defined on $\mathbb{R}_+$. This means that system (3) generates a semi-group dynamical system $(X, \mathbb{R}_+, \pi)$ on the space $X := \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$, where $(\mathcal{T}^m, \mathbb{R}, \sigma)$ is a dynamical system associated by equation

$$\theta' = \Phi(\theta), \tag{4}$$

$(\varphi(t, x_0, x_0', \theta), \varphi'(t, x_0, x_0', \theta))$ is a unique solution of equation

$$x'' + \nabla F(\sigma(t,\theta), x) = 0 \quad (\theta \in \mathcal{T}^m) \tag{5}$$

passing through the point (x_0, x_0') at the initial moment $t = 0$,

$$\pi(t, (x_0, x_0', \theta)) := (\varphi(t, x_0, x_0', \theta), \varphi'(t, x_0, x_0', \theta))$$

for all $(t, x_0, x_0', \theta) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$.

Remark 3. 1. By arguments above autonomous system (3) and non-autonomous equation (5) (in fact the family of non-autonomous equations depending on parameter $\theta \in \mathcal{T}^m$) are equivalent.

2. If equation (5) admits a trivial solution, then the set $\{(0,0)\} \times \mathcal{T}^m \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$ is an invariant subset (invariant torus) of system (3).

Definition. Recall that the trivial solution of equation (5) (or equivalently, the invariant torus of system (3)) is said to be uniformly (with respect to $\theta \in \mathcal{T}^m$) Lyapunov stable, if for arbitrary $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that $||x_0||^2 + ||x_0'||^2 < \delta^2$ implies $||\varphi(t, x_0, x_0', \theta)||^2 + ||\varphi'(t, x_0, x_0', \theta)||^2 < \varepsilon^2$ for all $t \in \mathbb{R}_+$ and $\theta \in \mathcal{T}^m$.

Denote by $\mathcal{K}$ the set of all continuous functions $a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ possessing the following properties: $a(0) = 0$; a is monotonically strictly increasing.

Theorem 2. *Suppose that following conditions hold:*

1) $F(0,\theta) = 0$ *and* $\nabla_x F(0, \theta) = 0$ *for all* $\theta \in \mathcal{T}^m$;

2) *there exists a function* $a \in \mathcal{K}$ *such that* $F(x, \theta) \geqslant a(||x||)$ *for all* $x \in \mathbb{R}^n$ *and* $\theta \in \mathcal{T}^m$;

3) $\langle \nabla_\theta F(x,\theta), \Phi(\theta)\rangle \leqslant 0$ *for all* $(x, \theta) \in \mathbb{R}^n \times \mathcal{T}^m$.

Then the trivial solution of equation (5) *is uniformly Lyapunov stable.*

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ON POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR SINGULAR IN PHASE VARIABLES NONLINEAR DIFFERENTIAL SYSTEMS

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In the finite interval $[a, b]$ we consider the problem

$$\frac{du_i}{dt} = f_i(t, u_1, \ldots, u_n) \quad (i = 1, \ldots, n), \tag{1}$$

$$u_i(a) = \varphi_i(u_i(b)) \quad (i = 1, \ldots, n), \tag{2}$$

where $f_i : [a,b] \times \mathbb{R}^n_{0+} \to \mathbb{R}$ $(i=1,\dots,n)$ and $\varphi_i : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ $(i=1,\dots,n)$ are continuous functions, $\mathbb{R}_{0+} =]0,+\infty[$ and $\mathbb{R}^n_{0+} = \big\{(x_i)_{i=1}^n \in \mathbb{R}^n : \ x_1 > 0, \dots, x_n > 0\big\}$. The particular cases of (2) are the boundary conditions

$$u_i(a) = u_i(b) \quad (i=1,\dots,n), \tag{2_1}$$

$$u_i(a) = \alpha_i u_i(b) \quad (i=1,\dots,n). \tag{2_2}$$

A solution $(u_i)_{i=1}^n : [a,b] \to \mathbb{R}^n_{0+}$ of the system (1) satisfying the boundary conditions (2) is called a positive solution of the problem (1), (2).

For singular in phase variables first and second order differential equations, problems on the existence of positive solutions, satisfying periodic type boundary conditions, are investigated in detail (see, e. g., [1, 4, 6]). As for the system (1), for it similar problems are studied only in a regular case (see, e. g., [2, 5] and the references therein).

Based on the a priori estimates, established in [3], we obtain unimprovable in a certain sense conditions guaranteeing the existence and uniqueness of a positive solution of the problem (1), (2). These results cover the case where the system under consideration has singularities in phase variables, in particular, the case where for arbitrary $i,k \in \{1,\dots,n\}$ and $x_j > 0$ $(j=1,\dots,n;$ $j \neq k)$ the equality

$$\lim_{x_k \to 0} \big|f_i(t,x_1,\dots,x_n)\big| = +\infty$$

is fulfilled.

We also use the following notation:

$$\mathbb{R}_+ = [0,+\infty[, \quad \mathbb{R}^n_+ = \{(x_i)_{i=1}^n \in \mathbb{R}^n : \ x_1 \geqslant 0, \dots, x_n \geqslant 0\};$$

$X = (x_{ik})_{i,k=1}^n$ is the $n \times n$ matrix with components $x_{ik} \in \mathbb{R}$ $(i,k=1,\dots,n)$, and $r(X)$ is the spectral radius of the matrix X. For any continuous function $p : [a,b] \to \mathbb{R}$ and number β, satisfying the condition

$$\Delta(p,\beta) = 1 - \beta \exp\bigg(\int_a^b p(s)\,ds\bigg) \neq 0,$$

we put

$$g(p,\beta)(t,s) = \begin{cases} \dfrac{1}{\Delta(p,\beta)} \exp\bigg(\displaystyle\int_s^t p(\tau)\,d\tau\bigg) & \text{for } \ a \leqslant s \leqslant t \leqslant b, \\[2ex] \dfrac{\beta}{\Delta(p,\beta)} \exp\bigg(\displaystyle\int_a^b p(\tau)\,d\tau + \int_s^t p(\tau)\,d\tau\bigg) & \text{for } \ a \leqslant t < s \leqslant b, \end{cases}$$

and

$$w(p,\beta)(t) = \frac{1}{\Delta(p,\beta)}\bigg[(1-\beta)\exp\bigg(\int_a^t p(s)\,ds\bigg) + \beta \exp\bigg(\int_a^b p(s)\,ds\bigg) - 1\bigg].$$

In Theorem 1 it is assumed that the functions f_i $(i=1,\dots,n)$ and φ_i $(i=1,\dots,n)$, respectively, on the sets $[a,b] \times \mathbb{R}^n_{0+}$ and $\mathbb{R}_{0+}$ satisfy the inequalities

$$q_i(t,x_i) \leqslant \sigma_i(f_i(t,x_1,\dots,x_n) - p_i(t)x_i) \leqslant$$
$$\leqslant \sum_{k=1}^n p_{ik}(t,x_1+\dots+x_n)x_k + q_0(t,x_1,\dots,x_n) \quad (i=1,\dots,n), \tag{3}$$

and

$$\sigma_i(\varphi_i(x) - \alpha_i x) \geqslant 0, \quad \sigma_i(\varphi_i(x) - \beta_i x) \leqslant \beta_0 \quad (i = 1, \dots, n). \tag{4}$$

Here,

$$\sigma_i \in \{-1, 1\}, \quad \alpha_i > 0, \quad \beta_i > 0, \quad \sigma_i(\beta_i - \alpha_i) \geqslant 0 \quad (i = 1, \dots, n), \quad \beta_0 \geqslant 0, \tag{5}$$

$p_i : [a, b] \to \mathbb{R}$ $(i = 1, \dots, n)$ are continuous functions, $p_{ik} : [a, b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ and $q_i : [a, b] \times \times \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i, k = 1, \dots, n)$ are nonincreasing in the second argument continuous functions, and $q_0 : [a, b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ is a nonincreasing in the last n arguments continuous function. Moreover, p_i and q_i $(i = 1, \dots, n)$ satisfy the conditions

$$\sigma_i\biggl(\beta_i \exp\biggl(\int_a^b p_i(s)\, ds\biggr) - 1\biggr) < 0 \quad (i = 1, \dots, n), \tag{6}$$

$$\max\{q_i(t, x) : \ a \leqslant t \leqslant b\} > 0 \ \text{ for } \ x > 0 \quad (i = 1, \dots, n). \tag{7}$$

Theorem 1. *Let the conditions* (3) – (7) *hold and there exist continuous functions* $\ell_i : [a, b] \to \to \mathbb{R}_{0+}$ $(i = 1, \dots, n)$ *such that the matrix function* $H(x) = (h_{ik}(x))_{i,k=1}^n$ *with the components*

$$h_{ik}(x) = \max\biggl\{\frac{1}{\ell_i(t)} \int_a^b \bigl|g(p_i, \beta_i)(t, s)\bigr| p_{ik}(s, x)\ell_k(s)\, ds : \ a \leqslant t \leqslant b\biggr\} \quad (i, k = 1, \dots, n)$$

satisfies the inequality

$$\lim_{x \to +\infty} r(H(x)) < 1.$$

Then the problem (1), (2) *has at least one positive solution.*

As an example, we consider the differential systems

$$\frac{du_i}{dt} = \sigma_i\biggl(\sum_{k=1}^n p_{ik} u_k + f_{0i}(t, u_1, \dots, u_n)\biggr) \quad (i = 1, \dots, n), \tag{8}$$

$$\frac{du_i}{dt} = \sigma_i\biggl(\sum_{k=1}^n \frac{|1 - \alpha_k| h_{ik}}{(1 - \alpha_k)(t - a) + \alpha_k(b - a)}\, u_k + f_{0i}(t, u_1, \dots, u_n)\biggr) \quad (i = 1, \dots, n), \tag{9}$$

where $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$, p_{ik} $(i, k = 1, \dots, n)$ and α_i $(i = 1, \dots, n)$ are constants satisfying the inequalities

$$p_{ii} < 0, \quad p_{ik} \geqslant 0 \quad (i \neq k; \quad i, k = 1, \dots, n), \tag{10}$$

$$\alpha_i > 0, \quad \sigma_i(\alpha_i - 1) < 0 \quad (i = 1, \dots, n), \tag{11}$$

h_{ik} $(i, k = 1, \dots, n)$ are nonnegative constants and $f_{0i} : [a, b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are continuous functions. Moreover, on the set $[a, b] \times \mathbb{R}^n_{0+}$ the inequalities

$$q_i(t, x_i) \leqslant f_{0i}(t, x_1, \dots, x_n) \leqslant q_0(t, x_1, \dots, x_n) \quad (i = 1, \dots, n) \tag{12}$$

are fulfilled, where $q_0 : [a, b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ is a nonincreasing in the last n arguments continuous function and $q_i : [a, b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are nonincreasing in the second argument continuous functions satisfying the conditions (7).

Corollary 1. *Let the conditions* (11) *and* (12) *be fulfilled. Then for the existence of at least one positive solution of the problem* (8), (2_1) *it is necessary and sufficient that the real parts of the eigenvalues of the matrix* $(p_{ik})_{i,k=1}^n$ *be negative.*

Corollary 2. *Let the conditions* (11) *and* (12) *be fulfilled. Then for the existence of at least one positive solution of the problem* (9), (2_2) *it is necessary and sufficient that the matrix* $H = (h_{ik})_{i,k=1}^n$ *satisfy the inequality*

$$r(H) < 1. \tag{13}$$

Remark 1. In the conditions of Corollaries 1 and 2 the functions f_{0i} $(i = 1, \dots, n)$ may have singularities of arbitrary order in the least n arguments. For example, in (8) and (9) we may assume that

$$f_{0i}(t, x_1, \dots, x_n) = \sum_{k=1}^n (q_{1ik}(t) x_k^{-\mu_{1ik}} + q_{2ik}(t) \exp(x_k^{-\mu_{2ik}})) \quad (i = 1, \dots, n),$$

where $\mu_{1ik} > 0$, $\mu_{2ik} > 0$ $(i, k = 1, \dots, n)$ and $q_{1ik} : [a, b] \to \mathbb{R}_{0+}$ and $q_{2ik} : [a, b] \to \mathbb{R}_+$ $(i, k = 1, \dots, n)$ are continuous functions such that $q_{1ii}(t) + q_{2ii}(t) \not\equiv 0$ $(i = 1, \dots, n)$.

The uniqueness of a positive solution of the problem (1), (2) can be proved only in the case where each function f_i has the singularity in the i-th phase variable only. More precisely, we consider the case when the system (1) has the following form

$$\frac{du_i}{dt} = p_i(t) u_i + \sigma_i (f_{0i}(t, u_1, \dots, u_n) + q_i(t, u_i)) \quad (i = 1, \dots, n). \tag{14}$$

The particular cases of (14) are the differential systems

$$\frac{du_i}{dt} = \sigma_i \bigg(\sum_{k=1}^n p_{ik} u_k + q_i(t, u_i) \bigg) \quad (i = 1, \dots, n), \tag{15}$$

$$\frac{du_i}{dt} = \sigma_i \bigg(\sum_{k=1}^n \frac{|1 - \alpha_k| h_{ik}}{(1 - \alpha_k)(t - a) + \alpha_k (b - a)} u_k + q_i(t, u_i) \bigg) \quad (i = 1, \dots, n). \tag{16}$$

Here $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$, $p_i : [a, b] \to \mathbb{R}$ and $f_{0i} : [a, b] \times \mathbb{R}_+^n \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are continuous functions, and $q_i : [a, b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are nonincreasing in the second argument continuous functions. Moreover, p_i and q_i $(i = 1, \dots, n)$ satisfy the conditions (6) and (7), p_{ik} and α_i $(i, k = 1, \dots, n)$ are constants satisfying the inequalities (10) and (11), and h_{ik} $(i, k = 1, \dots, n)$ are nonnegative constants.

Theorem 2. *Let on the sets* $[a, b] \times \mathbb{R}_+^n$ *and* $\mathbb{R}_+$ *the conditions*

$$\sigma_i (f_{0i}(t, x_1, \dots, x_n) - f_{0i}(t, y_1, \dots, y_n)) \operatorname{sgn}(x_i - y_i) \leqslant \sum_{k=1}^n p_{ik}(t) |x_k - y_k| \quad (i = 1, \dots, n)$$

and

$$\sigma_i (\varphi_i(x) - \alpha_i x) \geqslant 0, \quad \sigma_i [(\varphi_i(x) - \varphi_i(y)) \operatorname{sgn}(x - y) - \beta_i |x - y|] \leqslant 0 \quad (i = 1, \dots, n)$$

holds, where $p_{ik} : [a, b] \to \mathbb{R}_+$ $(i, k = 1, \dots, n)$ *are continuous functions. Let, moreover, there exist continuous functions* $\ell_i : [a, b] \to \mathbb{R}_{0+}$ $(i = 1, \dots, n)$ *such that the matrix* $H = (h_{ik})_{i,k=1}^n$ *with the components*

$$h_{ik} = \max \bigg\{ \frac{1}{\ell_i(t)} \int_a^b |g(p_i, \beta_i)(t, s)| p_{ik}(s) \ell_k(s) \, ds : \ a \leqslant t \leqslant b \bigg\} \quad (i, k = 1, \dots, n)$$

satisfies the inequality (13). *Then the problem* (14), (2) *has a unique positive solution.*

Theorem 2 results in the following corollaries.

Corollary 3. *For the existence of a unique positive solution of the problem* (15), (2_1) *it is necessary and sufficient that the real parts of the eigenvalues of the matrix* $(p_{ik})_{i,k=1}^n$ *be negative.*

Corollary 4. *For the existence of a unique positive solution of the problem* (16), (2_2) *it is necessary and sufficient that the matrix* $H = (h_{ik})_{i,k=1}^n$ *satisfy the inequality* (13).

Remark 2. In the conditions of Theorem 2 and its corollaries, the functions q_i $(i = 1, \dots, n)$ may have singularities of arbitrary order in the second argument. For example, in (14), (15) and (16) we may assume that

$$q_i(t, x) = q_{i1}(t)x^{-\mu_{i1}} + q_{i2}(t)\exp(x^{-\mu_{i2}}) \quad (i = 1, \dots, n),$$

where $\mu_{i1} > 0$, $\mu_{i2} > 0$ $(i = 1, \dots, n)$, and $q_{ik} : [a, b] \to \mathbb{R}_+$ $(i = 1, \dots, n;\ k = 1, 2)$ are continuous functions such that $q_{i1}(t) + q_{i2}(t) \not\equiv 0$ $(i = 1, \dots, n)$.

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MIRONENKO REFLECTING FUNCTION AND EQUIVALENCE OF DIFFERENTIAL SYSTEMS

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Theorem. *Let* $F(t, x)$ *be Mironenko reflecting function* [1, 2] *of the differential system* $\dot{x} = X(t, x)$ *and* $\Delta(t, x)$ *be a solution of the system* $\Delta_t + \Delta_x X(t, x) - X_x(t, x)\Delta = \mu\Delta$, *where* $\mu(t, x)$ *is a scalar function, for which* $\mu(-t, F(t, x)) + \mu(t, x) \equiv 0$. *Then for every scalar odd function* $\alpha(t)$ *system* $\dot{x} = X(t, x) + \alpha(t)\Delta(t, x)$ *has the same reflecting function* $F(t, x)$.

This theorem generalizes the theorem of V. V. Mironenko [3].

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