SOME PROPERTIES OF GERHOLD — GARRA — POLITO FUNCTION

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Recently S. Gerhold [1] and R. Garra — F. Polito [2] have introduced independently a new function

$$F^{(\gamma)}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(\alpha k+\beta)]^{\gamma}}, \quad z \in \mathbb{C}, \quad \alpha,\beta,\gamma > 0. \tag{1}$$

For positive values of parameters the function $F^{(\gamma)}_{\alpha,\beta}(z)$ is an entire function of the complex variable z. It belongs to the family of Mittag — Leffler functions (see the recent monograph [3]).

The function (1) is related to so-called Le Roy function

$$\sum_{k=0}^{\infty} \frac{z^k}{[(k+1)!]^{\gamma}}, \quad z \in \mathbb{C},$$

which was used in [4] at the study of asymptotics of the analytic continuation of the sum of power series.

In this report we present three results, namely, the integral representation of the Gerhold — Garra — Polito function, its asymptotics on the negative semi-axes and the Laplace transform of this function.

We consider the function $F^{(\gamma)}_{\alpha,\beta}(z)$ in the case of positive values of all parameters $(\alpha,\beta,\gamma > 0)$. In this case the function $\Gamma(\alpha s+\beta)$ is a meromorphic function of complex variable s which have negative simple poles at points $s = -(\beta+k)/\alpha$, $k = 0,1,2,\ldots$ We fix the principal branch of the multi-valued function $[\Gamma(\alpha s+\beta)]^{\gamma}$ by drawing the cut along the negative semi-axes starting from $-\beta/\alpha$, ending at $-\infty$ and by supposing that $[\Gamma(\alpha x+\beta)]^{\gamma}$ is positive for all positive x. Let also the function $(-z)^s$ be defined in the complex plane cut along negative semi-axis and

$$(-z)^s = \exp\{s[\log|z| + i\arg(-z)]\},$$

where $\arg(-z)$ is any arbitrary chosen branch of $\mathrm{Arg}(-z)$.

Theorem 1. *Let $\alpha,\beta,\gamma > 0$ and $[\Gamma(\alpha s+\beta)]^{\gamma}$, $(-z)^s$ be the described branches of the corresponding multivalued functions. Then the Gerhold — Garra — Polito function posesses the following integral representation of the Mellin — Barnes type*

$$F^{(\gamma)}_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}_{+\infty}} \frac{\Gamma(-s)\Gamma(1+s)}{[\Gamma(\alpha s+\beta)]^{\gamma}} (-z)^s\, ds + \frac{1}{[\Gamma(\beta)]^{\gamma}}, \tag{2}$$

where $\mathcal{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at the point $+\infty + i\varphi_1$ and terminating at the point $+\infty + i\varphi_2$, $-\infty < \varphi_1 < 0 < \varphi_2 < +\infty$, crossing the real line at a point c, $0 < c < 1$.

To get this representation we first show an existence of the integral in the right-hand side of (2), and the calculate it by using the Residue Theorem. Similar representation can be obtained for another type of the integration contour. In this case we consider the multivalued function $[\Gamma(\alpha(-s)+\beta)]^{\gamma}$ and fix its principal branch by drawing the cut along the positive semi-axis starting from β/α and ending at $+\infty$ and supposing that $[\Gamma(\alpha(-x)+\beta)]^{\gamma}$ for all negative x. We also define the function z^{-s} in the complex plane cut along negative semi-axis and

$$z^{-s} = \exp\{(-s)[\log|z| + i\arg z]\},$$

where $\arg z$ is any arbitrary chosen branch of $\operatorname{Arg} z$.

Theorem 2. *Let $\alpha, \beta, \gamma > 0$ and $[\Gamma(\alpha(-s)+\beta)]^\gamma$, z^{-s} be the described branches of the corresponding multivalued functions. Then the Gerhold — Garra — Polito function posesses the following integral representation of the Mellin — Barnes type*

$$F_{\alpha,\beta}^{(\gamma)}(z) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}_{-\infty}} \frac{\Gamma(s)\Gamma(1-s)}{[\Gamma(\alpha(-s)+\beta)]^\gamma} z^{-s}\, ds + \frac{1}{[\Gamma(\beta)]^\gamma},$$

where $\mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at the point $-\infty + i\varphi_1$ and terminating at the point $-\infty + i\varphi_2$, $-\infty < \varphi_1 < 0 < \varphi_2 < +\infty$, crossing the real line at a point c, $-1 < c < 0$.

In order to describe the asymptotics of the considered function we use representation (2) and move the contour $\mathcal{L}_{+\infty}$ to the left up to the contour $\mathcal{L}_{+\infty}^{(k)}$ lying in the same strip as that for $\mathcal{L}_{+\infty}$, but crossing the negative semi-axes at a point $c_k \in (-k-1, -k)$. Then the following theorem holds.

Theorem 3. *Let $\alpha, \beta > 0$, $\gamma = m \in \mathbb{N} \setminus \{1\}$. Then the Gerhold — Garra — Polito function has the following representation for negative real values of $z = -t$:*

$$\mathcal{F}_{\alpha,\beta}^{(m)}(-t) = -\sum_{j=1}^{k} \frac{(-t)^{-j}}{[\Gamma(-\alpha j+\beta)]^m} + O(t^{-k-1}), \quad t \to +\infty.$$

In applications it is important to know one or another form of the Laplace transform of the Gerhold — Garra — Polito function. Here we present two of the possible form of such transform.

Lemma 1. *Let $\alpha, \beta > 0$, $\gamma > 1$ be positive numbers, $\lambda \in \mathbb{C}$. The following formula is true for the Laplace transforms of the Gerhold — Garra — Polito function:*

$$(\mathcal{L} t^{\beta-1} \mathcal{F}_{\alpha,\beta}^{(\gamma)}(\lambda t^\alpha))(s) = \frac{1}{s^\beta} \mathcal{F}_{\alpha,\beta}^{(\gamma-1)}(\lambda s^{-\alpha}).$$

Lemma 2. *Let $\alpha, \beta > 0$, $\gamma = m \in \mathbb{N}$ be positive numbers. The following formula is true for the Laplace transforms of the Gerhold — Garra — Polito function:*

$$(\mathcal{L}\mathcal{F}_{\alpha,\beta}^{(m)}(t))(s) = \frac{1}{s}\, {}_2\Psi_m \left(\left[\begin{array}{c} (1,1),(1,1) \\ \underbrace{(\beta,\alpha),\ldots,(\beta,\alpha)}_{m\ times} \end{array} \right] \frac{1}{s} \right),$$

where ${}_p\Psi_q$ is the generalized Wright function (*see* [3, Appendix F])

$${}_p\Psi_q(z) = {}_p\Psi_q\left(\left[\begin{array}{c} (a_l,\alpha_l)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array} \right] z \right) = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p} \Gamma(a_l+\alpha_l k)}{\prod_{j=1}^{q} \Gamma(b_j+\beta_j k)} \frac{z^k}{k!}.$$

Refrences

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