3 + 8x_{1,3}^4 + 3x_{1,4}^1 + 3x_{1,4}^2 + 4x_{1,4}^3 + 7x_{2,1}^1 + 5x_{2,1}^2 + 2x_{2,1}^3 + 3x_{2,1}^4 + 7x_{2,3}^1 + 9x_{2,3}^4 + 2x_{2,4}^1 + 5x_{2,4}^2 + \\ +4x_{2,4}^3 + 3x_{2,4}^4 + 9x_{3,4}^1 + 4x_{3,4}^4 = 287; \end{gathered} \tag{20}$$

$$\begin{gathered} x_{21}^3 + x_{21}^4 \le 8, \; x_{21}^3 \ge 0, \; x_{21}^4 \ge 0, \; u_{21} \ge 0, \; h_{21} \ge 0, \\ x_{2,4}^1 + x_{2,4}^2 \le 15 + u_{24} - h_{24}, \; x_{2,4}^1 \ge 0, \; x_{2,4}^2 \ge 0, \; u_{24} \ge 0, \; h_{24} \ge 0; \end{gathered} \tag{21}$$

$$\begin{gathered} 0 \le x_{1,3}^3 \le 7, \quad 0 \le x_{1,3}^4 \le 10, \quad 0 \le x_{1,4}^1 \le 2, \quad 0 \le x_{1,4}^2 \le 1, \quad 0 \le x_{1,4}^3 \le 3, \\ 0 \le x_{2,1}^1 \le 2, \quad 0 \le x_{2,1}^2 \le 2, \quad 0 \le x_{2,1}^1 \le 9, \quad 0 \le x_{2,1}^4 \le 1, \quad 0 \le x_{2,3}^1 \le 5, \quad x_{2,3}^4 = 0, \\ 0 \le x_{2,4}^1 \le 10, \quad 0 \le x_{2,4}^2 \le 2, \quad 0 \le x_{2,4}^3 \le 8, \quad 0 \le x_{2,4}^4 \le 1, \quad 0 \le x_{3,4}^1 \le 5, \quad 0 \le x_{3,4}^4 \le 1, \end{gathered} \tag{22}$$

$$-5 \le x_1^2 \le 5, \quad -2 \le x_4^4 \le 2. \tag{23}$$

In the Table 1 presents an infeasible solution $x = (x_{ij}^k, (i, j) \in U, k \in K(i, j); x_i^k, i \in I_k^*, k \in K)$, where some restrictions (16) – (23) of the considered numerical example are not met for the known parameters. We shall point out the parameters for which are not met the restrictions (16) – (23). For the parameters $a_2^1 = 15$, $a_3^1 = \frac{16}{5}$, $a_1^2 = -\frac{1}{5}$, $a_1^3 = \frac{47}{5}$, $a_2^3 = 14$, $a_4^3 = -\frac{83}{10}$ and $a_4^4 = -\frac{17}{10}$ are not met the following restrictions:

$$x_{2,1}^1 + x_{2,3}^1 + x_{2,4}^1 = 15, \quad x_{3,4}^1 - \frac{9}{10}x_{2,3}^1 = \frac{16}{5}, \quad x_{1,4}^2 - \frac{3}{5}x_{2,1}^2 - x_1^2 = -\frac{1}{5}, \quad x_{1,3}^3 + x_{1,4}^3 - \frac{1}{10}x_{2,1}^3 = \frac{47}{5}$$
$$x_{2,1}^3 + x_{2,4}^3 = 14, \quad -\frac{1}{10}x_{1,4}^3 - x_{2,4}^3 = -\frac{83}{10}, \quad -\frac{4}{5}x_{2,4}^4 - \frac{9}{10}x_{3,4}^4 - x_4^4 = -\frac{17}{10}. \tag{24}$$

The restrictions (20) are not met for the parameters $\beta_1 = 234$ and $\beta_2 = 287$. The restriction $x_{2,1}^3 + x_{2,1}^4 \le 8$ is not met for the parameter $d_{2,1}^0 = 8$. The following restrictions are not satisfied for the known parameters $d_{1,4}^1 = 2$ and $d_{2,1}^1 = 2$: $0 \le x_{1,4}^1 \le 2$, $0 \le x_{2,1}^1 \le 2$. The restriction $-5 \le x_1^2 \le 5$ is not met for the parameters $b_{*1}^2 = -5$ and $b_1^{*2} = 5$.

Mathematical model of the inverse optimization problem for the modeling of the lower and upper bounds has the form (25) – (33).

$$\sum_{k=1,3,4}\sum_{i=1,2,3,4}(t_i^k + \psi_i^k) + \sum_{i=1,2,4}(t_i^2 + \psi_i^2) + \sum_{p=1}^{2}(\varphi_p + \delta_p) +$$
$$+(u_{2,1} + h_{2,1}) + (u_{2,4} + h_{2,4}) + \sum_{k=3,4}(m_{1,3}^k + n_{1,3}^k) + \sum_{k=1,2,3}(m_{1,4}^k + n_{1,4}^k) +$$
$$+\sum_{k=1,2,3,4}(m_{2,1}^k + n_{2,1}^k) + \sum_{k=1,2,3,4}(m_{2,4}^k + n_{2,4}^k) + \sum_{k=1,4}(m_{2,3}^k + n_{2,3}^k) + \sum_{k=1,4}(m_{3,4}^k + n_{3,4}^k) +$$
$$+(k_1^2 + s_1^2) + (u_1^2 + h_1^2) + (k_4^4 + s_4^4) + (u_4^4 + h_4^4) \longrightarrow \min, \tag{25}$$

$$x_{1,4}^1 - x_{2,1}^1 = t_1^1 - \psi_1^1,\ t_1^1 \ge 0,\ \psi_1^1 \ge 0,$$
$$x_{2,1}^1 + x_{2,3}^1 + x_{2,4}^1 = 15 + t_2^1 - \psi_2^1,\ t_2^1 \ge 0,\ \psi_2^1 \ge 0,$$
$$x_{3,4}^1 - \frac{9}{10}x_{2,3}^1 = \frac{16}{5} + t_3^1 - \psi_3^1,\ t_3^1 \ge 0,\ \psi_3^1 \ge 0,$$
$$-\frac{3}{5}x_{1,4}^1 - \frac{3}{10}x_{2,4}^1 - \frac{3}{10}x_{3,4}^1 = -\frac{63}{10} + t_4^1 - \psi_4^1,\ t_4^1 \ge 0,\ \psi_4^1 \ge 0, \tag{26}$$

$$x_{1,4}^2 - \frac{3}{5}x_{2,1}^2 - x_1^2 = -\frac{1}{5} + t_1^2 - \psi_1^2,\ t_1^2 \ge 0,\ \psi_1^2 \ge 0,$$
$$x_{2,1}^2 + x_{2,4}^2 = 4 + t_2^2 - \psi_2^2,\ t_2^2 \ge 0,\ \psi_2^2 \ge 0,$$
$$-\frac{7}{10}x_{1,4}^2 - \frac{1}{2}x_{2,4}^2 = -\frac{17}{10} + t_4^2 - \psi_4^2,\ t_4^2 \ge 0,\ \psi_4^2 \ge 0, \tag{27}$$

$$
\begin{aligned}
&x_{1,3}^3 + x_{1,4}^3 - \frac{1}{10}x_{2,1}^3 = \frac{47}{5} + t_1^3 - \psi_1^3,\ t_1^3 \ge 0,\ \psi_1^3 \ge 0,\\
&x_{2,1}^3 + x_{2,4}^3 = 14 + t_2^3 - \psi_2^3,\ t_2^3 \ge 0,\ \psi_2^3 \ge 0,\\
&-\frac{1}{2}x_{1,3}^3 = -\frac{7}{2} + t_3^3 - \psi_3^3,\ t_3^3 \ge 0,\ \psi_3^3 \ge 0,\\
&-\frac{1}{10}x_{1,4}^3 - x_{2,4}^3 = -\frac{83}{10} + t_4^3 - \psi_4^3,\ t_4^3 \ge 0,\ \psi_4^3 \ge 0,
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
&x_{1,3}^4 - \frac{3}{10}x_{2,1}^4 = \frac{57}{10} + t_1^4 - \psi_1^4,\ t_1^4 \ge 0,\ \psi_1^4 \ge 0,\\
&x_{2,1}^4 + x_{2,3}^4 + x_{2,4}^4 = 2 + t_2^4 - \psi_2^4,\ t_2^4 \ge 0,\ \psi_2^4 \ge 0,\\
&x_{3,4}^4 - \frac{7}{10}x_{1,3}^4 - \frac{4}{5}x_{2,3}^4 = -\frac{16}{5} + t_3^4 - \psi_3^4,\ t_3^4 \ge 0,\ \psi_3^4 \ge 0,\\
&-\frac{4}{5}x_{2,4}^4 - \frac{9}{10}x_{3,4}^4 - x_4^4 = -\frac{17}{10} + t_4^4 - \psi_4^4,\ t_4^4 \ge 0,\ \psi_4^4 \ge 0,
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
&3x_{1,3}^3 + 4x_{1,3}^4 + x_{1,4}^1 + 3x_{1,4}^2 + x_{1,4}^3 + 8x_{2,1}^1 + 5x_{2,1}^2 + 3x_{2,1}^3 + 4x_{2,1}^4 + x_{2,3}^1 +\\
&+6x_{2,3}^4 + 9x_{2,4}^1 + x_{2,4}^2 + 2x_{2,4}^3 + 9x_{2,4}^4 + 2x_{3,4}^1 + x_{3,4}^4 = 234 + \varphi_1 - \delta_1,\ \varphi_1 \ge 0,\ \delta_1 \ge 0,\\
&8x_{1,3}^3 + 8x_{1,3}^4 + 3x_{1,4}^1 + 3x_{1,4}^2 + 4x_{1,4}^3 + 7x_{2,1}^1 + 5x_{2,1}^2 + 2x_{2,1}^3 + 3x_{2,1}^4 + 7x_{2,3}^1 + 9x_{2,3}^4 + 2x_{2,4}^1 + 5x_{2,4}^2 +\\
&+4x_{2,4}^3 + 3x_{2,4}^4 + 9x_{3,4}^1 + 4x_{3,4}^4 = 287 + \varphi_2 - \delta_2,\ \varphi_2 \ge 0,\ \delta_2 \ge 0;
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
&x_{2,1}^3 + x_{2,1}^4 \le 8 + u_{2,1} - h_{2,1},\ x_{2,1}^3 \ge 0,\ x_{2,1}^4 \ge 0,\ u_{2,1} \ge 0,\ h_{2,1} \ge 0,\\
&x_{2,4}^1 + x_{2,4}^2 \le 15 + u_{2,4} - h_{2,4},\ x_{2,4}^1 \ge 0,\ x_{2,4}^2 \ge 0,\ u_{2,4} \ge 0,\ h_{2,4} \ge 0;
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
&0 \le x_{1,3}^3 \le 7 + m_{13}^3 - n_{13}^3,\ \ m_{1,3}^3 \ge 0,\ \ n_{1,3}^3 \ge 0,\quad 0 \le x_{1,3}^4 \le 10 + m_{1,3}^4 - n_{1,3}^4,\ \ m_{1,3}^4 \ge 0,\ \ n_{1,3}^4 \ge 0,\\
&0 \le x_{1,4}^1 \le 2 + m_{1,4}^1 - n_{1,4}^1,\ \ m_{1,4}^1 \ge 0,\ \ n_{1,4}^1 \ge 0,\quad 0 \le x_{1,4}^2 \le 1 + m_{1,4}^2 - n_{1,4}^2,\ \ m_{1,4}^2 \ge 0,\ \ n_{1,4}^2 \ge 0,\\
&0 \le x_{1,4}^3 \le 3 + m_{1,4}^3 - n_{1,4}^3,\ \ m_{1,4}^3 \ge 0,\ \ n_{1,4}^3 \ge 0,\quad 0 \le x_{2,1}^1 \le 2 + m_{2,1}^1 - n_{2,1}^1,\ \ m_{2,1}^1 \ge 0,\ \ n_{2,1}^1 \ge 0,\\
&0 \le x_{2,1}^2 \le 2 + m_{2,1}^2 - n_{2,1}^2,\ \ m_{2,1}^2 \ge 0,\ \ n_{2,1}^2 \ge 0,\quad 0 \le x_{2,1}^3 \le 9 + m_{2,1}^3 - n_{2,1}^3,\ \ m_{2,1}^3 \ge 0,\ \ n_{2,1}^3 \ge 0,\\
&0 \le x_{2,1}^4 \le 1 + m_{2,1}^4 - n_{2,1}^4,\ \ m_{2,1}^4 \ge 0,\ \ n_{2,1}^4 \ge 0,\quad 0 \le x_{2,3}^1 \le 5 + m_{2,3}^1 - n_{2,3}^1,\ \ m_{2,3}^1 \ge 0,\ \ n_{2,3}^1 \ge 0,\\
&0 \le x_{2,3}^4 \le m_{2,3}^4 - n_{2,3}^4,\ \ m_{2,3}^4 \ge 0,\ \ n_{2,3}^4 \ge 0,\quad 0 \le x_{2,4}^1 \le 10 + m_{2,4}^1 - n_{2,4}^1,\ \ m_{2,4}^1 \ge 0,\ \ n_{2,4}^1 \ge 0,\\
&0 \le x_{2,4}^2 \le 2 + m_{2,4}^2 - n_{2,4}^2,\ \ m_{2,4}^2 \ge 0,\ \ n_{2,4}^2 \ge 0,\quad 0 \le x_{2,4}^3 \le 8 + m_{2,4}^3 - n_{2,4}^3,\ \ m_{2,4}^3 \ge 0,\ \ n_{2,4}^3 \ge 0,\\
&0 \le x_{2,4}^4 \le 1 + m_{2,4}^4 - n_{2,4}^4,\ \ m_{2,4}^4 \ge 0,\ \ n_{2,4}^4 \ge 0,\quad 0 \le x_{3,4}^1 \le 5 + m_{3,4}^1 - n_{3,4}^1,\ \ m_{3,4}^1 \ge 0,\ \ n_{3,4}^1 \ge 0;\\
&0 \le x_{3,4}^4 \le 1 + m_{3,4}^4 - n_{3,4}^4,\ \ m_{3,4}^4 \ge 0,\ \ n_{3,4}^4 \ge 0;
\end{aligned} \tag{32}
$$

$$-5+k_1^2-s_1^2 \le x_1^2 \le 5+u_1^2-h_1^2,\;\; k_1^2\ge 0,\;\; s_1^2\ge 0,\;\; u_1^2\ge 0,\;\; h_1^2\ge 0,$$
$$-2+k_4^4-s_4^4 \le x_4^4 \le 2+u_4^4-h_4^4,\;\; k_4^4\ge 0,\;\; s_4^4\ge 0,\;\; u_4^4\ge 0,\;\; h_4^4\ge 0. \tag{33}$$

The new parameters (8) for the problem (25) – (33) computed as follows:

$$\widetilde{a}_i^k = a_i^k + t_i^k - \psi_i^k,\ t_i^k\ge 0,\ \psi_i^k\ge 0,\ i\in I^k,\ k\in K=\{1,2,3,4\};\quad \widetilde{\alpha}_p=\alpha_p+\varphi_p-\delta_p,\ \varphi_p\ge 0,\ \delta_p\ge 0,\ p=\overline{1,2};$$
$$\widetilde{d}_{ij}^0 = d_{ij}^0+u_{ij}-h_{ij},\ u_{ij}\ge 0,\ h_{ij}\ge 0,\ (i,j)\in U_0=\{(2,1),(2,4)\};$$

$$\widetilde{d}_{ij}^k = d_{ij}^k+m_{ij}^k-n_{ij}^k,\ m_{ij}^k\ge 0,\ n_{ij}^k\ge 0,\ (i,j)^k\in T,$$
$$T\{(1,3)^k, k=3,4; (1,4)^k, k=1,2,3; (2,1)^k, k=1,2,3,4; (2,4)^k, k=1,2,3,4; (2,3)^k, k=1,4; (3,4)^k, k=1,4\};$$
$$\widetilde{b}_{*ik} = b_{*i}^k + k_i^k - s_i^k,\ k_i^k\ge 0,\ s_i^k\ge 0,\qquad \widetilde{b}_i^{*k} = b_i^{*k}+u_i^k-h_i^k,\ u_i^k\ge 0,\ h_i^k\ge 0,\ i\in I_k^*,\ \ I_2^*=\{1\},\ \ I_4^*=\{4\}.$$

The values increase and decrease of each known parameter of lower and upper bounds are the unknowns of the problem (25) – (33). Decrease and increase of each parameter can not both be positive.

As a result of solving the inverse optimization problem (25) – (33) we obtained the following changes of the parameters :

$$t_2^1=3,\quad \psi_3^1=\frac{27}{10},\quad \psi_1^2=6,\quad \psi_1^3=\frac{3}{10},\quad t_2^3=3,\quad t_4^3=\frac{107}{10},\quad \psi_4^4=1,$$
$$\varphi_1=18,\quad \varphi_2=42,\quad u_{2,1}=2,\quad m_{1,4}^1=1,\quad m_{2,1}^1=1,\quad u_1^2=1.$$

The parameters of lower and upper bounds adjusted as little as possible so that an infeasible solution $x = (x_{ij}^k,\ (i,j)\in U,\ k\in K(i,j); x_i^k,\ i\in I_k^*,\ k\in K)$ becomes the feasible solution for the new parameters. For the vector x (see Table 1) fulfilled all restrictions (16) – (23), therefore x is a feasible solution for the new parameters in the considered numerical example.

INVERSE OPTIMIZATION PROBLEM: MODELING PARAMETERS OF THE OBJECTIVE FUNCTION

For the primal problem (1) – (7) we form the dual problem (34) – (38):

$$\sum_{k\in K}\sum_{i\in I^k} a_i^k u_i^k + \sum_{p=1}^{q}\beta_p r_p - \sum_{(i,j)\in U_0} d_{ij}^0 v_{ij} - \sum_{(i,j)\in U}\sum_{k\in K_1(i,j)} d_{ij}^k\omega_{ij}^k + \sum_{k\in K}\sum_{i\in I_k^*} b_{*i}^k\omega_i^k - \sum_{k\in K}\sum_{i\in I_k^*} b_i^{*k}t_i^k \longrightarrow \max, \tag{34}$$

$$u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{q}\lambda_{ij}^{kp} r_p - v_{ij} \le c_{ij}^k,\ v_{ij}\ge 0,\ k\in K_0(i,j),\ (i,j)\in U_0, \tag{35}$$

$$u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{q}\lambda_{ij}^{kp} r_p - \omega_{ij}^k \le c_{ij}^k,\ \omega_{ij}^k\ge 0,\ k\in K_1(i,j),\ (i,j)\in U, \tag{36}$$

$$u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{q}\lambda_{ij}^{kp} r_p \le c_{ij}^k,\ k\in K(i,j)\setminus K_1(i,j),\ (i,j)\in U\setminus U_0, \tag{37}$$

$$-u_i^k + \sum_{p=1}^{q}\lambda_i^{kp} r_p + \omega_i^k - t_i^k = c_i^k,\ \omega_i^k\ge 0,\ t_i^k\ge 0,\ i\in I_k^*, k\in K, \tag{38}$$

where the vector $\lambda = \Big(u_i^k,\ k \in K,\ i \in I^k;\ r_p,\ p = \overline{1,l};\ \nu_{ij} \geq 0,\ (i,j) \in U_0;\ \omega_{ij}^k \geq 0,\ (i,j) \in U,\ k \in K_1(i,j);\ \omega_i^k \geq 0,\ t_i^k \geq 0,\ i \in I_k^*,\ k \in K\Big)$ is a feasible solution of the dual problem (34) – (38).

Now we formulate the inverse optimization problem for the modeling of the parameters of the objective function for the primal problem (1) – (7). Let $x = (x_{ij}^k,\ (i,j) \in U,\ k \in K(i,j); x_i^k,\ i \in I_k^*,\ k \in K)$ is a feasible solution of the problem (1) – (7). It is necessary to adjust the values of the components of the cost vector $c = (c_{ij}^k,\ (i,j) \in U,\ k \in K(i,j); c_i^k,\ i \in I_k^*,\ k \in K)$ so that the feasible solution x becomes the optimal solution of the problem (1) – (7) for new values of the cost vector. Depending on its configured of values x_{ij}^k, $(i,j) \in U$, $k \in K(i,j)$ and x_i^k, $i \in I_k^*$, $k \in K$ of the given feasible solution x of the problem (1) – (7) we define the sets:

$$B_1 = \Big\{(i,j)^k, k \in K_0(i,j), (i,j) \in U_0 : x_{ij}^k = 0, \quad \sum_{k \in K_0(i,j)} x_{ij}^k = d_{ij}^0\Big\},$$

$$B_2 = \Big\{(i,j)^k, k \in K_0(i,j), (i,j) \in U_0 : x_{ij}^k = 0, \quad \sum_{k \in K_0(i,j)} x_{ij}^k < d_{ij}^0\Big\},$$

$$B_3 = \Big\{(i,j)^k, k \in K_0(i,j), (i,j) \in U_0 : x_{ij}^k \neq 0, \quad \sum_{k \in K_0(i,j)} x_{ij}^k = d_{ij}^0\Big\},$$

$$B_4 = \Big\{(i,j)^k, k \in K_0(i,j), (i,j) \in U_0 : x_{ij}^k \neq 0, \quad \sum_{k \in K_0(i,j)} x_{ij}^k \neq d_{ij}^0\Big\};$$

$$R_1 = \Big\{(i,j)^k, k \in K_1(i,j),\ (i,j) \in U : x_{ij}^k = 0\Big\},$$

$$R_2 = \Big\{(i,j)^k, k \in K_1(i,j),\ (i,j) \in U : x_{ij}^k = d_{ij}^k\Big\},$$

$$R_3 = \Big\{(i,j)^k, k \in K_1(i,j),\ (i,j) \in U : 0 < x_{ij}^k < d_{ij}^k\Big\};$$

$$L_1 = \Big\{(i,j)^k, k \in K(i,j) \setminus K_1(i,j),\ (i,j) \in U \setminus U_0 : x_{ij}^k = 0\Big\},$$

$$L_2 = \Big\{(i,j)^k, k \in K(i,j) \setminus K_1(i,j),\ (i,j) \in U \setminus U_0 : x_{ij}^k > 0\Big\};$$

$$M_1 = \{i \in I_k^*,\ k \in K : x_i^k = b_{*i}^k\},$$

$$M_2 = \{i \in I_k^*,\ k \in K : x_i^k = b_i^{*k}\},$$

$$M_3 = \{i \in I_k^*,\ k \in K : b_{*i}^k < x_i^k < b_i^{*k}\}.$$

Applying the principle of inverse optimization we change the parameters of the objective function of the problem (1) – (7) in accordance with the following norm l_1 a sum of vectors:

$$l_1 = \sum_{(i,j) \in U} \sum_{k \in K(i,j)} |\alpha_{ij}^k - \beta_{ij}^k| + \sum_{k \in K} \sum_{i \in I_k^*} |\alpha_i^k - \beta_i^k| = \sum_{(i,j) \in U} \sum_{k \in K(i,j)} (\alpha_{ij}^k + \beta_{ij}^k) + \sum_{k \in K} \sum_{i \in I_k^*} (\alpha_i^k + \beta_i^k),$$

where values increase α_{ij}^k, decrease β_{ij}^k of each parameter c_{ij}^k and increase α_i^k, decrease β_i^k of each parameter c_i^k of the objective function (1) can not be positive numbers simultaneously, where $\alpha_{ij}^k \geq 0,\ \beta_{ij}^k \geq 0,\ (i,j) \in U,\ k \in K(i,j)$, $\alpha_i^k \geq 0,\ \beta_i^k \geq 0,\ i \in I_k^*,\ k \in K$.

Mathematical model of inverse optimization problem for modeling new parameters $\widetilde{c} = (\widetilde{c}_{ij}^k, (i,j) \in U, k \in K(i,j);\ \widetilde{c}_i^k, i \in I_k^*, k \in K)$, $\widetilde{c}_{ij}^k = c_{ij}^k + \alpha_{ij}^k - \beta_{ij}^k$, $\widetilde{c}_i^k = c_i^k + \alpha_i^k - \beta_i^k$ in accordance with the norm l_1 has the form:

$$\sum_{(i,j)\in U} \sum_{k\in K(i,j)} (\alpha_{ij}^k + \beta_{ij}^k) + \sum_{k\in K} \sum_{i\in I_k^*} (\alpha_i^k + \beta_i^k) \longrightarrow \min \tag{39}$$

$$\begin{gathered}
u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{q} \lambda_{ij}^{kp} r_p - \nu_{ij} \leq c_{ij}^k + \alpha_{ij}^k - \beta_{ij}^k,\ \nu_{ij} \geq 0 \ \textit{for the arcs}\ (i,j)^k \in B_1; \\
u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{q} \lambda_{ij}^{p} r_p \leq c_{ij}^k + \alpha_{ij}^k - \beta_{ij}^k \ \textit{for the arcs}\ (i,j)^k \in B_2; \\
u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{q} \lambda_{ij}^{p} r_p - \nu_{ij} = c_{ij}^k + \alpha_{ij}^k - \beta_{ij}^k,\ \nu_{ij} \geq 0 \ \textit{for the arcs}\ (i,j)^k \in B_3; \\
u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{q} \lambda_{ij}^{p} r_p = c_{ij}^k + \alpha_{ij}^k - \beta_{ij}^k \ \textit{for the arcs}\ (i,j)^k \in B_4;
\end{gathered} \tag{40}$$

$$\begin{gathered}
u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{q} \lambda_{ij}^{p} r_p - \omega_{ij}^k \leq c_{ij}^k + \alpha_{ij}^k - \beta_{ij}^k \ \ \omega_{ij}^k \geq 0 \ \textit{for the arcs}\ (i,j)^k \in R_1; \\
u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{q} \lambda_{ij}^{p} r_p - \omega_{ij}^k = c_{ij}^k + \alpha_{ij}^k - \beta_{ij}^k,\ \omega_{ij}^k \geq 0 \ \textit{for the arcs}\ (i,j)^k \in R_2; \\
u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{q} \lambda_{ij}^{p} r_p = c_{ij}^k + \alpha_{ij}^k - \beta_{ij}^k \ \textit{for the arcs}\ (i,j)^k \in R_3;
\end{gathered} \tag{41}$$

$$\begin{gathered}
u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{q} \lambda_{ij}^{p} r_p \leq c_{ij}^k + \alpha_{ij}^k - \beta_{ij}^k,\ \ \textit{for the arcs}\ (i,j)^k \in L_1; \\
u_i^k - \mu_{ij}^k u_j^k + \sum_{p=1}^{q} \lambda_{ij}^{p} r_p = c_{ij}^k + \alpha_{ij}^k - \beta_{ij}^k,\ \ \textit{for the arcs}\ (i,j)^k \in L_2;
\end{gathered} \tag{42}$$

$$\begin{gathered}
-u_i^k + \sum_{p=1}^{q} \lambda_i^{kp} r_p + \omega_i^k = c_i^k + \alpha_i^k - \beta_i^k,\ \omega_i^k \geq 0,\ \alpha_i^k \geq 0,\ \beta_i^k \geq 0 \ \textit{for the nodes}\ i \in M_1; \\
-u_i^k + \sum_{p=1}^{q} \lambda_i^{kp} r_p - t_i^k = c_i^k + \alpha_i^k - \beta_i^k,\ t_i^k \geq 0,\ \alpha_i^k \geq 0,\ \beta_i^k \geq 0 \ \textit{for the nodes}\ i \in M_2; \\
-u_i + \sum_{p=1}^{q} \lambda_i^{kp} r_p = c_i^k + \alpha_i^k - \beta_i^k,\ \alpha_i^k \geq 0,\ \beta_i^k \geq 0 \ \textit{for the nodes}\ i \in M_3.
\end{gathered} \tag{43}$$

The unknowns of the inverse optimization problem (39) – (43) are the values of the increase and the decrease of each parameter of the objective function (1). Decrease and increase of each parameter can not be positive numbers simultaneously. The parameters of the objective function (1) adjusted as little as possible in accordance with the following norm l_1 a sum of vectors so that the feasible solution $x = (x^k_{ij}, (i, j) \in U, k \in K(i, j); x^k_i, i \in I^*_k, k \in K)$ becomes the optimal solution of the problem (1) – (7) for the new parameters $\widetilde{c} = (\widetilde{c}^k_{ij}, (i, j) \in U, k \in K(i, j);\ \widetilde{c}^k_i, i \in I^*_k, k \in K)$.

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