PARTIAL ANALYTICAL SOLUTION TO 2D KIRCHHOFF SYSTEM AND APPLICATION TO NUMERICAL SIMULATION OF CILIA CARPET

D.A. Lyakhov[1], D.L. Michels[2], V.P. Gerdt[3], A.G. Weber[4], G.A. Sobottka[4]

[1] National Academy of Sciences of Belarus, Belarus, lyakhovda@gmail.com
[2] Stanford University, USA, michels@cs.stanford.edu
[3] Joint Institute for Nuclear Research, Russia, gerdt@jinr.ru
[4] University of Bonn, Germany, {weber,sobottka}@cs.uni-bonn.de

The research object of Special Cosserat rod theory is fiber based systems (e.g. tresses of human hair, wool pillow filling, spaghetti, cable looms, bristles of a toothbrush, fur, algae carpets and etc). It describes spatiotemporal evolution of a rod and takes into account its bending, twisting, shearing and longitudinal dilation deformation [1]. In this contribution we derive a combined analytical and numerical scheme to solve the two-dimensional differential Kirchhoff system

$$\rho A\partial_t\vec{v} = \partial_s\vec{n} + \vec{f}, \quad \rho I\partial_t\vec{\omega} = \partial_s\vec{m} + \operatorname{adiag}(1,-1)\,\vec{n} + \vec{l}, \tag{1}$$

$$\partial_t\vec{\kappa} = \partial_s\vec{\omega}, \quad \partial_s\vec{v} = \operatorname{adiag}(1,-1)\,\vec{\omega}, \tag{2}$$

$$\vec{\omega}\,\operatorname{adiag}(1,-1)\,\vec{\kappa}^{\mathsf{T}} = 0, \quad \vec{v}\,\operatorname{adiag}(1,-1)\,\vec{\kappa}^{\mathsf{T}} = 0. \tag{3}$$

Here the object is to obtain an accurate as well as an efficient solution process. Purely numerical algorithms typically have the disadvantage that the quality of solutions decreases enormously with increasing temporal step sizes, which results from the numerical stiffness of the underlying partial differential equations. To prevent that, we apply a differential Thomas decomposition [2] and a Lie symmetry analysis to derive explicit analytical solutions to specific parts of the Kirchhoff system [3]

$$\vec{\kappa} = -\frac{A^2(u)C'(u)}{A'(u)s+1}\begin{pmatrix}\cos(C(u))\\ \sin(C(u))\end{pmatrix}, \quad \vec{\omega} = \frac{A(u)C'(u)}{F'(A(u)s+u)(A'(u)s+1)))}\begin{pmatrix}\cos(C(u))\\ \sin(C(u))\end{pmatrix},$$

$$\vec{v} = \frac{1}{F'(A(u)s+u)}\begin{pmatrix}\cos(C(u))\\ \sin(C(u))\end{pmatrix}, \quad t = F(A(u)\,s+u).$$

These solutions are general and depend on three arbitrary functions, which we set up according to the numerical solution of the remaining parts. In contrast to a purely numerical handling, this reduces the numerical solution space and prevents the system from becoming unstable. The differential Kirchhoff equation describes the dynamic equilibrium of one-dimensional continua, i. e. slender structures like fibers. We evaluate the advantage of our method by simulating a cilia carpet.

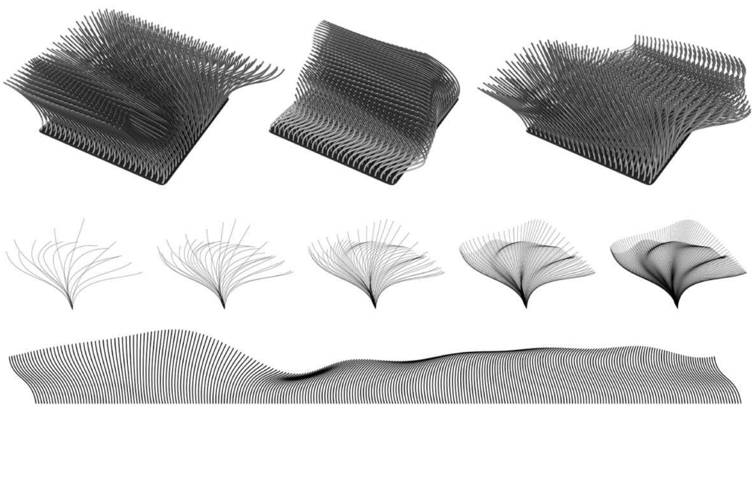

Fig. Simulation of a cilia carpet (top) composed of multiple cilia beating in a metachronal rhythm (middle). This produces the appearance of a wave (below).

Compared to a purely numerical handling using a discretization of the equations (1)–(3) in a similar way, the step size can be increased by three orders of magnitude, which leads to an acceleration of two orders of magnitude. This is of special interest for complex systems like cilia carpets, in which multiple cilia are beating in parallel with phase differences in order to produce the appearance of a wave.

References

1. Antman S. *Nonlinear Problems of Elasticity.* Applied Mathematical Sciences. Vol. 107. Berlin and New York: Springer-Verlag, 1995.

2. Bachler T., Gerdt V. P., Lange-Hegermann M., Robertz D. *Algorithmic Thomas decomposition of algebraic and differential systems* // J. Symb. Comput. 2012. Vol. 47. P. 1233–1266.

3. Michels D. L., Lyakhov D. A., Gerdt V. P., Sobottka G. A., Weber A. G. *On the Partial Analytical Solution of the Kirchhoff Equation* // Lecture Notes in Computer Science. 2015. Vol. 9301. P. 322–333.