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Quantum Oscillator Problem on $SO(2,2)$ Hyperboloid

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In this note the harmonic oscillator system on the hyperboloid H_2^2: $z_0^2 + z_1^2 - z_2^2 - z_3^2 = R^2$ has been considered.

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1. Introduction

Quantum mechanics on spaces with constant curvature (negative and positive) has always drawn considerable attention for its many peculiarities and tangible difficulties, including the nontrivial quantization of particle dynamics, that distinguishes these spaces from the Euclidean space. On the other hand the negative curvature spaces are the model of relativistic space time with a constant curvature (de Sitter and anti de - Sitter spaces), which is a crucial point for its wide applications in relativistic field theories [1] and quantum gravity [2]. Among other applications we can mention also the quantum dots [3] and the quantum Hall effect [4].

In this short note we discuss the harmonic oscillator problem on the configuration space based on the surface of three-dimensional hyperboloid H_2^2: $z_0^2 + z_1^2 - z_2^2 - z_3^2 = R^2$, $R > 0$. The main aim of our investigations is to describe features which curvature of space on the dynamics of these systems introduces. In our recent work we have already considered the Kepler-Coulomb problem on the H_2^2 [6] and have obtain that, as in Euclidean space, the energy spectrum splits to scattering and bound states, but the number of bound states is finite (really very large $\sim \sqrt{R}$) and degenerate on angular quantum number and infinitely degenerate on azimuthal one.

We recall that the first investigations of Hydrogen atom in quantum mechanics probably were given in the famous works of Schrödinger [7], who in the framework of the factorization method studied this system on the three-dimensional sphere and then by Stevenson [8] and by Infeld and Shild [9], who solved the Schrödinger equation for the same problem on the sphere and two-sheeted hyperboloid. More recently various aspects of Kepler-Coulomb problem and harmonic oscillator, including its superintegrable generalization, on the curved spaces have been considered in the papers [10, 11].

2. Oscillator eigenfunctions and eigenvalues

Our interest lies in investigation of the Schrödinger equation on H_2^2: $z_0^2+z_1^2-z_2^2-z_3^2 = R^2$ hyperboloid ($\hbar = m = 1$)

$$-\frac{1}{2}\Delta_{LB}\Psi + U(\mathbf{z})\Psi = E\Psi \qquad (1)$$

for the potential of the harmonic oscillator - $U(\mathbf{z})$, introduced recently in the article [6]. In the pseudo spherical coordinates ($r > 0$, $\tau \in$

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$(-\infty, \infty)$, $\varphi \in [0, 2\pi))$:

$$z_0 = R\cosh r, \quad z_1 = R\sinh r \sinh\tau,$$
$$z_2 = R\sinh r\cosh\tau\cos\varphi, \; z_3 = R\sinh r\cosh\tau\sin\varphi \tag{2}$$

this potential has the form

$$U(\mathbf{z}) = \frac{\omega^2 R^2}{2}\left(\frac{z_2^2 + z_2^2 - z_1^2}{z_0^2}\right) = \frac{\omega^2 R^2}{2}\,\tanh^2 r \tag{3}$$

where ω is a constant. The Laplace–Beltrami operator Δ_{LB} up to the R^2 factor is connected with the Casimir operator for the group SO(2,2) (which is an isometry group of the H_2^2 hyperboloid) has the form

$$\begin{aligned}\mathcal{C}_2 = R^2\Delta_{LB} = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial\eta^i}\sqrt{|g|}\,g^{ik}\frac{\partial}{\partial\eta^k} \\ = \mathbf{N}^2 + \mathbf{L}^2, \\ g = \det\,(g_{ik}), \quad g^{ij}g_{jk} = \delta^i_k\end{aligned} \tag{4}$$

where $\mathbf{N}^2 = -N_1^2 + N_2^2 + N_3^2$ and $\mathbf{L}^2 = -L_1^2 + L_2^2 + L_3^2$. The six generators L_i, N_i $(i = 1, 2, 3)$ of the SO(2,2) group are given with the following formulas

$$L_1 = z_2\partial_3 - z_3\partial_2, \quad -N_1 = z_0\partial_1 - z_1\partial_0,$$
$$L_2 = z_1\partial_3 + z_3\partial_1, \quad N_2 = z_0\partial_2 + z_2\partial_0,$$
$$-L_3 = z_1\partial_2 + z_2\partial_1, \quad N_3 = z_0\partial_3 + z_3\partial_0.$$

Choosing now the wave function Ψ in the Schrödinger equation (1) as

$$\Psi(r, \tau, \varphi) = (\sinh r)^{-1} R(r)\, \mathcal{Y}_\ell^m(\tau, \varphi) \tag{5}$$

where the pseudospherical functions on the two-dimensional one-sheeted hyperboloid $\mathcal{Y}_\ell^m$ are the common eigenfunctions of the pair of operators $\mathbf{L}^2\mathcal{Y}_\ell^m = \ell(\ell+1)\mathcal{Y}_\ell^m$ and $L_1^2\mathcal{Y}_\ell^m = m^2\mathcal{Y}_\ell^m$. The spectrum of ℓ, according to the irreducible representations of the $SO(2,1)$ group, splits into the following classes[a]: ℓ is integer and $m = \ell+1, \ell+2, \ldots$ for the positive discrete series; ℓ is integer and $m = -(\ell+1), -(\ell+2), \ldots$ for negative discrete series;
and $\ell = -1/2 + i\rho$ and $m = 0, \pm1, \pm2, \ldots$ for the continuous principal series.

[a] We not consider here the continuous supplementary series when $-1/2 < \ell < 0$ and $m = 0, \pm1, \pm2, \ldots$.

1. First, let us consider the case when ℓ is integer: $\ell = 0, 1, 2 \ldots.$ Then the pseudospherical functions $\mathcal{Y}_\ell^m$, normalized with respect to the invariant measure on the one-sheeted hyperboloid $\cosh\tau d\tau d\varphi$, have the form

$$\begin{aligned}\mathcal{Y}_\ell^m(\tau, \varphi) = \frac{2^\ell \ell!}{\pi}\sqrt{\frac{(2\ell+1)\,(|m|-\ell-1)!}{2(|m|+\ell)!}} \\ \times(\cosh\tau)^{-\ell-1}\, C^{\ell+1}_{|m|-\ell-1}(\tanh\tau)\, e^{im\varphi}\end{aligned} \tag{6}$$

Putting in Eq. (1) the wave function in the form of (5), we arrive at the differential equation in modified Poeschl–Teller form

$$\frac{d^2R}{dr^2} + \left[\mathcal{E} + \frac{\nu^2 - 1/4}{\cosh^2 r} - \frac{(\ell+1/2)^2 - 1/4}{\sinh^2 r}\right] R = 0 \tag{7}$$

where $\mathcal{E} = 2R^2E - \omega^2R^4 - 1$ and $\nu = \sqrt{\omega^2R^4 + \frac{1}{4}}$. The spectrum of the equation (7) contains a finite number of bound states at $\nu > \ell + 3/2$ and is described with the following wave function normalized on $[0, \infty)$ [12]:

$$R_{n_r\ell}\,(r) = N_{n_r\ell}\,(\sinh r)^{\ell/2}(\cosh r)^{1/2-\nu}$$
$$\times {}_2F_1\left(-n_r, n_r + \ell + 3/2 - \nu; \ell + 3/2; -\sinh^2\tau\right),$$

$$N_{n_r\ell} = \frac{1}{\Gamma(\ell+3/2)}$$
$$\times\sqrt{\frac{2(\nu-\ell-2n-3/2)\,\Gamma(\ell+n_r+3/2)\,\Gamma(\nu-n_r)}{n_r!\,\Gamma(\nu-n_r-\ell-1/2)}} \tag{8}$$

where $n_r = 0, 1, \ldots; [\frac{1}{2}(\nu - \ell - 3/2)]$ is a radial quantum number and ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function. These also may be expressed in terms of the Jacobi polynomials of order n_r and argument $\cosh 2r$. The normalized total wave function $\Psi(r,\tau,\varphi) \equiv \Psi_{n_r\ell m}(r,\tau,\varphi;\nu)$ is given by the formulas (5), (6) and (8). The quantized energy is

$$E_N(\nu,R) = -\frac{(N+1)(N+3)}{2R^2} + \frac{\nu+1/2}{R^2}\left(N+\frac{3}{2}\right). \tag{9}$$

Here $N = \ell + 2n_r$ is the principal quantum number and the bound states occur for $N = 0,1,...,N_{max} = [\nu - 3/2]$ ([x] is an integer part of x). Therefore, there is a finite number of positive discrete energy state of oscillator problem on SO(2,2) hyperboloid, the energy of the ground state is equal to $E_0 = 3(\nu - 1/2)/2R^2$ and the last higher excited state is situated around $E_{N_{max}} \sim (\nu^2 + 3/4)/2R^2$. For a fixed quantum number N all level are degenerated for the quantum number ℓ and n_r, for N even and odd correspondingly $(N+2)/2$ and $(N+1)/2$ times. Apart from this, all states at a fixed value of ℓ are infinite times degenerated for the azimuthal quantum number m: $m = \pm(\ell+1),\ \pm(\ell+2),\ \ldots$.

The continuous states are described by the wave functions [12]

$$R_{p\ell}(r) = N_{p\ell}\,(\sinh r)^{\ell/2}(\cosh r)^{1/2-\nu}{}_2F_1\left(\frac{\ell+3/2-\nu+ip}{2},\frac{\ell+3/2-\nu-ip}{2};\ell+\frac{3}{2};-\sinh^2\tau\right), \tag{10}$$

$$N_{p\ell} = \frac{1}{\Gamma(\ell+3/2)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\left|\Gamma\left\{\frac{\ell+3/2+\nu+ip}{2}\right\}\Gamma\left\{\frac{\ell+3/2-\nu+ip}{2}\right\}\right| \tag{11}$$

where the energy of the states of continuous spectrum are $E = (\nu^2 + 3/4 + p^2/4)/2R^2$, $p \in \mathbf{R}$ with the minimum state: $E_{min} = (\nu^2 + 3/4)/2R^2$. The normalized total wave functions $\Psi(r,\tau,\varphi) \equiv \Psi_{p\ell m}(r,\tau,\varphi;\nu)$ are given by the formulas (5) and (10). Thus, we obtain, that for the integer value of the quantum number ℓ the energy spectrum is positive and splits into two part, discrete part $3(\nu - 1/2)/2R^2 < E < (\nu^2 + 3/4)/2R^2$ and continuous one for $(\nu^2 + 3/4)/2R^2 \le E$.

2. For the case when $\ell = -1/2 + i\rho$, the potential in the equation (7) takes the form of the attractive potential $V(r) = -(\nu^2 - 1/4)\cosh^2 r - (\rho^2 + 1/4)\sinh^2 r$ which is a singular as $\sim r^{-2}$ at $r \sim 0$ and thus the corresponding eigenvalue problem is singular at the beginning of the interval $r \in (0,\infty)$. Each of linearly independent solutions of (7) is a square integrable, so the spectrum is discrete for each self-adjoint extension. However, the direct calculation of orthonormal bases of eigenfunctions is a complicated task and we omit it here.

3. Conclusion

In this note we have discussed the harmonic oscillator on the surface of the hyperboloid H_2^2: $z_0^2 + z_1^2 - z_2^2 - z_3^2 = R^2$. We have found the exact solution of the Schrödinger equation in pseudo-spherical coordinates (2) and have shown that the spectrum of the harmonic oscillator on H_2^2, as in case of two-sheeted hyperboloid, contains the scattering states and a finite number of bound states. Each of the energy levels is degenerated by radial and angular quantum number and infinitely degenerated by azimuthal quantum number m. The finite degeneration of energy levels is connected with the existence of an additional (to the angular momentum operator $\mathbf{L}^2$) integral of motion, specific for the harmonic oscillator operator, the so-called Demkov tensor: $D_{ik} = (N_i N_k + N_k N_i)/2R^2 - \omega^2 R^2 z_i z_k / z_0^2$.

As it was shown in the work of Kalnins and Miller [13] there exist 71 orthogonal systems of coordinates, which admit the separation of variables in the Helmholtz equation (free Schrödinger equation) on SO(2,2) hyperboloid. Therefore it is interesting to consider a more general problem connected to the construction of normalized solutions of harmonic oscillator potential in all separation on SO(2,2) hyperboloid. We plan to consider this question in the near future.

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