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Complex Masses of Resonances in the Potential Approach

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Complex masses of resonances in the quark potential model are obtained. Two exact asymptotic solutions for the QCD motivated potential are used to derive the resonance complex-mass formula. The centered masses and total widths of mesonic resonances are calculated. A possible origin of the dark matter is discussed.

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Resonance is the tendency of a system to oscillate at a greater amplitude at some frequencies. Most particles listed in the Particle Data Group tables [1] are unstable. A thorough understanding of the physics summarized by the PDG is related to the concept of a resonance.

Operators in Quantum Mechanics are Hermitian and the corresponding eigenvalues are real. However, in scattering experiment, the wave function requires different boundary condition, that is why the complex energy is required [2, 3].

The concept of a purely outgoing wave belonging to the complex eigenvalue $\mathcal{E}_n = E_n - i\Gamma_n/2$ was introduced in 1939 by Siegert [4]. It is an appropriate tool in studying of resonances. The complex eigenvalue also corresponds to a first-order pole of the S-matrix [5].

Resonances in QFT are described by the complex-mass poles of the scattering matrix [3]. Resonance is presented as *transient oscillations* associated with metastable states. The masses of the states develop imaginary masses from loop corrections [6, 7]. In this case, the probability density comes from the particle's propagator, with the complex mass,

$$\mathcal{M} = M - i\Gamma/2. \tag{1}$$

This formula is related to the particle's decay rate by the optical theorem [3, 8].

Fundamentals of scattering theory and strict mathematical definition of resonances in QM was considered in [9, 10]. The rigorous QM definition of a resonance requires determining the pole position in the second Riemann sheet of the analytically continued partial-wave scattering amplitude in the complex Mandelstam s variable plane [11]. This definition has the advantage of being quite universal regarding the pole position, but can only be applied if the amplitude can be analytically continued in a reliable way.

In this work, in contrast to the usual analysis of the scattering amplitude, we consider mesonic resonances in bound state region, i.e., quasi-bound states of the meson constituents. In traditional approach to investigate resonances one deals with the scattering theory, exploring the properties of S-matrix and partial amplitudes. In contrast to the usual analysis, we consider mesonic resonances to be the transient excited states of a quark-antiquark system. We consider the bound-state problem using the potential approach and analyze the mass spectrum generated from the solution of the relativistic wave equation for the Cornell potential.

We use the advantage of analyzing the system in the complex plane (the complex-mass scheme) that has important features such as a simpler and more general framework [6, 7]. We show, that two asymptotic components of the Cornell potential, the Coulombic term and linear one, yield the complex masses of resonances that allows to calculate in a unified way their centered masses and total widths.

In this work, we consider the famous Cornell

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$q\bar{q}$ potential [12–14],

$$V(r) = V_S(r) + V_L(r) \equiv -\frac{4}{3}\frac{\alpha_s}{r} + \sigma r. \quad (2)$$

All phenomenologically acceptable QCD-inspired potentials are only variations around this potential. The potential (2) is one of the most popular in hadron physics and incorporates in clear form the basic features of the strong interaction. In hadron physics, the nature of the potential is very important. There are normalizable solutions for scalarlike potentials, but not for vectorlike ones. The effective interaction has to be Lorentz-scalar in order to confine quarks and gluons [15], i.e., we take the potential (2) to be Lorentz-scalar.

Our aim is to find in analytic form the energy/mass eigenvalues for the potential (2). This problem is not easy if one uses known relativistic wave equations. However, this aim can be achieved with the use of the semi-classical wave equation [16]. An important feature of this equation is that, for two and more turning-point problems, it can be solved exactly by the conventional WKB method [16].

There is another approach to achieve the aim. Using the two-point Padé approximant, we joined two exact solutions obtained separately for the $V_S(r)$ and $V_L(r)$ of the potential (2). As a result we obtained the interpolating mass formula [17, 18],

$$M_n^2 = 4\left[2\sigma\tilde{N} - \left(\frac{\tilde{\alpha}m}{N}\right)^2 + m^2 - 2\tilde{\alpha}\sigma\right] \quad (3)$$

where $\tilde{\alpha} = (4/3)\alpha_s$, $\tilde{N} = N + n_r + 1/2$, $N = n_r + J + 1$ and m is the constituent quark mass.

The simple mass formula (3) describes equally well the mass spectra of all $q\bar{q}$ and $Q\bar{Q}$ mesons ranging from the $u\bar{d}$ ($d\bar{d}$, $u\bar{u}$, $s\bar{s}$) states up to the heaviest known $b\bar{b}$ systems [17, 18].

The obtained from Eq. (3) "saturating" Regge trajectories [17, 18] were applied with success to the photoproduction of vector mesons that provide an excellent simultaneous description of the high and low $-t$ behavior of the $\gamma p \to p\rho$, ω, ϕ cross sections [19, 20], given an appropriate choice of the relevant coupling constants (JML-model) [21, 22]. It was shown that the hard-scattering mechanism is incorporated in an effective way by using the "saturated" Regge trajectories that are independent of t at large momentum transfers [17, 18].

The universal formula (3) has been used to calculate the glueball masses and Regge trajectories including the Pomeron [23, 24]. It appears to be successful in many applications and can be justified with the use of the complex-mass scheme.

Resonances are complex-mass values and can be described by complex numbers. These numbers are important even if one wants to find real solutions of the problem. Using complex numbers, we are getting more than what we insert.

The mass formula (3) is very transparent physically, as well as the potential (2) (Coulomb + linear). This formula contains in a hidden form some important information; we can get it in the following way. It is easy to see that Eq. (3) is the real part of the complex equation,

$$\mathcal{M}_n^2 = 4\left[\left(\sqrt{2\sigma\tilde{N}} - i\frac{\tilde{\alpha}m}{N}\right)^2 + \left(m + i\sqrt{2\tilde{\alpha}\sigma}\right)^2\right]. \quad (4)$$

This expression has the form of the equation, $\mathcal{M}_n^2 = 4[(\pi_n)^2 + \mu^2]$, for two free relativistic particles with the complex momentum, $\pi_n = (2\sigma\tilde{N})^{1/2} - i\tilde{\alpha}m/N$, and mass, $\mu = m + i(2\tilde{\alpha}\sigma)^{1/2}$. The complex-mass expression (4) contains additional information, but let us give some ground to our consideration.

An important hint we observe studying the hydrogen atom problem. The *total* energy eigenvalues for the non-relativistic Coulomb problem can be written with the use of complex quantities in the form of kinetic energy for a free particle,

$$E_n = \frac{p_n^2}{2m}, \quad p_n = \frac{i\alpha m}{n_r + l + 1}. \quad (5)$$

Here $p_n = mv_n$ is the electron's momentum eigenvalue with the imaginary discrete velocity,

$v_n = i\alpha/(n_r+l+1)$. This means, that the motion of the electron in a hydrogen atom is free, but restricted by the "walls" of the potential [16, 25].

The Cornell potential (2) is unique in that sense, it yields the *complex* masses of resonances. To show that, analyze the eigenvalues obtained separately for two components of the potential (2), i.e., the Coulombic term and the linear one.

Relativistic two-body Coulomb problem for particles of equal masses can be solved analytically [25]. The exact expression for the c.m. energy squared is well known and can be written in the form of two free relativistic particles as [23, 24]:

$$E_n^2 = 4[(i\mathrm{Im}\,\pi_n)^2 + m^2], \quad \mathrm{Im}\,\pi_n = -\tilde{\alpha}m/N \quad (6)$$

where N is given above. Here we have introduced the *imaginary* momentum eigenvalues, $i\mathrm{Im}\,\pi_n$.

The linear term of the Cornell potential (2) can be dealt with analogously. In this case the exact solution is also well known [23–25]:

$$E_n^2 = 8\sigma\tilde{N}, \quad \tilde{N} = N + n_r + 1/2. \quad (7)$$

This expression does not contain the mass term and can be written in the similar to (6) relativistic form,

$$E_n^2 = 4(\mathrm{Re}\,\pi_n)^2, \quad \mathrm{Re}\,\pi_n = \sqrt{2\sigma\tilde{N}} \quad (8)$$

where $\mathrm{Re}\,\pi_n$ is the *real* momentum eigenvalue.

Thus, two asymptotic additive terms of the potential (2), $V_S(r)$ and $V_L(r)$, separately, yield the imaginary (6) and real (8) momentum eigenvalues. These terms of the potential represent two "different physics" (Coulombic OGE and linear string tension), therefore, two different realms of the interaction. Each of these two expressions, (6) and (8), is exact and was obtained independently, therefore, we can consider the complex sum, $\pi_n = \mathrm{Re}\,\pi_n + i\mathrm{Im}\,\pi_n$. Here we accept the complex momentum eigenvalues, that means the total energy and mass should be complex, too.

It is an experimental fact that the dependence $M_n^2(J)$ is linear for light mesons [26]. However, at present, the best way to reproduce the experimental masses of particles is to rescale the entire spectrum given by (7) assuming that the masses of the mesons are expressed by the relation [15]

$$M_n^2 = E_n^2 - C^2 \quad (9)$$

where C is a constant energy (shift parameter). Relation (9) is used to shift the spectra and appears as a means to simulate the effects of unknown structure approximately. But, if we rewrite (9) in the usual relativistic form,

$$M_n^2 = 4\left[(\mathrm{Re}\,\pi_n)^2 + (\pm i\mu_{\mathrm{I}})^2\right] \quad (10)$$

where $\mathrm{Re}\,\pi_n$ is given by Eq. (8), we come to the concept of the imaginary mass, μ_{I}. Here in (10) we have introduced the notation, $4(\pm i\mu_{\mathrm{I}})^2 = -C^2$. What is the mass μ_{I}?

The required shift of the spectra naturally follows from the asymptotic solution of the semi-classical wave equation for the potential (2) [23–25, 27]. To show that, we account for the "weak coupling effect", i.e., together with the linear dependence in (8) we should include the contribution of the Coulombic term, $-\alpha_s/r$, of the potential.

This kind of calculations results in the asymptotic expression similar to (9) [23, 25],

$$M_n^2 = 8\sigma(\tilde{N} - \tilde{\alpha}). \quad (11)$$

The additional term, $-8\tilde{\alpha}\sigma$, arises from the interference of the Coulombic and linear components of the Cornell potential (2). Comparing (11) with (10), we obtain

$$\mu_{\mathrm{I}} = \pm\sqrt{2\tilde{\alpha}\sigma}. \quad (12)$$

The interference term $-8\tilde{\alpha}\sigma$ in (11) contains only the parameters of the potential (2) and is Lorentz-scalar, i.e., additive to the particle masses. This is why, we accept the last term in (11) to be the mass term, i.e., Eq. (12) is the imaginary-part mass generated by the interference term of the Cornell potential (2).

Thus, we have the particle real-part (constituent) mass, $\mu_{\rm R} = m$, and the imaginary-part mass (12). As in case of the eigen-momenta, we introduce the *complex* mass, $\mu = \mu_{\rm R} + i\mu_{\rm I}$, which enters in Eq. (4).

In the pole approach, the parameters of resonances are defined in terms of the pole position s_p in the complex s-plane as [9, 10, 28, 29]

$$s_p = M_p^2 - iM_p\Gamma_p \tag{13}$$

where $s = \mathcal{M}^2$ is the two-particle c.m. energy squared. The complex-mass Eq. (4) can be written in the similar form,

$$\mathcal{M}_n^2 = \mathrm{Re}\,\mathcal{M}_n^2 + i\mathrm{Im}\,\mathcal{M}_n^2 \tag{14}$$

where $\mathrm{Re}\,\mathcal{M}_n^2$ exactly coincides with the interpolating mass formula (3) [17, 18, 23, 24], and the imaginary part is

$$\mathrm{Im}\,\mathcal{M}_n^2 = 8m\mu_{\rm I}\left(1 - \sqrt{\tilde{\alpha}\tilde{N}}/N\right). \tag{15}$$

Comparing (14) and (13), we obtain the centered mass squared, M_n^2, given by Eq. (3), and the total width,

$$\Gamma_n^{\rm TOT} = 8m\mu_{\rm I}M_n^{-1}\left(\sqrt{\tilde{\alpha}\tilde{N}}/N - 1\right). \tag{16}$$

In general (mathematically), the S-matrix is a meromorphic function of complex variable $\mathcal{M} = \pm\sqrt{s}$ where the complex s-plane is replaced by the two-sheet Riemann surface, $\pm\sqrt{s}$, made up of two sheets R_0 and R_1, each cut along the positive real axis, $\mathrm{Re}\mathcal{M}$, and with R_1 placed in front of R_0 [9, 11]. The square root of the complex expression (14) gives $(\mathcal{M}_n^2)^{1/2} = \pm[\mathrm{Re}\,\mathcal{M}_n + i\xi\mathrm{Im}\,\mathcal{M}_n]$ where

$$\mathrm{Re}\,\mathcal{M}_n = \left[\left(|\mathcal{M}_n^2| + \mathrm{Re}\,\mathcal{M}_n^2\right)/2\right]^{1/2}, \tag{17}$$

$$\mathrm{Im}\,\mathcal{M}_n = \left[\left(|\mathcal{M}_n^2| - \mathrm{Re}\,\mathcal{M}_n^2\right)/2\right]^{1/2}, \tag{18}$$

$|\mathcal{M}_n^2| = \left[(\mathrm{Re}\,\mathcal{M}_n^2)^2 + \mathrm{Im}\,\mathcal{M}_n^2)^2\right]^{1/2}$, $\xi = \mathrm{sgn}(\mathrm{Im}\,\mathcal{M}_n^2)$. The expressions (17) and (18) define the resonance position in the Riemann $\mathcal{M}$-surface. The centered mass, $M_n^R = \mathrm{Re}\,\mathcal{M}_n$, and the total width, $\Gamma_n^{\rm TOT} = -2\mathrm{Im}\,\mathcal{M}_n$, are process independent parameters of the resonance.

Resonance masses arise in complex conjugate pairs. Poles in the left half-plane correspond to either bound or anti-bound states [2, 8]. If $\mathcal{M}_n = M_n - i\Gamma_n/2$ is a pole in the fourth quadrant of the surface $\pm\sqrt{s}$, then $\mathcal{M}_n = -M_n - i\Gamma_n/2$ is also a pole, but in the third quadrant (antiparticle) [2].

There is the lack of a precise definition of what is meant by mass and width of resonance. Comprehensive definitions of these resonance's parameters require further investigations. An alternate definition of the resonance's width can be obtained from Eq. (15). According to definition (1) the width is given by the *imaginary-part* mass of the resonance's complex mass, $\mathcal{M}_n$. Dividing Eq. (15) by $8m$ (exclusion of the real-mass term), we come to the following expression:

$$\Gamma_n^{\rm TOT} = \mu_{\rm I}\left(1 - \sqrt{\tilde{\alpha}\tilde{N}}/N\right). \tag{19}$$

This total width is restricted by the maximum possible value $\Gamma_n^{\max} = \mu_{\rm I}$ for highest excitations (resonances).

As an example, consider the ρ-family resonances of the leading Regge trajectory, $\alpha_\rho(s)$. Calculation results are in Table 1 where masses and widths are in MeV.

Table 1. Masses and total widths of the ρ-family resonances.

Meson	J^{PC}	M_n^{ex}	M_n^{th}	Γ_n^{ex}	$\Gamma_n^{th}(18)$	$\Gamma_n^{th}(19)$
$\rho\ (1S)$	1^{--}	776	775	149	150	75
$a_2(1P)$	2^{++}	1318	1323	107	108	93
$\rho_3(1D)$	3^{--}	1689	1689	161	170	188
$a_4(1F)$	4^{++}	1996	1985	255	194	249
$\rho_5(1G)$	5^{--}	–	2234	–	202	294
$a_6(1H)$	6^{++}	–	2462	–	205	328

Parameters values in these calculations are found from the best fit to the available data [1]: $\alpha_s = 1.463$, $\sigma = 0.134\,\mathrm{GeV}^2$, $m = 193$ MeV. The widths $\Gamma_n^{th}(18)$ and $\Gamma_n^{th}(19)$ are calculated with the use of Eqs. (18) and (19). The maximum width $\Gamma_n^{\max} = \mu_{\rm I} = 0.723$ GeV.

More accurate calculations require accounting for the spin corrections, i.e., spin-spin and spin-orbit interactions. The spin-dependent corrections to the potential (2) have been considered in [17].

Location of the ρ-family resonances in the complex $\mathcal{M}$-surface is shown in Fig. 1. Note a

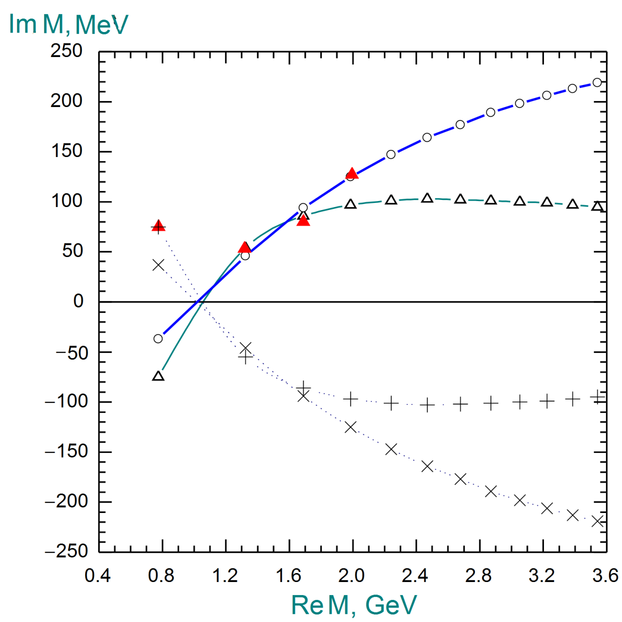

FIG. 1. The complex Riemann $\mathcal{M} = \pm\sqrt{s}$-surface. Triangles show the location of the complex-mass resonances relating to the leading ρ Regge trajectory; the imaginary-part component is the resonance half-width. Crosses show calculation results with the use of Eqs. (18) and (19).

feature of the ρ-family resonance data. There is a dip ($\Gamma_n^{\mathrm{TOT}} = 107$ MeV) for the $a_2(1320)$ resonance (see Fig. 1). This dip is described in our scheme and has the following explanation. According to definition the imaginary-part mass $\mathrm{Im}\,\mathcal{M}_n$ can be positive and negative, i.e., poles can be located below and above of the real $\mathcal{M}$-axis in the complex $\mathcal{M}$-surface (bound state or scattering region). This choice depends on boundary conditions.

We have studied mesonic resonances to be the quasi-bound eigenstates of two interacting quarks using the Cornell potential. Using the complex analysis, we have derived the meson complex-mass formula, in which the real and imaginary parts are exact expressions. This approach has allowed us to simultaneously describe in the unified way the centered masses and total widths of the the ρ-family resonances.

The complex masses and energies are not observable directly but may have relation to the "Missing Mass" and "Dark Matter". Imaginary mass is contained in the magnitude of the complex mass, $|\mathcal{M}| = (\mathcal{M}_R^2 + \mathcal{M}_I^2)^{1/2}$, and give contribution to observables. "Missing Mass" can perhaps be measured, and mass or energy apparently vanishing from a region of space-time may be taken as an indication that something is leaving that region, perhaps along another perpendicular axis, the imaginary one.

As shown in this work the energy, momentum and mass of particles can be complex. Reggeons in Regge theory are imaginary-mass particles hypothetical objects. But they descrive a real interaction. Imaginary mass may have the same possibility to exist as real one.

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