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SPARSE LINEAR SYSTEMS: THEORY OF DECOMPOSITION, METHODS, TECHNOLOGY, APPLICATIONS AND IMPLEMENTATION IN WOLFRAM MATHEMATICA

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Abstract. In this paper we propose the theory of decomposition, methods, technologies, applications and implementation in Wolfram Mathematica for the constructing the solutions of the sparse linear systems. One of the applications is the Sensor Location Problem for the symmetric graph in the case when split ratios of some arc flows can be zeros. The objective of that application is to minimize the number of sensors that are assigned to the nodes. We obtain a sparse system of linear algebraic equations and research its matrix rank. Sparse systems of these types appear in generalized network flow programming problems in the form of restrictions and can be characterized as systems with a large sparse sub-matrix representing the embedded network structure.

INTRODUCTION

In this work we consider the application of the graph theory for construction the solutions of linear systems with rectangular sparse matrices, namely of linear underdetermined sparse systems. The decomposition theory for a graph or a multigraph will be applied to construct the solutions of linear systems with rectangular sparse matrices with different types of sparsity. Sparse systems of these types appear in non-homogeneous network flow programming problems in the form of restrictions and can be characterized as systems with a large sparse sub-matrix representing the embedded network structure. We develop direct methods for finding solutions of systems of these types. These algorithms are based on the theoretic-graph specificities of the structure of the support for the graph and on the properties of the basis of the solution space of homogeneous sparse systems of special types.

SPARSE LINEAR HOMOGENEOUS SYSTEM WITH BLOCK-DIAGONAL RECTANGULAR MATRIX

For the finite oriented connected loopless multigraph $G = (I, U)$ with set of nodes I and set of multiarcs U we consider the problem in unknown multiflow $x = (x^k, k \in K)$, $x^k = (x^k_{ij}, (i, j)^k \in U^k; x^k_i, i \in I^*_k)$ satisfying the following sparse underdetermined system of linear algebraic equations:

$$\sum_{j \in I^+_i(U^k)} x^k_{ij} - \sum_{j \in I^-_i(U^k)} \mu^k_{ji} x^k_{ji} = \begin{cases} x^k_i, & i \in I^*_k; \\ 0, & i \in I^k \setminus I^*_k, \end{cases} \quad k \in K, \tag{1}$$

Analogously to [1], we represent G as $|K|$ connected networks $G^k = (I^k, U^k)$, $K = \{1, \ldots, |K|\}$, where I^k, U^k are the sets of nodes and arcs respectively, through which the flow of type k is transported, $k \in K$. For each node $i \in I$

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as well as for each multiarc $(i, j) \in U$ we introduce the sets of flow types $K(i) = \{k \in K : i \in I^k\}$, $K(i, j) = \{k \in K : (i, j)^k \in U^k\}$. Here $I_i^+(U^k) = \{j : (i, j)^k \in U^k\}$, $I_i^-(U^k) = \{j : (j, i)^k \in U^k\}$, $I_k^* \subseteq I^k$ is the set of nodes with variable intensities, x_i^k is the unknown intensity of node $i \in I_k^*$, $k \in K$. The matrix of the system (1) has the following block structure:

$$A = \begin{bmatrix} T & B \end{bmatrix}, \tag{2}$$

where T corresponds to the left-hand side of (1) and B – to the right-hand side of (1). The matrix T is block diagonal with non-square $|I^k| \times |U^k|$-blocks T_k. Each column of the matrix T_k corresponds to the arc $(i, j)^k$, and the nonzero elements of the specified column are the two elements: element of the row with the number i, equal to 1, and element of the row with the number j, equal to $-\mu_{ij}^k$. Analogously B is block diagonal with $|I^k| \times |I_k^*|$-blocks B_k, $k \in K$. For each $k \in K$, there is a single non-zero element per column in B_k. This element equals (-1) and is located at the intersection of the row and the column both corresponding to the node $i \in I_k^*$. In [2, 3, 4] for fixed k was research the rank of the matrix A of system (1) in the Sensor Location Problem for symmetric generalized graph. Combinatorial aspects of the Sensor Location Problem are considered in [5]. The methods of decomposition and the theory of graphs partitioning are applied for constructing the general solutions of the systems with special sparse matrices [6, 7]. These systems arise in the Sensor Location Problem for one new application connected to the optimal sensors location in the nodes of a generalized graph.

THE USE OF A PRIORI INFORMATION ABOUT MULTINODES WITH SENSORS FOR EXCLUSION UNKNOWNS

In [6] was considered one of applications of sparse underdetermined system (1). The objective of that application is to minimize the number of sensors that are assigned to the nodes for the symmetric graph in the case when split ratios of some arc flows can be zeros [7]. We obtain a new sparse system of linear algebraic equations and research its matrix rank. To get the a priori information about some unknowns x_{ij}^k, $(i, j)^k \in U^k$ and x_i^k, $i \in I_k^*$, $k \in K$ we locate *sensors* at multinodes $(i, K(i))$. If a multinode $(i, K(i))$ is monitored, i.e. $(i, K(i)) \in M$, then the values of flows for all outgoing and all incoming arcs of this multinode, i.e. x_{ij}^k, x_{ji}^k, $k \in K(i)$, are considered known:

$$\begin{aligned} &x_{ij}^k = f_{ij}^k,\ j \in I_i^+(U^k), \\ &x_{ji}^k = f_{ji}^k,\ j \in I_i^-(U^k), \quad k \in K(i),\ (i, K(i)) \in M. \end{aligned} \tag{3}$$

Besides, if a set M_k, $k \in K(i)$, includes some nodes i from the set I_k^*, then the variable intensities x_i^k, $k \in K(i)$, $i \in M_k \bigcap I_k^*$, are considered known, too:

$$x_i^k = f_i^k, \quad k \in K(i),\ i \in M_k \bigcap I_k^*. \tag{4}$$

Consider a multinode $(i, K(i))$. For each outgoing arc $(i, j)^k \in U^k$ for the node i we introduce a real number $p_{ij}^k \in [0, 1]$, that is called *split ratio* and denotes the corresponding part (for the given flow type k) of the total outgoing flow $\sum\limits_{j \in I_i^+(U^k)} x_{ij}^k$. Obviously, $\sum\limits_{j \in I_i^+(U^k)} p_{ij}^k = 1$. That is,

$$x_{ij}^k = p_{ij}^k \cdot \sum_{j \in I_i^+(U^k)} x_{ij}^k, \quad 0 \le p_{ij}^k \le 1, \quad \sum_{j \in I_i^+(U^k)} p_{ij}^k = 1. \tag{5}$$

It worth mentioning, that earlier in [1, 2] we considered only nonzero split ratios.

For every node $i \in I^k$, if $|I_i^+(U^k)| \ge 2$ then we can express arc flows x_{ij}^k along all arcs going out of the node i in terms of just a single outgoing arc $(i, v_i)^k$, $v_i \in I_i^+(U^k)$, provided $p_{i,v_i}^k > 0$:

$$x_{ij}^k = \frac{p_{ij}^k}{p_{i,v_i}^k} x_{i,v_i}^k, \quad p_{i,v_i}^k > 0,\ j \in I_i^+(U^k) \setminus v_i,\ |I_i^+(U^k)| \ge 2,\ k \in K(i),\ i \in I. \tag{6}$$

Let's state the Sensor Location Problem for the multigraph with zero split ratios of some arc flows: *find the minimal number $|M|$ of monitored multinodes such that the system (1) given the constraints (6) is uniquely solvable and obtain at least one variant of sensor placement.*

Combinatory properties of algorithms of solving the Sensor Location Problem with nonzero split ratios only and for graph, i.e. for the case $|K| = 1$, are considered in [1, 2].

To solve the formulated problem, we substitute the a priori information (3) and (4) into the system (1). If $|I_i^+(U^k)| \geq 2$ for the node $i \in I^k$ then one can write the flow along all outgoing arcs from node i in terms of a single known outgoing arc flow f^k_{i,v_i} for the arc $(i, v_i)^k, v_i \in I_i^+(U^k)$, where x^k_{i,v_i} is known and equals f^k_{i,v_i}:

$$x^k_{ij} = \frac{p^k_{ij}}{p^k_{i,v_i}} f^k_{i,v_i}, \quad p^k_{i,v_i} > 0,\ j \in I_i^+(U^k) \setminus v_i,\ |I_i^+(U^k)| \geq 2,\ k \in K(i),\ i \in I. \tag{7}$$

And also we substitute the known arcs flows (7) into the system (1).

Let's remove from the graphs $G^k = (I^k, U^k)$, $k \in K$, arcs (but not their nodes) $(i, j)^k$, for which the constraints (3) are stated, and all the monitored nodes $i \in M_k$ from every graph G^k, $k \in K$. Likewise we remove from these graphs the arcs $(i, j)^k$ for which arc flows x^k_{ij} are expressed through (7).

Thus, we have a new multigraph $\overline{G} = (\overline{I}, \overline{U})$, which consists of the set of graphs

$$\overline{G}^k = (\overline{I}^k, \overline{U}^k),\ \overline{I}^k \subseteq I^k,\ \overline{U}^k \subseteq U^k,\ k \in \overline{K} \subseteq K,$$

where each $\overline{G}^k = (\overline{I}^k, \overline{U}^k)$ is, in general, a disconnected graph, corresponding to a certain type of flow $k \in \overline{K}$. We introduce for each multiarc $(i, j) \in \overline{U}$ of multigraph $\overline{G}$ the set $\overline{K}(i, j) = \{k \in \overline{K} : (i, j)^k \in \overline{U}^k\}$ of flow types, transported through it, and, analogously, for each node $i \in \overline{I}$ we denote the set of flow types, transported through that node, with $\overline{K}(i) = \{k \in \overline{K} : i \in \overline{I}^k\}$. Obviously, $|\overline{K}| \leq |K|$ in general case: for some flow types $k \in K$ we could have obtained the complete information, i.e. the arc flows x^k_{ij} and the variable intensities x^k_i for the whole network G^k (in that case $\overline{G}^k$, $k \in K \setminus \overline{K}$, don't exist).

For the multigraph $\overline{G}$ the homogeneous sparse system (1) is transformed into the following inhomogeneous one:

$$\sum_{j \in \overline{I}_i^+(\overline{U}^k)} x^k_{ij} - \sum_{j \in \overline{I}_i^-(\overline{U}^k)} \mu^k_{ji} x^k_{ji} = \begin{cases} x^k_i + b^k_i, & i \in \overline{I}^*_k; \\ b^k_i, & i \in \overline{I}^k \setminus \overline{I}^*_k, k \in \overline{K}, \end{cases} \tag{8}$$

where $I_i^+(\overline{U}^k) = \{j : (i, j)^k \in \overline{U}^k\}$, $I_i^-(\overline{U}^k) = \{j : (j, i)^k \in \overline{U}^k\}$, $\overline{I}^*_k \subseteq \overline{I}^k$, $b^k_i \in \mathbf{R}$.

Some connectivity components of the multigraph $\overline{G}$ may not contain such multinodes $(i, \overline{K}(i))$ that $i \in \bigcup_{k \in \overline{K}(i)} \overline{I}^*_k$, where $\overline{I}^*_k$ denotes the set of nodes with variable intensities of graph $\overline{G}^k$, $k \in \overline{K}$.

Relations (6) hold for the new multigraph $\overline{G}$:

$$x^k_{ij} = \frac{p^k_{ij}}{p^k_{i,v_i}} x^k_{i,v_i}, \quad p^k_{i,v_i} > 0,\ j \in I_i^+(\overline{U}^k) \setminus v_i,\ |I_i^+(\overline{U}^k)| \geq 2,\ i \in \overline{I},\ k \in \overline{K}(i). \tag{9}$$

Let's enumerate the equalities (9): $r = 1, 2, \ldots, q$, where r stands for an equality number and q is the total number of equalities. Now we collect all the x-terms in the left-hand sides of the equalities and, thus, have linear forms of the unknown vector $x = (x^k, k \in \overline{K})$, $x^k = (x^k_{ij}, (i, j)^k \in \overline{U}^k; x^k_i, i \in \overline{I}^*_k)$, equated to zero. We denote the coefficients of these forms with $\lambda^{k,r}_{ij}$ (since the coefficients of x^k_i-terms are deliberately zero, we don't use any denotation for them) and obtain the equations

$$\sum_{(i,j) \in \overline{U}} \sum_{k \in \overline{K}(i,j)} \lambda^{k,r}_{ij} x^k_{ij} = 0,\ r = 1, 2, \ldots, q, \tag{10}$$

where for each fixed r all $\lambda^{k,r}_{ij}$ are zero except for some two of them (one of which equals 1 and the other one is of kind $-\frac{p^k_{ij}}{p^k_{i,v_i}}$).

DECOMPOSITION OF THE INHOMOGENEOUS UNDERDETERMINED SPARSE SYSTEM AND COMPUTATION OF MATRIX RANK

We consider the inhomogeneous underdetermined sparse system (8), (10) in unknowns x_{ij}^k, $(i,j)^k \in \overline{U}^k$; $x_i^k, i \in \overline{I}_k^*$, $k \in \overline{K}$. Based on the developed in [1, 2] decomposition theory the linear system (8) is not changed. The new multigraph $\overline{G} = (\bar{I}, \overline{U})$ consists from the set of graphs

$$\overline{G}^k = (\overline{I}^k, \overline{U}^k),\ \overline{I}^k \subseteq I^k,\ \overline{U}^k \subseteq U^k,\ k \in \overline{K} \subseteq K.$$

Each $\overline{G}^k = (\overline{I}^k, \overline{U}^k)$ is, in general, a disconnected graph, corresponding to a certain type of flow $k \in \overline{K}$. The general solutions of sparse underdetermined systems (8) with rectangular matrices were obtained in [4] for fixed k. For the blocks of the sparse matrix of the underdetermined system (8) we apply the fundamental results of the theory of flows in networks, as well as advancements in the technology of construction their analytical and numerical solutions. Having the ranks of matrices, corresponding to connectivity components of the graphs $\overline{G}^k$, $k \in \overline{K}$, computed, one obtains the rank of the matrix of the whole system (8). The unknown (9) may be substituted into (8) to eliminate the unknowns, but such an approach (considered for the case $|\overline{K}| = 1$ in [5]) leads a new system with matrix of specific structure.

We will describe the our approach. Substituting into (10) the general solutions of sparse underdetermined systems (8) with rectangular matrices respect to a given support [4] we obtained the special systems. Matrices of such systems are composed of determinants $\Lambda_{\tau\rho}^{k,r}$, $\Lambda_{\gamma}^{k,r}$ [1, 2, 8, 9] of the structures entailed by the arcs $\overline{U}^k \backslash \overline{U}_R^k, k \in \overline{K}$ and by the nodes $i \in \overline{I}^k \backslash \overline{I}_k^*, k \in \overline{K}$, which not included in a given support $\{U_R^k, I_R^{*k}, k \in K\}$. Here

$$\Lambda_{\tau\rho}^{k,r} = \lambda_{\tau\rho}^{k,r} + \sum_{(i,j)^k \in U_R^k} \lambda_{ij}^{k,r} \delta_{ij}^k(\tau,\rho) + \sum_{i \in I_R^{*k}} \lambda_i^{k,r} \delta_i^k(\tau,\rho),\quad \Lambda_{\gamma}^{k,r} = \lambda_{\gamma}^{k,r} + \sum_{(i,j)^k \in U_R^k} \lambda_{ij}^{k,r} \delta_{ij}^k(\gamma) + \sum_{i \in I_R^{*k}} \lambda_i^{k,r} \delta_i^k(\gamma),$$

where $\delta^k(\tau,\rho)$ and $\delta^k(\gamma)$ are characteristic vectors [1, 2, 8, 9], entailed by the elements which not included in a given support $\{U_R^k, I_R^{*k}, k \in K\}$, r is the number of equation of the system (10). Let rank of the matrix of determinants is equal n (n is the highest order nonzero minor of that matrix). The rank of the entire underdetermined sparse system (8), (10) is equal $m + n$, where m is the rank of the matrix of the system (8) [1, 2, 4, 9].

As result, based on the theory of decomposition of multigraph support [1, 2] and using the general solutions of sparse systems with rectangular matrices [4] for fixed k we obtained efficient algorithms for computing the matrix rank of the system (8), (10). The system (8), (10) has a unique solution for the given set M in the Sensor Location Problem for generalized multigraph if and only if the rank $m + n$ of its matrix equals the number $\sum_{k \in \overline{K}} \left(|\overline{U}^k| + |\overline{I}_k^*| \right)$ of unknowns.

FUNDAMENTAL SYSTEM OF ELEMENTS: TECHNOLOGIES OF CONSTRUCTION IN WOLFRAM MATHEMATICA

In the [1, 2] presented implementation in CAS *Wolfram Mathematica* of decomposition algorithms for constructing the general solutions of the sparse underdetermined system (8), (10) in the case $I_k^* = \emptyset, k \in K$. We used in [1, 2] at realization in *Wolfram Mathematica* the fundamental results of the theory of flows in networks, as well as advancements in the technology of construction their analytical and numerical solutions.

We compute a basis of the solution space of the corresponding homogeneous system for the sparse underdetermined system (8) and interpret the basis vectors as characteristic vectors, entailed by non-support arcs and nodes with variables intensities. Effective algorithm for constructing a characteristic vector of the network part of the sparse linear non-homogeneous system is obtained. Also the effective algorithms for finding a partial solution of the network part of the sparse linear non-homogeneous system is obtained. We use the technologies for presentation a spanning tree [1, 2], which allows to compute the nonzero components of every characteristic vector $\delta(\tau,\rho)$ in $O(m)$ complexity in the worst case, where $m = |I^n|$.

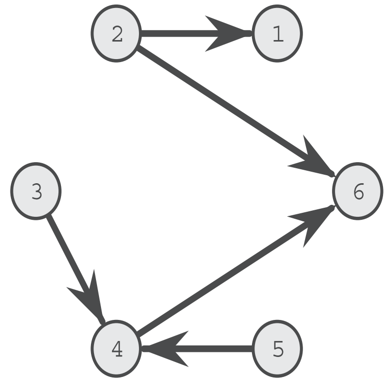

FIGURE 1. Spanning tree U_R^n of the graph $G^n = (I^n, U^n)$, where $I^n = \{1, 2, 3, 4, 5, 6\}$, $U^n = \{(1, 3), (2, 1), (2, 6), (3, 4), (3, 6), (4, 6), (5, 4), (6, 5)\}$.

In *Listing 1* we present the implementation in *Wolfram Mathematica* of the algorithm for computing of the components of the characteristic vector $\delta(\tau, \rho) = \delta(3, 6)$, entailed by the arc $(\tau, \rho) = (3, 6)$ with respect the spanning tree U_R^n =(2, 1),(2, 6),(3, 4),(4, 6),(5, 4) (see Figure 1) of the graph $G^n = (I^n, U^n)$, where $I^n = \{1, 2, 3, 4, 5, 6\}$, $U^n = \{(1, 3), (2, 1), (2, 6), (3, 4), (3, 6), (4, 6), (5, 4), (6, 5)\}$ with the given estimate $O(m)$ for computing of the non-zero components of the characteristic vector $\delta(\tau, \rho)$ for fixed n, where $m = |I^n|$.

Listing 1

```
depth={0,1,4,3,4,2};
p={0,1,4,6,4,2};
d={0,-1,-1,-1,-1,1};
τ =3;
ρ =6;
δ_{3,6} = {x_{3,6} → 1,  x_{6,5} → 0,  x_{1,3} → 0,  x_{5,4} → 0,  x_{2,6} → 0,  x_{2,1} → 0};
i = τ; j = ρ;
While[i ≠ j,
  If [depth[[i]] > depth[[j]],
  {
  If[d[[i]] == 1,
  δ_{3,6} = Join[δ_{3,6}, {x_{p[[i]],i} → 1}],
  δ_{3,6} = Join[δ_{3,6}, {x_{i,p[[i]]} → −1}],
  ];
  i = p[[i]];
  },
  If [depth[[j]] > depth[[i]],
  {
  If[d[[j]] == 1,
  δ_{3,6} = Join[δ_{3,6}, {x_{p[[j]],j} → −1}],
  δ_{3,6} = Join[δ_{3,6}, {x_{j,p[[j]]} → 1}],
  ];
  j = p[[j]];
  },
  {
  If[d[[i]] == 1,
  δ_{3,6} = Join[δ_{3,6}, {x_{p[[i]],i} → 1}],
  δ_{3,6} = Join[δ_{3,6}, {x_{i,p[[i]]} → −1}],
  ];
```

```
If[d[[j]] == 1,
δ_{3,6} = Join[δ_{3,6}, {x_{p[[j]],j} → −1}],
δ_{3,6} = Join[δ_{3,6}, {x_{j,p[[j]]} → 1}],
];
i = p[[i]];
j = p[[j]];
}
]
];
];
```

The implementation in *Wolfram Mathematica* of the algorithm for computing of the components of the characteristic vector $\delta(\tau,\rho)$, entailed by the arc (τ,ρ) (see *Listing 1*) with respect to given spanning tree U_R^n (see Figure 1) of the graph $G^n = (I^n, U^n)$ use in different structures of the support after applying of the methods of decomposition and the theory of graphs partitioning: forest of trees with special properties and the connected components, every of which contains at least one non-degenerate cycle [4] and etc.

EXAMPLE OF APPLYING THE THEORY OF DECOMPOSITION TO THE CONSTRUCTION OF THE SOLUTIONS OF SPARSE UNDERDETERMINED SYSTEMS

For the graph $G = (I,U)$ (see Figure 2) we consider the sparse underdetermined system (11) – (12), where $I = \{1,2,3,4,5,6,7,8\}$ is the set of nodes of graph G, $I^* = \{1,4,5,7,8\}$ is the set of nodes with variable intensities. The set of arcs of the graph G is: $U = \{(1,2),(1,8),(2,8),(3,1),(3,7),(4,3),(4,6),(6,5),(6,7),(7,4),(7,8),(8,5)\}$.

$$
\begin{gathered}
x_{1,2} + x_{1,8} - \frac{1}{4}x_{3,1} = x_1, \quad x_{2,8} - \frac{1}{2}x_{1,2} = 5, \\
x_{3,1} + x_{3,7} - \frac{1}{7}x_{4,3} = 10, \quad x_{4,3} + x_{4,6} - \frac{2}{5}x_{7,4} = x_4, \\
-\frac{3}{5}x_{6,5} - \frac{1}{4}x_{8,5} = x_5, \quad x_{6,5} + x_{6,7} - x_{4,6} = -15, \\
x_{7,4} + x_{7,8} - \frac{2}{3}x_{3,7} - \frac{3}{4}x_{6,7} = -x_7, \quad x_{8,5} - \frac{1}{3}x_{1,8} - \frac{1}{5}x_{2,8} - \frac{4}{5}x_{7,8} = -x_8
\end{gathered}
\tag{11}
$$

$$
\begin{gathered}
x_{1,2} + 10x_{1,8} + 2x_{2,8} + 2x_{3,1} + x_{3,7} + x_{4,3} + 4x_{4,6} + x_{6,5} + \\
+3x_{6,7} + 13x_{7,4} + 2x_{7,8} + x_{8,5} + 7x_1 + 12x_4 + 19x_5 - 2x_7 + 13x_8 = 37 \\
2x_{1,2} + 4x_{1,8} + x_{3,1} + 3x_{3,7} + 7x_{4,3} + 2x_{4,6} + 5x_{6,5} + \\
+8x_{6,7} + x_{7,4} + 2x_{7,8} + 10x_{8,5} - 8x_1 + x_4 + 3x_5 - 3x_7 - 4x_8 = 53
\end{gathered}
\tag{12}
$$

Let the aggregate of sets $K = \{U_K, I_K^*\}$ is a support of the graph $G = \{I, U\}$ for system (11), (12). It is any graph, which includes a support $R = \{U_R, I_R^*\}$ of the graph G for the system (11) and the aggregate of sets $W = \{U_W, I_W^*\}$. For example, the support $R = \{U_R, I_R^*\}$ can be a forest of trees and each tree of the forest has exactly one node from the set I_R^*. Supporting elements that correspond to the aggregate R cover all the nodes of the set I of the graph $G = \{I, U\}$. In that example of support (a forest of trees), each tree of the forest $R = \{U_R, I_R^*\}$ contains only one node from set I_R^*.

In Figure 3 a support $R = \{U_R, I_R^*\}$ of the graph $G = (I,U)$ for the system (11) is presented: that is a forest from $|K| = 3$ trees U_R, where $U_R = \{U_T^k, k \in K\} : U_T^1 = \{(1,2),(1,8)\}$, $U_T^2 = \{(3,7)\}$, $U_T^3 = \{(4,6),(6,5)\}$, $I_R^* = \{1, 5, 7\}$. Each tree from forest $\{U_T^k, k \in K\}$, $K = \{1,2,3\}$ has only single node from the set $I_R^* = \{1,5,7\}$: $|I(U_T^k) \bigcap I_R^*| = 1$, $I(U_T^1) \bigcap I_R^* = \{1\}$, $I(U_T^2) \bigcap I_R^* = \{7\}$, $I(U_T^3) \bigcap I_R^* = \{5\}$. We chose a support $R = \{U_R, I_R^*\}$ of the graph $G = (I,U)$ for the sparse system (11) (see Figure 3). We compute nonzero components of each characteristic vector $\delta(\tau,\rho) = (\delta_{ij}^{\tau\rho}, (i,j) \in U; \delta_i^{\tau\rho}, i \in I^*), (\tau,\rho) \in U \setminus U_R$ and $\delta(\gamma)$ where $\delta(\gamma) = (\delta_{ij}^{\gamma}, (i,j) \in U; \delta_i^{\gamma}, i \in I^*), \gamma \in I^* \setminus I_R^*$.

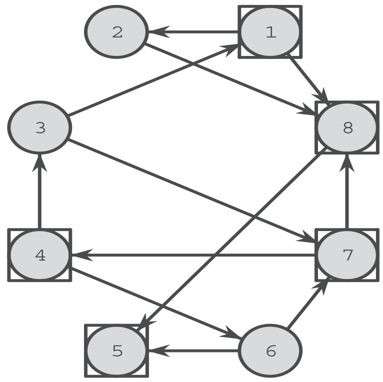

FIGURE 2. Finite connected directed graph G.

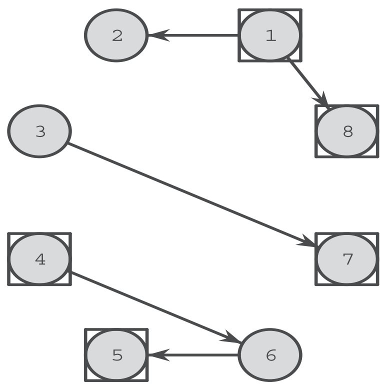

FIGURE 3. Graph support $R = \{U_R, I_R^*\}$, $U_R = U_T^1 \bigcup U_T^2 \bigcup U_T^3, I_R^* = \{1,\ 5,\ 7\}$.

Nonzero components of the characteristic vectors are equal:

$$\delta_{2,8}^{2,8} = 1,\ \ \delta_{1,8}^{2,8} = -\frac{3}{5},\ \ \delta_{1,2}^{2,8} = 2,\ \ \delta_{1}^{2,8} = \frac{7}{5};\quad \delta_{3,1}^{3,1} = 1,\ \ \delta_{3,7}^{3,1} = -1,\ \ \delta_{7}^{3,1} = -\frac{2}{3},\ \ \delta_{1}^{3,1} = -\frac{1}{4};$$

$$\delta_{4,3}^{4,3} = 1,\ \ \delta_{4,6}^{4,3} = -1,\ \ ,\delta_{6,5}^{4,3} = -1\ \ \delta_{3,7}^{4,3} = \frac{1}{7},\ \ \delta_{5}^{4,3} = \frac{3}{5},\ \ \delta_{7}^{4,3} = \frac{2}{21};$$

$$\delta_{6,7}^{6,7} = 1,\ \ \delta_{6,5}^{6,7} = -1,\ \ \delta_{5}^{6,7} = \frac{3}{5},\ \ \delta_{7}^{6,7} = \frac{3}{4};\quad \delta_{7,4}^{7,4} = 1,\ \ \delta_{4,6}^{7,4} = \frac{2}{5},\ \ ,\delta_{6,5}^{7,4} = \frac{2}{5}\ \ \delta_{5}^{7,4} = -\frac{6}{25},\ \ \delta_{7}^{7,4} = -1;$$

$$\delta_{7,8}^{7,8} = 1,\ \ \delta_{1,8}^{7,8} = -\frac{12}{5},\ \ \delta_{1}^{7,8} = -\frac{12}{5},\ \ \delta_{7}^{7,8} = -1;\quad \delta_{8,5}^{8,5} = 1,\ \ \delta_{1,8}^{8,5} = 3,\ \ \delta_{1}^{8,5} = 3,\ \ \delta_{5}^{8,5} = -\frac{1}{4}.$$

$$\delta_{4}^{4} = 1,\ \ \delta_{4,6}^{4} = 1,\ \ \delta_{6,5}^{4} = 1,\ \ \delta_{5}^{4} = -\frac{3}{5};\quad \delta_{8}^{8} = 1,\ \ \delta_{1,8}^{8} = 3,\ \ \delta_{1}^{8} = 3.$$

Using the formulas $\Lambda_{\tau\rho}^p = \lambda_{\tau\rho}^p + \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{\tau\rho} + \sum_{i\in I_R^*} \lambda_i^p \delta_i^{\tau\rho}$ and $\Lambda_{\gamma}^p = \lambda_{\gamma}^p + \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{\gamma} + \sum_{i\in I_R^*} \lambda_i^p \delta_i^{\gamma}$ we compute the numbers $\Lambda_{\tau\rho}^p$ – determinants of the structures, entailed by the arcs $(\tau,\rho) \in U \setminus U_R$, $p = 1,2$ and determinants of the structures, entailed by the nodes $\gamma \in I^* \setminus I_R^*$, $p = 1,2$, where the sets $U \setminus U_R$, $I^* \setminus I_R^*$ are defined as follows: $U \setminus U_R = \{(2,8),\ (3,1),\ (4,3),\ (6,7),\ (7,4),\ (7,8),\ (8,5)\}$, $I^* \setminus I_R^* = \{4,\ 8\}$. The nonzero components of a particular solution $\tilde{x} = (\tilde{x}_{ij},\ (i,j) \in U;\ \tilde{x}_i,\ i \in I^*)$ of system (11) are equal: $\tilde{x}_{6,5} = -15$, $\tilde{x}_5 = 9$, $\tilde{x}_{3,7} = 10$, $\tilde{x}_7 = \frac{20}{3}$, $\tilde{x}_{1,2} = -10$, $\tilde{x}_1 = -10$.

The numbers A_p, $p=\overline{1,q}$ we compute as follows:

$$A_p=\alpha_p-\sum_{(i,j)\in U_R}\lambda_{ij}^p\widetilde{x}_{ij}-\sum_{i\in I_R^*}\lambda_i^p\widetilde{x}_i. \tag{13}$$

Using the formulas (13) we calculate the numbers of A_1 and A_2: $A_1=-\frac{107}{3}$, $\quad A_2=31$. To form the matrix D of determinants of structures, generated by the elements of sets W with respect to the equations (12) with the numbers $p=1,2$, elements of the set $W=\{U_W,I_W^*\}$, $U_W=\{(3,1)\}$, $I_W^*=\{4\}$ must be arbitrarily numbered according to: $t(4)=1$, $t(3,1)=2$. The matrix D has the form:

$$D=\begin{pmatrix}\Lambda_4^1 & \Lambda_{3,1}^1\\ \Lambda_4^2 & \Lambda_{3,1}^2\end{pmatrix}=\begin{pmatrix}\dfrac{28}{5} & \dfrac{7}{12}\\ \dfrac{31}{5} & 2\end{pmatrix},\ \det D\neq 0.$$

We calculate the determinants of the structures generated by the arcs of the set U_N where $U_N=U\setminus(U_R\bigcup U_W)$ and nodes $\gamma\in I_N^*=I^*\setminus(I_R^*\bigcup I_W^*)$: $\Lambda_8^1=64$, $\Lambda_{2,8}^1=\frac{39}{5}$, $\Lambda_{4,3}^1=\frac{772}{105}$, $\Lambda_{6,7}^1=\frac{119}{10}$, $\Lambda_{7,4}^1=\frac{311}{25}$, $\Lambda_{7,8}^1=-\frac{184}{5}$, $\Lambda_{8,5}^1=\frac{189}{4}$, $\Lambda_8^2=-16$, $\Lambda_{2,8}^2=-\frac{48}{5}$, $\Lambda_{4,3}^2=\frac{68}{35}$, $\Lambda_{6,7}^2=\frac{51}{20}$, $\Lambda_{7,4}^2=\frac{152}{25}$, $\Lambda_{7,8}^2=\frac{73}{5}$, $\Lambda_{8,5}^2=-\frac{11}{4}$.

Using [1] we compute the components of the vector $\beta'=(\beta_1,\beta_2)$:

$$\beta=\begin{pmatrix}-\frac{107}{3}-64x_8-\frac{39}{5}x_{2,8}-\frac{772}{105}x_{4,3}-\frac{119}{10}x_{6,7}-\frac{311}{25}x_{7,4}+\frac{184}{5}x_{7,8}+\frac{11}{4}x_{8,5}\\ 31+16x_8+\frac{48}{5}x_{2,8}-\frac{68}{35}x_{4,3}-\frac{51}{20}x_{6,7}-\frac{152}{25}x_{7,4}-\frac{73}{5}x_{7,8}+\frac{11}{4}x_{8,5}\end{pmatrix}.$$

Since the matrix D is non-singular, we compute from the system $Dx_W=\beta$ the unknown of the desired solution of underdetermined sparse system (11) – (12) that corresponds to the components of the vector $x_W=(x_{3,1},x_4)$:

$$x_{3,1}=\frac{1}{15925}(828940+1021440x_8+214452x_{2,8}+72880x_{4,3}+124950x_{6,7}+90468x_{7,4}-650832x_{7,8}+647535x_{8,5}),$$

$$x_4=\frac{1}{12740}(-150220-230720x_8-35616x_{2,8}-22800x_{4,3}-37485x_{6,7}-35840x_{7,4}+137956x_{7,8}-161455x_{8,5}).$$

We substitute the components of the vector x_W in the system (11). Using the graph theoretic properties of the support for the sparse system (11), we calculate unknown system that correspond to the arcs U_R of forest trees and unknown system that correspond to the nodes I_R^*. The independent variables $x_{\tau,\rho}$, $(\tau,\rho)\in U_N$ and x_γ, $\gamma\in I_N^*$ in the sparse underdetermined system (11) – (12) are: x_8, $x_{2,8}$, $x_{4,3}$, $x_{6,7}$, $x_{7,4}$, $x_{7,8}$, $x_{8,5}$.

Thus, the general solution of the system (11) – (12) for the elements of the aggregate $R=\{U_R,I_R^*\}$ has the form

$$x_1=\frac{1}{63700}(-1465940-830340x_8-125272x_{2,8}-72880x_{4,3}-124950x_{6,7}-90468x_{7,4}+497952x_{7,8}-456435x_{8,5}),$$

$$x_{1,2}=2(-5+x_{2,8}),\qquad x_{1,8}=\frac{3}{5}(5x_8-x_{2,8}-4x_{7,8}+5x_{8,5}),$$

$$x_{3,7}=-\frac{3}{15925}(223230+340480x_8+71484x_{2,8}+23535x_{4,3}+41650x_{6,7}+30156x_{7,4}-216944x_{7,8}+215845x_{8,5}),$$

$$x_{4,6}=-\frac{1}{12740}(150220+230720x_8+35616x_{2,8}+35540x_{4,3}+37485x_{6,7}+30744x_{7,4}-137956x_{7,8}+161455x_{8,5}),$$

$$x_5=\frac{1}{63700}(1023960+692160x_8+106848x_{2,8}+106620x_{4,3}+150675x_{6,7}+92232x_{7,4}-413868x_{7,8}+468440x_{8,5}),$$

$$x_{6,5}=\frac{1}{12740}(-341320-230720x_8-35616x_{2,8}-35540x_{4,3}-50225x_{6,7}-30744x_{7,4}+137956x_{7,8}-161455x_{8,5}),$$

$$x_7=\frac{1}{63700}(-1785840-2723840x_8-571872x_{2,8}-188280x_{4,3}-285425x_{6,7}-$$
$$-304948x_{7,4}+1671852x_{7,8}-1726760x_{8,5}).$$

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