БИБЛИОГРАФИЧЕСКИЙ СПИСОК

1. Cramer R., Damgard I., Nielsen J. Multiparty Computation from Threshold Homomorphic Encryption // LNCS. 2001. Vol. 2045. P. 280–300.
2. Bethencourt J., Sahai A., Waters B. Ciphertext-policy attribute-based encryption // Proc. of IEEE Symposium on Security and Privacy. California, 2007. P. 321–334.
3. Benaloh J. Secret sharing homomorphisms: keeping shares of a secret secret // LNCS. 1987. Vol. 263. P. 251–260.
4. Blum M., Feldman P., Micali S. Non Interactive Zero-Knowledge and Its Applications // Proc. of the 20th ACM Symposium on Theory of Computing. ISSN for Comp. Machinery. New York, 1988. P. 103–112.
5. Feldman P. A practical scheme for non-interactive verifiable secret sharing // IEEE Symposium on Foundations of Computer Science. Washington, 1987. P. 427–437.
6. Asmuth C. A., Bloom J. A modular approach to key safeguarding // IEEE Transactions on Information Theory. 1983. Vol. 29. P. 156–169.
7. Galibus T., Matveev G., Shenets N. Some structural and security properties of the modular secret sharing // Proc. of SYNASC’08. Los Alamitos, 2009. P. 197–200.
8. Galibus T., Matveev G. Generalized Mignotte Sequences in Polynomial Rings // ENTCS. 2007. Vol. 186.
9. Iftene S. Secret sharing schemes with applications in security protocols: technical report TR 07-01 / University Alexandru Ioan Cuza of Iasi, Faculty of Computer Science. Iasi, 2007.
10. A non-interactive modular verifiable secret sharing scheme / L. Qiong [et al.] // Proc. of ICCCAS’05. Hong Cong, 2005. P. 84–87.
11. Kaya K., Selcuk A. Verifiable Secret Sharing Scheme Based on the Chinese Remainder Theorem // LNCS. 2008. Vol. 5365. P. 288–305.
12. Mignotte M. How to share a secret // LNCS. 1982. Vol. 149. P. 371–375.
13. Quisquater M., Preneel B., Vandewalle J. On the security of the threshold scheme based on the Chinese remainder theorem // LNCS. 2002. Vol. 2274. P. 199–210.
14. Галибус Т. В. Разделение секрета над полиномиальными кольцами // Вестн. БГУ. Сер. 1, Физика. Математика. Информатика. 2006. № 2. С. 97–100.

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ON THE PSEUDO-STABILITY OF SEMIDYNAMICAL SYSTEMS

Введено свойство псевдоустойчивости как необходимое условие орбитальной устойчивости замкнутых инвариантных множеств полудинамических систем на произвольном метрическом пространстве. Приведена классификация устойчивоподобных свойств в форме диаграммы, которая отражает взаимоотношения псевдоустойчивости и равномерной псевдоустойчивости с известными характеристиками качественной теории устойчивости полудинамических систем (инвариантность, устойчивость, притяжение и их модификации). Установлена связь между понятием псевдоустойчивости и определением первого интеграла полудинамической системы. Сформулированы критерии псевдоустойчивости в форме достаточных условий с использованием определенно положительных и знакопостоянных функций Ляпунова. Даны комментарии к результатам на иллюстративных примерах.

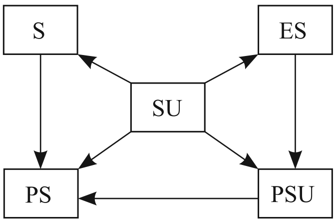
Ключевые слова: полудинамическая система; положительно инвариантное множество; устойчивость; псевдоустойчивость; первый интеграл; функция Ляпунова.

Pseudo-stability property is introduced as a necessary condition of the orbital stability of closed positively invariant sets of semidynamical systems defined on an arbitrary metric space. We give a classification of stability-like properties in the form of a diagram. The diagram reflects the relationship between pseudo-stability and uniform pseudo-stability with known characteristics of qualitative stability theory of motion of semidynamical systems (invariance, stability, attraction and their modifications). We establish particular connection between the pseudo-stability notion and first integrals of semidynamical systems. The criteria of pseudo-stability are formulated and sufficient conditions for this property with positive definite and semidefinite Lyapunov functions are provided. We also give comments on the results with a number of illustrating examples.

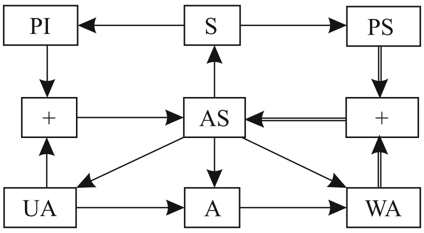
Key words: semidynamical system; positively invariant set; stability; pseudo-stability; first integral; Lyapunov function.

Overview of results

Over the past 50 years development of the qualitative theory of dynamical systems has contributed greatly to the generation of methods of topological dynamics with respect to objectives of stability theory of movement [1]. In V. I. Zubov monograph [2] author presents the research methods for problems of the stability theory of dynamical systems $(X, \mathbb{R}, \pi)$ defined on a metric space (X, d). The author gives the basics of direct Lyapunov’s method and lays the foundation of qualitative study of the structure of closed invariant sets’ neighbourhoods in terms of their stability properties. In [3–5] authors introduce an interesting idea of splitting the Lyapunov's definition of asymptotic stability into two components: stability and attraction. The foundations of the latter were laid by the examples of dynamic processes with the attracting, but not stable, equilibrium. Thereafter, attracting invariant sets (attractors) were thoroughly studied. The concepts of uniform attractor [6], weak attractor [7], etc., were introduced. This in turn has led to identification and studying of semi-asymptotic stability and uniform asymptotic stability of invariant sets by method of Lyapunov functions [7]. For locally compact dynamical systems the author in his work [8] reveals the reason for the existence of attraction properties for invariant sets in the absence of the stability property. By introducing a new definition of «pseudo-stability», a criterion is proved [8, p. 129, theorem 3.5], stating that a missing component of the weak attraction property of a compact invariant set to constitute an asymptotic stability property is actually pseudo-stability. This result was further developed for dynamical systems on an arbitrary metric space in [9]. A system of differential equations in the plane $\dot{x} = y$, $\dot{y} = 0$ gives us a simple example of pseudo-stable, but not stable, stationary point (0, 0). The pseudo-stability property is a necessary condition for stability in the sense of Lyapunov. This can be easily shown on a diagram 1 [8] (S – stability, SU – uniform stability, PS – pseudo-stability, PSU – uniform pseudo-stability, ES – equi-stability) and diagram 2 (PI – positive invariance, A – attraction, UA – uniform attraction, WA – weak attraction, AS – asymptotic stability).



Diagr. 1. Stability and pseudo-stability



Diagr. 2. Attraction and pseudo-stability

This paper contributes to further development of the theory of pseudo-stability for semidynamical systems on an arbitrary metric space.

Definitions and notations

We recall the definitions and some properties semidynamical systems. Let X be a metric space with a function of distance $d: X \times X \to \mathbb{R}^+$. A semidynamical system on X is the triplet $\left(X, \mathbb{R}^+, \pi\right)$, where π is the phase map $\left(\pi\,(x, t) = xt, \forall x \in X \text{ and } \forall t \in \mathbb{R}^+\right)$ satisfying the following axioms:

1) $\pi\,(x, 0) = x, \forall x \in X$;
2) $\pi\,(\pi\,(x, t), \tau) = \pi\,(x, t + \tau), \forall x \in X$ and $\forall t, \tau \in \mathbb{R}^+$;
3) π – is continuous.

Given a dynamical system on X, the space X and the map $x: t \to xt$ $\left(x \in X \text{ и } t \in \mathbb{R}^+\right)$ are respectively called the phase space and the movement of the semidynamical system. In line with this notation, if $N \subset X$ and $I \subset \mathbb{R}^+$, then NI is the set $NI = \{x \in X: x \in N, t \in I\}$. For any $x \in X$, the set $\gamma^+(x) = x\mathbb{R}^+$ is called the positive semi-trajectory through x (or of x). The set $N \subset X$ is called positively invariant if $N\mathbb{R}^+ = N$. The set N is invariant if simultaneously N and $X \setminus N$ are positively invariant sets.

We use the following concepts [10, 11]. Let $\Delta(x)$ be an interval of the existence of motion $x: t \to xt$. The map $\bar{x}: t \to \bar{x}t$ is called an extension of the motion $x: t \to xt$ if $\Delta(\bar{x}) \supset \Delta(x)$ and $xt = \bar{x}t$ on $\Delta(x)$. The motion $x: t \to xt$ is called maximal motion if for every extension y of this motion: $\Delta(y) = \Delta(x)$ (and hence $yt = xt$ on $\Delta(x)$). It is clear that the extension of motion $x: t \to xt$, $t \in \mathbb{R}^+$, in the negative direction is not unique. In this case for every $t \in \mathbb{R}^-$ with defined motion we pose $xt = \{y \in X : x \in y(-t)\}$.

The following result is known [10, ch. I, theorem 4.5]. Let $x: t \to xt$ be the maximal motion through $x \in X$. Then one of the following statements holds:

1) $\Delta(x) = \mathbb{R}$;
2) $\Delta(x) = [-a_x, \infty[$ for some number $a_x \in \mathbb{R}^+$;
3) $\Delta(x) =]-a_x, \infty[$ for some number $a_x > 0$.

For any maximal motion $x: t \to xt$ in accordance with the statement of the [10, ch. I, theorem 2] we pose:

$$\gamma(x) = \left\{y \in X: y = xt, t \in \mathbb{R}^-, \text{ if } \Delta(x) = \mathbb{R}\right\};$$

$$\gamma^-(x) = \gamma(x) \setminus \gamma^+(x);$$

$$\Gamma(x) = \left\{y \in X: y = xt, t \in \left[-a_x, 0\right], \text{if } \Delta(x) = \left[-a_x, +\infty\right[\right\};$$

$\Gamma^*(x) = \{y \in X: y = xt, t \in]-a_x, 0]$, if $\Delta(x) =]-a_x, +\infty[\}$.

The sets $\gamma(x)$, $\Gamma(x)$ and $\Gamma^*(x)$ are called, respectively, the full funnel, the ultimate negative funnel and the local negative funnel of point x.

The concept of the ω-limit set $L^+(x)$ [10, 11] is introduced for each motion $x: t \to xt$ of semidynamical system, just as for the dynamic system [7].

The identification of the α-limit set $L^-(x)$ is only appropriate for principal movements [12]. We introduce it in the following way.

A point $x \in X$ is called α-limit point of the principal movement $x: t \to xt$, $t \in \mathbb{R}^+$, if there are such sequences $(x_n) \subset \gamma(x)$ and $(t_n) \subset \mathbb{R}^-$ that the following conditions are true:

1) $x_n(-t_n) = x\ \forall n \geq 1$;

2) $t_n \to -\infty$ as $n \to +\infty$;

3) $x_n \to p$ as $n \to +\infty$.

We denote by $\bar{N}$ and FrN, respectively, the closure and the boundary of the set $N \subset X$. Let $B(N, \alpha) = \{x \in X: d(N, x) < \alpha\}$, where $\alpha > 0$, and we recall the following definitions [7].

A closed set $M \subset X$ is called:

- stable, if $(\forall \varepsilon > 0)(\forall m \in M)(\exists \delta = \delta(\varepsilon, m) > 0) \Rightarrow B(m, \delta)\mathbb{R}^+ \subset B(M, \varepsilon)$;
- uniformly stable, if $(\forall \varepsilon > 0)(\exists \delta > 0) \Rightarrow B(M, \delta)\mathbb{R}^+ \subset B(M, \varepsilon)$.

Pseudo-stability

In this section, we introduce the notions of pseudo-stability for closed positively invariant sets.

Definition 1. A set $M \subset X$ is said to be [8, 12]:

- pseudo-stable, if $(\forall x \notin M)(\forall m \in M)(\exists \delta = \delta(x, m) > 0) \Rightarrow x \notin B(m, \delta)\mathbb{R}^+$;
- uniformly pseudo-stable, if $(\forall x \notin M)(\exists \delta = \delta(x) > 0) \Rightarrow x \notin B(M, \delta)\mathbb{R}^+$.

Remark 1. The introduced notions of pseudo-stability do not depend on whether a set M is closed or compact. The connection between the concept of positive invariance, as well as the property of being an open or closed set, and the notion of pseudo-stability is discussed further below. Examples of stable (and thus pseudo-stable), but not uniformly pseudo-stable, as well as uniformly pseudo-stable, but not stable invariant sets are given in [8, 12].

Example 1. Consider the set $X = \{a\varphi: a \in \mathbb{R}\}$ of continuous functions φ: $\mathbb{R}^+ \to \mathbb{R}$ where $\varphi(t) = \exp(t^2)$, $\forall t \in \mathbb{R}$. We enter in X the topology of uniform convergence on compacts $\mathbb{R}^+$ using metric of Bebutov [13]:

$$d(\varphi_1, \varphi_2) = \sup_{T>0} \min\left\{\max_{0 \leq t \leq T} |\varphi_1(t) - \varphi_2(t)|, \frac{1}{T}\right\}, \quad \varphi_1, \varphi_2 \in X, \quad T \in \mathbb{R}^+.$$

We denote $X = \{\varphi_\tau: \tau \in \mathbb{R}^+\}$ and define semidynamical system $(X, \mathbb{R}^+, \pi)$, where $\varphi_\tau(y, t) = \varphi(y, t + \tau)$ is the translation of function $\varphi(y, t)$ on τ in X. Then it is easy to show that the equilibrium $\theta(t) \equiv 0$, $\forall t \notin \mathbb{R}^+$ is pseudo-stable. It is enough to use [13, p. 76, lemma 1.20] stating that for Bebutov's metric the inequality $d(\varphi_1, \varphi_2) \leq \sigma$ is equivalent to the condition $\max\limits_{0 \leq t \leq 1/\sigma} |\varphi_1(t) - \varphi_2(t)| \leq \sigma$. One can show that the point $\theta(t) \equiv 0$ is not stable in the space X.

Example 2. Consider the linear system of differential equations $\dot{x} = Ax$, $x \in \mathbb{R}^n$, with constant $(n \times n)$ matrix A and singleton set $M = \{0\}$. The set M is pseudo-stable if and only if the real parts of all eigenvalues of matrices A are not positive.

A number of pseudo-stability properties of compact invariant set M is provided in the author's monograph [8] for a dynamical system $(X, \mathbb{R}, \pi)$ given in the locally compact metric space (X, d). It should be noted that part of them is true for common dynamic processes studied in this article. We formulate the ones that you can easily apply in case of a semidynamical system $(X, \mathbb{R}^+, \pi)$ on an arbitrary metric space relative to the closed positively invariant set $M \subset X$, namely:

(*i*) if the set M is pseudo-stable, then it is positively invariant;

(*ii*) the open set M is pseudo-stable if and only if it is positively invariant;

(*iii*) if the set M is uniformly pseudo-stable, then it is closed;

(*iv*) if the set M is stable (uniformly stable) then it is pseudo-stable (respectively, uniformly pseudo-stable).

In [8] the classification of stable-like properties is given, combining traditional classic characteristics of invariant sets in terms of their stability properties, however, with the introduced new concept of pseudo-stability.

There are diagr. 1 and 2 for the case of general semidynamic system $(X, \mathbb{R}^+, \pi)$ given on an arbitrary metric space (X, d).

Criteria pseudo-stability

The following theorems are pseudo-stability criteria.

Theorem 1. *A set M is pseudo-stable if and only if the following conditions are true:*

1) *M is positively invariant*;

2) $L^{-}(x)\bigcap M=\varnothing,\ \forall x\in X\setminus M.$

P r o o f. Suppose that for a set $M\subset X$ conditions 1) and 2) are met. We assume here that M is not pseudo-stable. Then there is an element $m\in M$ such that for any $\delta>0$ there is $x\in B(m,\delta)\mathbb{R}^{+}$. In other words, you can specify sequences $(x_n)\subset X\setminus M$, $x_n\to m$, and $(t_n)\subset\mathbb{R}^{+}$ such that $x_nt_n=m,\ \forall n\geq 1$. Since M is positively invariant, we can show that $t_n\to+\infty$ as $n\to+\infty$. Indeed, if it is not, then there exists a bounded subsequence $(t_{n_k})_{k\geq 1}\subset(t_n)$ such that $t_{n_k}\to t^{*}\in\mathbb{R}^{+}$ if $k\to+\infty$. In this case, the continuity of the phase map π gives: $mt^{*}=x$. However, this contradicts the positive invariance of the set M. On this basis, it can be argued that there is a negative semi-trajectory $\gamma^{-}(x)$. Now, since $m\in\gamma^{-}(x)$, the set $L^{-}(x)$ is non-empty. Consequently, $m\in L^{-}(x)$ and this contradicts 2). Thus, M is pseudo-stable.

Conversely, if M is pseudo-stable, then it is positively invariant (property (i)). If the hypothesis 2) of the theorem does not hold, then there exists a point $x\in X\setminus M$ with a non-empty set $L^{-}(x)$. Subsequently, you can specify a point $m\in L^{-}(x)\bigcap M$. Then, by definition, $\forall\delta>0,\ \exists\ \tau>0$ for which $m\tau\in B(x,\delta)\mathbb{R}^{+}$, and by the arbitrariness of $\delta>0$ it is contrary to pseudo-stability of M. Theorem 1 is proved.

Theorem 2. *A subset $M\subset X$ is uniformly pseudo-stable if and only if there is a neighborhood U of M, such that the following conditions are met:*

1) *M is positively invariant*;

2) $d\left(\gamma^{-}(x),\ M\right)>0,\ \forall x\in U\setminus M.$

The proof of theorem 2 is similar to the proof of theorem 1.

Pseudo-stability and first integrals

We now proceed to establish some relationship between pseudo-stability and presence of first integral of semidynamical systems.

Definition 2. A continuous function $\varphi\colon X\times\mathbb{R}^{+}\to\mathbb{R}$ is called a first integral of semi-dynamic system $\left(X,\mathbb{R}^{+},\pi\right)$ if $\varphi(xt,t)\equiv\varphi(x,0)$ for any movement $x\colon t\to xt,\ t\geq 0.$

Theorem 3. *A closed set $M\subset X$ is uniformly pseudo-stable if there is a neighborhood U of M and a first integral $\varphi\colon U\times\mathbb{R}^{+}\to\mathbb{R}$ such that the following conditions are met:*

1) $\varphi(x,t)=0,\ \forall x\in M\ \ u\ \ \forall t\geq 0;$

2) *M is uniformly pseudo-stable with respect to set* $Y_0=\left\{x\in U\colon\varphi(x,t)=0,\forall t\geq 0\right\}.$

P r o o f. Suppose that the semi-dynamic system $\left(X,\mathbb{R}^{+},\pi\right)$ has a first integral with the specified properties and, however, M is not uniformly pseudo-stable. Then under the assumption 2), there exist a point $x\in Y_0\setminus M$ in U and the sequence (x_n), $x_n\to M$ such that $x\in\gamma^{+}(x_n),\ \forall n\geq 1$. This means that there exists a sequence $(t_n)\subset\mathbb{R}^{+}$ such that $x_nt_n=x,\ \forall n\geq 1$. In this case, according to the definition of the first integral, we have $\varphi(x,0)=\varphi\left(x_nt_n\right)=\varphi\left(x_n,0\right),\ \forall n\geq 1$ and $\forall t\geq 0$. Moving here to the limit as $n\to+\infty$, we obtain $\varphi(x,0)=0$. In other words, $x\in M$. However, the latter contradicts the choice of point x. Thus, M is uniformly pseudo-stable.

Consider a dynamical system $\dot{x}=x,\ \dot{y}=-y,\ (x,y)\in\mathbb{R}^{2}$. The set $M=\left\{(x,y)\in X=\mathbb{R}^{2}:y=0\right\}$ is pseudo-stable, since the first integral $\varphi(x,y)=xy$ is equal to zero outside M, where $x=0$. The set M is uniformly stable with respect to points where $x=0$, and hence uniformly pseudo-stable. According to theorem 1, M is uniformly pseudo-stable.

Pseudo-stability and Lyapunov's direct method

For pseudo-stability and uniform pseudo-stability we can state the following criterions in the terms of Lyapunov's functions.

Theorem 4. *A positively invariant subset $M\subset X$ is uniformly pseudo-stable if and only if there exists a function $V\colon X\to\mathbb{R}$ such that:*

1) $V(x)>0,\ \forall x\notin M$ *and* $V\left(x_n\right)\to 0$ *as* $d\left(x_n,M\right)\to 0;$

2) $V(xt)\leq V(x),\ \forall t\geq 0$ *and* $\forall x\in X.$

P r o o f. Suppose that the conditions 1) and 2) are met and M is not uniformly pseudo-stable. Then, there exists $x\in X\setminus M$ contained in $B(M,\delta)\mathbb{R}^{+}$ for any $\delta>0$. Consequently, $\forall\left(\delta_n\right)_{n\geq 1}\left(\delta_n>0,\delta_n\to 0\right),\exists\left(x_n\right)\subset X\setminus M$

$(x_n \to M)$ and $\exists (t_n) \subset \mathbb{R}^+$ such that $x_n t_n = x$, $\forall n \geq 1$. According to hypothesis 1) and 2), we have $0 < V(x_n t_n) \leq V(x_n)$. Therefore, $V(x_n) \to 0$ as $d(xt_n, M) \to 0$, which implies that $V(x) = 0$. This is impossible in accordance with the assumption.

Conversely, suppose that M is uniformly pseudo-stable. Then, by assertion (*iii*), the set M is closed. We pose:

$$V(x) = \begin{cases} d\left(\gamma^-(x), M\right) \ \forall x \in X \setminus M, \text{ if } \Delta(x) = \mathbb{R}, \\ d(\Gamma(x), M) \ \forall x \in X \setminus M, \text{ if } \Delta(x) = [-a_x, +\infty[, \\ d\left(\Gamma^*(x), M\right) \ \forall x \in X \setminus M, \text{ if } \Delta(x) =]-a_x, +\infty[. \end{cases} \tag{1}$$

Where $\Gamma(x)$ and $\Gamma^*(x)$ are defined above. It is easy to see that $V(x_n) \to 0$, if $d(x_n, M) \to 0$. We show that $V(x) > 0$, $\forall x \in X \setminus M$. In fact, if it is not true, there exists $x \in X \setminus M$, such that $V(x) = 0$. Note, that if there is semi-trajectory $\gamma^-(x)$, then $\gamma^-(x) \cap M = \varnothing$ since $M\mathbb{R}^+ = M$, and consequently the equality $V(x) = 0$ implies the existence of sequence $(t_n) \subset \mathbb{R}^-$, for which $d(xt_n, M) \to 0$. Moreover, since $M\mathbb{R}^+ = M$, $t_n \to -\infty$ (see the proof of theorem 2). In this case $d\left(\gamma^-(x), M\right) > 0$, which is contrary to theorem 3, if we take into consideration that M is uniformly pseudo-stable. The theorem is proved.

Hence follows the sufficient condition of pseudo-stability.

Consequence 1. *A positively invariant set $M \subset X$ is uniformly pseudo-stable if there is a function $V: X \to \mathbb{R}$ such that:*

1) $V(x) \geq 0$, $\forall x \in X \setminus M$ *and* $V(x_n) \to 0$ *if* $d(x_n, M) \to 0$;

2) $V(xt) \leq V(x)$, $\forall t \geq 0$ *and* $\forall x \in X \setminus M$;

3) $\{y \in X \setminus M: V(y) = 0\} \cap \{y \in X \setminus M: d\left(\gamma^-(x), M\right) > 0\} = \varnothing$.

Similarly, we have the following assertions.

Theorem 5. *A positively invariant set $M \subset X$ is pseudo-stable if and only if there is a function $V: X \to \mathbb{R}$ such that:*

1) $V(x, m) \geq 0$, $\forall x \in X \setminus M$ *and* $\forall m \in M$; $V(x_n, m) \to 0$, *if* $d(x_n, m) \to 0$ *for* $m \in M$;

2) $V(xt, m) \leq V(x, m)$, $\forall t \geq 0$, $\forall x \in X \setminus M$ *and* $\forall m \in M$.

Consequence 2. *A positively invariant set $M \subset X$ is pseudo-stable if there is a function $V: X \to \mathbb{R}$ such that:*

1) $V(x, m) \geq 0$, $\forall x \notin M$ *and* $\forall m \in M$; $V(x_n, m) \to 0$, *if* $d(x_n, m) \to 0$ *as* $n \to +\infty$, $\forall m \in M$;

2) $V(xt, m) \leq V(x, m)$, $\forall t \geq 0$, $\forall x \in X \setminus M$ *and* $\forall m \in M$;

3) $L^-(x) \cap M = \varnothing$, $\forall m \in M$ *and* $\forall x \in \{y \in X \setminus M: V(y, m) = 0\}$.

Example 3. For the system of differential equations in the plane $\mathbb{R}^2$ $\dot{x} = y$, $\dot{y} = -y$ the closed set $M = \{(x, y) \in \mathbb{R}^2: y = 0\}$ is uniformly pseudo-stable by theorem 3, if we pose $V(x, y) = y^2$. Note that for this system with compact set $M = (0, 0)$, the function (1) is defined by the condition:

$$V(x, y) = \sqrt{x^2 + y^2}, \text{ if } y(y - x) \geq 0 \text{ and } V(x, y) = |x + y| / \sqrt{2}, \text{ if } xy(y - x) \leq 0.$$

The function $V(x, y)$ guarantees the property of pseudo-stability for point (0, 0) and is not continuous.

Remark 2. 1. If M is singleton, the difference between the hypothesis of theorem 4 (on the uniform pseudo-stability) and hypothesis of [13, p. 111, theorem 1.26] (on the Lyapunov stability) is that function V in theorem 4 can be equal to zero outside the set M. This circumstance applies to the comparison of the consequence 1 with the theorem of stability in the method of semidefinite Lyapunov's functions [14].

2. Another situation occurs when we compare hypotheses of theorem 5 (on the pseudo-stability) with those of [2, theorem 12] (on the uniform stability) in case of closed invariant set M. Here the hypotheses of [2, theorem 12] are more severe than the hypotheses of theorem 5.

BIBLIOGRAPHY

1. Liapunov A. M. Stability of motion. New York, 1966.
2. Zubov V. I. The Method of Liapunov and their Applications. Leningrad, 1964.
3. Vinograd R. E. The inadequacy of the method of characteristic exponents for the study of nonlinear differential equations // Mat. sbornic. 1957. Vol. 41 (83). № 4. P. 431–438 (Russian).
4. Mendelson B. P. On unstable attractors // Bol. Math. Mexicana. 1960. Vol. 5. P. 270–276.
5. Antosiewicz H. A. A survey of Liapunov's second metod // Contribution to the Theory of nonlinear Oscillation. Princeton, 1958. Vol. 4, № 2. P. 141–166.

6. M a r k o v A. Sur une proprietè générale des ensembles minimaux de Birkhoff // C. R. Acad. Sci. 1931. Vol. 193. P. 823–825.
7. B h a t i a N. P., S z e g ö G. Dynamical Systems: Stability Theory and Applications: Lecture Notes in Mathematics. Berlin ; Heidelberg ; New York, 1967. Vol. 35.
8. K a l i t i n e B. S. Qualitative stability theory of movement of dynamical systems. Saarbrücken, 2012 (Russian).
9. A r r e d o n d o J. H. R., S e i b e r t P. On a Characterization of Asymptotic Stability // Aportaciones Matemáticas. Ser. Comunicaciones. 2001. № 29. P. 11–16.
10. S a p e r s t o n e S. H. Semidynamical Systems in Infinite Dimentional Space. Berlin, 1981.
11. S i b i r s k y K. S., S h u b e A. S. Semidynamical Systems (Topological Theory). Kishinev, 1987 (Russian).
12. K a l i t i n e B. S. Pseudo-stabilité et classification des ensembles fermés invariants // Prepublication USTHB, B. P. 9. Dar El-Beida, 1983. № 9.
13. S i b i r s k y K. S. Introduction in topological dynamics. Leyden, 1975.
14. K a l i t i n e B. Stability of differential equations (Method of semidefinite Lyapunov's functions). Saarbrücken, 2012 (Russian).

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МОДЕЛИРОВАНИЕ НАПРЯЖЕННО-ДЕФОРМИРОВАННОГО СОСТОЯНИЯ ПЕРИОДОНТАЛЬНОЙ СВЯЗКИ ПРИ НАЧАЛЬНЫХ ПЕРЕМЕЩЕНИЯХ КОРНЯ ЗУБА

Определены гидростатические напряжения, возникающие в тканях периодонта, при поступательных перемещениях корня зуба. Внешняя поверхность корня зуба и внутренняя поверхность периодонтальной связки описываются уравнением двуполостного гиперболоида. Толщина периодонта по нормали к поверхности является постоянной величиной. Корень зуба предполагается абсолютно твердым телом.

На основании значений, благоприятных для перестройки костной ткани напряжений, установлены диапазоны нагрузки для поступательного ортодонтического перемещения зубов. Показано, что полученные значения нагрузки приводят к деформациям тканей периодонта, которые соответствуют линейно-упругой модели периодонтальной связки. Проведен сравнительный анализ результатов расчета нормальных напряжений на основании аналитической и конечно-элементной моделей.

Ключевые слова: периодонтальная связка; корень зуба; двуполостный гиперболоид; начальное перемещение; гидростатические напряжения; нормальные деформации; метод конечных элементов.

Hydrostatic stresses in the periodontal ligament under translational displacement of the tooth root were defined. The external surface of the tooth root and the internal surface of the periodontal ligament were described by the equation of a two-sheet hyperboloid. The thickness of the periodontal ligament along normal to the tooth root surface is constant. The tooth root was assumed to be rigid. The system of equations for the translational displacements and rotation angles of the tooth root in periodontal ligament was formulated.

Load ranges for translational orthodontic tooth movement were defined based on the magnitudes of stresses favorable to bone remodelling. The obtained values of the load lead to the appearance of the periodontal tissue strains, corresponding to a linear elastic model of the periodontal ligament. Comparative analysis of the results of the calculation of normal stresses on the basis of analytical and finite element models was carried out.

Key words: periodontal ligament; root of the tooth; two-sheet hyperboloid; initial displacement; hydrostatic stresses; normal strain; finite element method.

Ортодонтическое лечение неправильного прикуса и аномального расположения зубов является сложной стоматологической процедурой, которая, как правило, подразумевает последовательное выполнение большого количества терапевтических этапов. В зависимости от величины и продолжительности силового воздействия могут возникать начальные и ортодонтические перемещения зубов. Начальные смещения зубов вызываются кратковременной нагрузкой, после снятия которой зуб возвращается на прежнее место. При этом дегенеративные и необратимые изменения периодонтальной ткани отсутствуют, деформация альвеолярного отростка обратима и имеет низкую амплитуду [1, 2]. Если отклонение зуба сохраняется в течение долгого промежутка времени, напряжения и деформации в периодонтальной связке вызывают процесс перестройки костной ткани, который приводит к ортодонтическому движению корня зуба и изменению его положения [2–5]. Основываясь на более высокой упругости периодонта по сравнению с костными структурами и зубами, как правило, предполагают, что периодонт определяет величину начального перемещения зуба [3, 6].

Установлению начальной подвижности однокоренных и многокоренных зубов посвящены многочисленные исследования, основанные на использовании метода конечных элементов [1, 6–10] или аналитических моделей [11–14]. Полученные результаты используются в ходе клинической терапии для виртуального планирования ортодонтического движения зубов без потерь времени и дискомфорта для пациента. Подходы к компьютерному моделированию долгосрочного и краткосрочного перемещений зубов представлены в работах [15–18]. Настоящее исследование развивает это актуальное направление.