

Massless and Massive Gauge-Invariant Fields in the Theory of Relativistic Wave Equations

V. A. Pletyukhov*

Brest State University, 21 Kosmonavtov Ave., Brest 224016, BELARUS

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In this work massless and massive gauge-invariant spin 0 and spin 1 fields (particles) have been considered within the scope of the theory of generalized relativistic wave equations with an extended set of the Lorentz group representations. The results obtained may be useful as regards the application of the relativistic wave-equation theory in modern field models.

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1. Introduction

One of the most extensively used ways to describe fundamental particles and fields is still a theory of relativistic wave equations (RWE), the foundations of which have been established by Dirac [1], Fierz and Pauli [2, 3], Bhabha [4, 5], Harish-Chandra [6, 7], Gel'fand and Yaglom [8], Fedorov [9, 10]. This theory has been advanced from the assumption that a relativistic-invariant description of both massive and massless particles (fields) may always be reduced to a system of the first-order differential equations with constant coefficients, in the matrix form being given as follows:

$$(\gamma_\mu \partial_\mu + \gamma_0) \psi(x) = 0 \quad (\mu = 1, \dots, 4). \quad (1)$$

Here $\psi(x)$ is a multicomponent wave function transformed in terms of some reducible Lorentz group representation T , γ_μ and γ_0 are square matrices.

In the case when the matrix γ_0 is nonsingular ($\det \gamma_0 \neq 0$), the equation (1) describing a massive particle may be reduced to the following form by multiplication into $m\gamma_0^{-1}$:

$$(\tilde{\gamma}_\mu \partial_\mu + mI) \psi(x) = 0, \quad (2)$$

where m is a parameter associated with the mass, I is the unity matrix.

The choice of the matrices γ_μ in the equations (1) and (2) is limited by the following requirements (e.g., see [8, 9]):

- i) invariance of the equation with respect to the transformations of the proper Lorentz group;
- ii) invariance with respect to space reflections;
- iii) possibility for derivation of the equation from the variational principle.

Equations of the form (2) meeting requirements i)–iii) are known as relativistic wave equations (RWE); equations of the form (1) with the same requirements are known as generalized RWE [9].

From requirement i) and the condition of indecomposability of the equation with respect to the Lorentz group it follows that the function ψ must be transformed by some set of linked irreducible Lorentz-group representations, forming what is known as a scheme for linking. The representations $\tau \sim (l_1, l_2)$ and $\tau' \sim (l'_1, l'_2)$ are referred to as linking if $l'_1 = l_1 \pm \frac{1}{2}$, $l'_2 = l_2 \pm \frac{1}{2}$.

Aside from the choice of the transformation law of the wave function ψ , in definition of possible spin and mass states of the particle, the explicit form of the matrices γ_4 and γ_0 is of particular importance. Properties of the matrix γ_4 are discussed comprehensively in [8]. The structure of the matrix γ_0 is determined in [5, 9]. Specifically, requirement i) results in reducibility of γ_0 to the diagonal form, the matrix

*E-mail: pletyukhov@yandex.by

being composed of independent scalar blocks corresponding to the irreducible representations of τ . If $\det \gamma_0 = 0$, then some of these blocks are zero. As follows from the requirement ii), nonzero elements a_τ of the matrix γ_0 satisfy the relation

$$a_\tau = a_{\dot{\tau}}, \quad (3)$$

where $\dot{\tau}$ is the representation conjugated to τ with respect to the spatial reflection, i.e., if $\tau \sim (l_1, l_2)$, we have $\dot{\tau} \sim (l_2, l_1)$. In case of the finite-dimensional representations requirement iii) also leads to the relation (3).

A distinctive feature of well-known RWE of the form (2) (Dirac equation for spin $\frac{1}{2}$, Duffin-Kemmer equations for spins 0 and 1, Fierz-Pauli equation for spin $\frac{3}{2}$) is the fact that they involve a set of the Lorentz group representations *minimally necessary* for constructing the theory of a corresponding spin.

Such an approach in the case of $\det \gamma_0 = 0$, results in equations for zero-mass particles (e.g., Maxwell equations). Because of this, the choice of $\det \gamma_0 = 0$ (also including $\gamma_0 = 0$) in a theory of RWE is associated with a description of massless particles [9, 11].

It is known that in the theory of massless particle with integer spin some of the wave-function components are unobservable (so called potentials) and others are observable (so called intensities). In consequence, for the potentials one can define gauge transformations and impose additional requirements excluding “superfluous” components of ψ . But for the description of massive particles by RWE of the form (2), the above-mentioned differentiation of the wave-function components is not the case. In other words, the notion of the gauge invariance of RWE (1) is usually used for massless theories.

At the same time, there are papers, where the so-called *massive gauge-invariant theories* are considered within other approaches. Illustrative examples are furnished by Stückelberg approach to the description of a massive spin 1 particle (see [12] and references herein) and by a $\widehat{B} \wedge \widehat{F}$ -theory [13–16] claiming for describing the string interactions in 4-dimensional (4d) space and

suggesting a mechanism (differing from Higgs’s) of the mass generation due to gauge-invariant mixing of electromagnetic and massless vector fields with zero helicity. In the literature, this field is called the Kalb-Ramond field [15, 16] or as the notoph [17]. Because of this, one should clear the status of massive gauge-invariant fields in the theory of RWE.

Another feature of well-known RWE is the fact that on going from the equation (2) for a massive spin S particle to its massless analog (1), by making the substitution $mI \rightarrow \gamma_0, \det \gamma_0 = 0$, not all of the helicity values from $+S$ to $-S$ are retained, a part of them is lost. For instance, when passing from the Duffin-Kemmer equation for spin 1 to Maxwell equations one loses the zero helicity. In some modern models, there exist necessity for simultaneous description of different massless fields [18]. Within the scope of the theory of RWE, it seems possible to solve this problem by the development of a scheme for passing from (2) to (1) with the singular matrix γ_0 retaining not only maximal but also intermediate helicity values.

In author’s opinion, the solution of the stated problems is important in the context of the applying the well-developed apparatus of RWE in the modern field theory, in particular including the phenomenological description of strings and superstrings in 4d space.

2. Gauge-invariant theories for massive spin 0 and 1 particles

Let us consider the following set of the Lorentz group irreducible representations in a space of the wave function ψ :

$$(0, 0) \oplus \left(\frac{1}{2}, \frac{1}{2} \right) \oplus (0, 1) \oplus (1, 0). \quad (4)$$

The most general form of the corresponding tensor system of the equations meeting the

requirements i) – iii) is given by

$$\alpha \partial_\mu \psi_\mu + a \psi_0 = 0, \quad (5a)$$

$$\beta^* \partial_\nu \psi_{\mu\nu} + \alpha^* \partial_\mu \psi_0 + b \psi_\mu = 0, \quad (5b)$$

$$\beta (-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu) + c \psi_{\mu\nu} = 0. \quad (5c)$$

Here ψ_0 is a scalar; ψ_μ is a vector, $\psi_{\mu\nu}$ is an antisymmetric second-rank tensor; α, β are arbitrary complex parameters; a, b, c are real nonnegative parameters. Writing the system (5) in the matrix form (1), in the basis

$$\psi = (\psi_0, \psi_\mu, \psi_{\mu\nu}) \quad (6)$$

we obtain for the matrix γ_0 the following expression:

$$\gamma_0 = \begin{pmatrix} a & & \\ & bI_4 & \\ & & cI_6 \end{pmatrix}. \quad (7)$$

Matrices γ_μ are not given as they are of no use in further consideration.

In general case, when none of the parameters in (5) is zero, this system describes a particle with a set of spins 0, 1 and with two masses

$$m_1 = \frac{\sqrt{ab}}{|\alpha|}, \quad m_2 = \frac{\sqrt{bc}}{|\beta|}, \quad (8)$$

the mass m_1 being associated with spin 0 and m_2 with spin 1. Omitting cumbersome calculations, we will verify this below in particular cases.

Imposing on the parameters of the system (5) the requirement

$$\frac{\sqrt{a}}{|\alpha|} = \frac{\sqrt{c}}{|\beta|}, \quad (9)$$

we obtain equation for a particle with spins 0, 1 and one mass $m = m_1 = m_2$. At $\alpha = 0$, the system (5) coincides with the Duffin-Kemmer equation of a particle with spin 1 and mass $m = m_2$

$$\beta^* \partial_\nu \psi_{\mu\nu} + b \psi_\mu = 0, \quad (10a)$$

$$\beta (-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu) + c \psi_{\mu\nu} = 0. \quad (10b)$$

Finally, by setting $\beta = 0$, we arrive at the Duffin-Kemmer equation for a particle with spin 0 and mass $m = m_1$:

$$\alpha \partial_\mu \psi_\mu + a \psi_0 = 0, \quad (11a)$$

$$\alpha^* \partial_\mu \psi_0 + b \psi_\mu = 0. \quad (11b)$$

Now we consider the case most interesting for us, when the parameters a, b, c by turns are zeros.

From (5), setting

$$a = 0, \quad (12)$$

we have the following system:

$$\partial_\mu \psi_\mu = 0, \quad (13a)$$

$$\beta^* \partial_\nu \psi_{\mu\nu} + \alpha^* \partial_\mu \psi_0 + b \psi_\mu = 0, \quad (13b)$$

$$\beta (-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu) + c \psi_{\mu\nu} = 0, \quad (13c)$$

that, being written in the matrix form (1), corresponds to the singular matrix γ_0

$$\gamma_0 = \begin{pmatrix} 0 & & \\ & bI_4 & \\ & & cI_6 \end{pmatrix}. \quad (14)$$

From the system (13), one can easily derive the second-order equations

$$\square \psi_0 = 0 \quad (15)$$

$$\square \psi_\mu - \frac{c\alpha^*}{|\beta|^2} \partial_\mu \psi_0 - \frac{bc}{|\beta|^2} \psi_\mu = 0. \quad (16)$$

As regards the scalar function ψ_0 governed by the equation (15), the following aspects must be taken into account. The system (13) is invariant with respect to the gauge transformations

$$\psi_0 \rightarrow \psi_0 - \frac{1}{\alpha^*} \Lambda, \quad \psi_\mu \rightarrow \psi_\mu + \frac{1}{b} \partial_\mu \Lambda, \quad (17)$$

where the gauge function Λ is limited by the constraint

$$\square \Lambda = 0. \quad (18)$$

From the comparison (18) with (15) it follows that the function ψ_0 acts as a gauge function and

hence it does not correspond to any physical field. In other words, the gauge transformations (17) and (18) make it possible to impose an additional condition

$$\psi_0 = 0. \quad (19)$$

In this case, the system (13) is transformed to (10) describing a massive spin 1 particle, whereas the equation (16), considered simultaneously with (13a), reduces to an ordinary Proca equation. In this way, the gauge invariance of the system (13), as compared to (5), leads to decrease in physical degrees of freedom from four to three, excluding the spin 0 state.

Note that a similar result may be obtained without use of gauge invariance arguments. By introducing

$$\varphi_\mu = \psi_\mu + \frac{\alpha^*}{b} \partial_\mu \psi_0, \quad (20)$$

the system (13) may be directly reduced to the form

$$\beta^* \partial_\nu \psi_{\mu\nu} + b \varphi_\mu = 0, \quad (21a)$$

$$\beta (-\partial_\mu \varphi_\nu + \partial_\nu \varphi_\mu) + c \psi_{\mu\nu} = 0 \quad (21b)$$

which coincides with (10). This variant of the gauge-invariant theory is known [12] as a Stueckelberg approach to the description of a massive spin 1 particle.

Now, in (5), let us set

$$c = 0. \quad (22)$$

Then the equations (5) take the form

$$\alpha \partial_\mu \psi_\mu + a \psi_0 = 0, \quad (23a)$$

$$\beta^* \partial_\nu \psi_{\mu\nu} + \alpha^* \partial_\mu \psi_0 + b \psi_\mu = 0, \quad (23b)$$

$$-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu = 0. \quad (23c)$$

According to (8), the last system should describe a particle with the mass $m_1 = \frac{\sqrt{ab}}{|\alpha|}$ and with spin 0. By convolution of the equation (23b) with the operator ∂_μ we have

$$\square \psi_0 + \frac{b}{\alpha^*} \partial_\mu \psi_\mu = 0. \quad (24)$$

Comparing (24) with (23a), we arrive at the equation

$$\square \psi_0 - \frac{ab}{|\alpha|^2} \psi_0 = 0, \quad (25)$$

that says that the theory describes a massive particle with spin zero. The states associated with spin 1 disappear due to the invariance of the system (23) with respect to the gauge transformations

$$\psi_{\mu\nu} \rightarrow \psi_{\mu\nu} - \frac{1}{\beta^*} \Lambda_{\mu\nu}, \quad \psi_\mu \rightarrow \psi_\mu + \frac{1}{b} \partial_\nu \Lambda_{\mu\nu}, \quad (26)$$

where the gauge function $\Lambda_{\mu\nu}$ is constrained by

$$\partial_\alpha \partial_\nu \Lambda_{\mu\nu} - \partial_\mu \partial_\nu \Lambda_{\alpha\nu} = 0. \quad (27)$$

On the other hand, as follows from (23b) and (23c), a similar equation

$$\partial_\alpha \partial_\nu \psi_{\mu\nu} - \partial_\mu \partial_\nu \psi_{\alpha\nu} = 0 \quad (28)$$

is satisfied by the tensor $\psi_{\mu\nu}$. Consequently, the freedom in $\Lambda_{\mu\nu}$ permits the following additional constraint

$$\partial_\nu \psi_{\mu\nu} = 0. \quad (29)$$

In this case the system (23) takes the form (11), i.e. it actually describes a massive spin 0 particle.

Also note that the system (23) may be reduced to the form

$$\alpha \partial_\mu \varphi_\mu + a \psi_0 = 0, \quad (30a)$$

$$\alpha^* \partial_\mu \psi_0 + b \varphi_\mu = 0 \quad (30b)$$

similar to (11) by the use of the vector

$$\varphi_\mu = \psi_\mu + \frac{\beta^*}{b} \partial_\nu \psi_{\mu\nu}. \quad (31)$$

Thus, the considered variant of the massive gauge-invariant theory is an analog for the Stückelberg approach but for the spin 0 particle.

In the matrix formalism (1), this theory is consistent with the matrix γ_0 of the form

$$\gamma_0 = \begin{pmatrix} a & & \\ & bI_4 & \\ & & 0_6 \end{pmatrix}. \quad (32)$$

Now, we consider the following set of representations

$$\left(\frac{1}{2}, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, \frac{1}{2}\right)' \oplus (0, 1) \oplus (1, 0), \quad (33)$$

where the representation $\left(\frac{1}{2}, \frac{1}{2}\right)'$ conforms to the pseudovector (to the absolutely antisymmetric third-rank tensor). The most general form of a tensor system based on the set (33) and meeting the requirements i)–iii) is given by

$$\alpha \partial_\nu \psi_{\mu\nu} + a \psi_\mu = 0, \quad (34a)$$

$$\beta \partial_\nu \tilde{\psi}_{\mu\nu} + b \tilde{\psi}_\mu = 0, \quad (34b)$$

$$\alpha^* (-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\psi}_\beta + c \psi_{\mu\nu} = 0. \quad (34c)$$

Here

$$\tilde{\psi}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \psi_{\alpha\beta}, \quad \tilde{\psi}_\mu = \frac{1}{6} \varepsilon_{\mu\nu\alpha\beta} \psi_{\nu\alpha\beta},$$

$\varepsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor ($\varepsilon_{1234} = -i$); $\psi_{\nu\alpha\beta}$ is the antisymmetric third-rank tensor; α, β are arbitrary complex parameters; a, b, c are mass parameters.

Writing the system (34) in the form (1), where $\Psi = (\psi_\mu, \tilde{\psi}_\mu, \psi_{\mu\nu})$ is a column, for the matrix γ_0 we get the expression

$$\gamma_0 = \begin{pmatrix} aI_4 & & \\ & bI_4 & \\ & & cI_6 \end{pmatrix}. \quad (35)$$

Now we elaborate on massive gauge-invariant theories obtainable from (34).

Let us take the case

$$a = 0. \quad (36)$$

In this case we have the system of equations

$$\partial_\nu \psi_{\mu\nu} = 0, \quad (37a)$$

$$\beta \partial_\nu \tilde{\psi}_{\mu\nu} + b \tilde{\psi}_\mu = 0, \quad (37b)$$

$$\alpha^* (-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\psi}_\beta + c \psi_{\mu\nu} = 0 \quad (37c)$$

which is associated with the singular matrix γ_0

$$\gamma_0 = \begin{pmatrix} 0_4 & & \\ & bI_4 & \\ & & cI_6 \end{pmatrix}. \quad (38)$$

From (37) we can obtain the second-order equations

$$\left(\square - \frac{bc}{|\beta|^2}\right) \tilde{\psi}_\mu = 0, \quad (39)$$

$$\partial_\mu \tilde{\psi}_\mu = 0, \quad (40)$$

$$\square \psi_\mu - \partial_\mu \partial_\nu \psi_\nu = 0. \quad (41)$$

The equations (39) and (40) mean that the system (37) involves the description of a massive spin 1 particle. In addition, as seen from (41), the system (37) describes a massless field with the potential ψ_μ . The latter assumes the existence of the gauge transformation

$$\psi_\mu \rightarrow \psi_\mu + \partial_\mu \Lambda \quad (42)$$

with respect to which the system (37) and the equation (41) are invariant. The gauge invariance means that this massless field is of Maxwell-type field with helicity ± 1 .

Thus, the gauge-invariant system (37) provides us with simultaneous description of a massive spin 1 particle and of a massless field with helicity ± 1 . In other words, here we deal with a massive-massless gauge-invariant theory rather than massive theory as is in (13) and (23).

A similar result can be obtained if we set in (37)

$$b = 0. \quad (43)$$

Then we have

$$\gamma_0 = \begin{pmatrix} aI_4 & & \\ & 0_4 & \\ & & cI_6 \end{pmatrix}, \quad (44)$$

and the second-order equations following from the corresponding system

$$\alpha \partial_\nu \psi_{\mu\nu} + a \psi_\mu = 0, \quad (45a)$$

$$\partial_\nu \tilde{\psi}_{\mu\nu} = 0, \quad (45b)$$

$$\alpha^* (-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\psi}_\beta + c \psi_{\mu\nu} = 0 \quad (45c)$$

are of the form

$$\left(\square - \frac{ac}{|\alpha|^2}\right)\psi_\mu = 0, \quad (46)$$

$$\partial_\mu \psi_\mu = 0, \quad (47)$$

$$\square \tilde{\psi}_\mu - \partial_\mu \partial_\nu \tilde{\psi}_\nu = 0. \quad (48)$$

The equation (48) and the system (45) are invariant with respect to the gauge transformations

$$\tilde{\psi}_\mu \rightarrow \tilde{\psi}_\mu + \partial_\mu \tilde{\Lambda}. \quad (49)$$

Thus, here we deal again with a gauge-invariant massive-massless spin 1 theory.

Let us consider another set of representations

$$(0,0)' \oplus \left(\frac{1}{2}, \frac{1}{2}\right)' \oplus (0,1) \oplus (1,0), \quad (50)$$

where $(0,0)'$ is associated with the absolutely antisymmetric four-rank tensor $\psi_{\mu\nu\alpha\beta}$. The most general tensor formulation of the system referring to the set (50) takes the form

$$\alpha \partial_{[\mu} \psi_{\nu\alpha\beta]} + a \psi_{\mu\nu\alpha\beta} = 0, \quad (51a)$$

$$\alpha^* \partial_{[\nu} \psi_{\alpha\beta]} + \beta^* \partial_\mu \psi_{\mu\nu\alpha\beta} + b \psi_{\nu\alpha\beta} = 0, \quad (51b)$$

$$\beta \partial_\nu \psi_{\nu\alpha\beta} + c \psi_{\alpha\beta} = 0, \quad (51c)$$

where the following notations are used:

$$\partial_{[\nu} \psi_{\alpha\beta]} \equiv \partial_\nu \psi_{\alpha\beta} + \partial_\beta \psi_{\nu\alpha} + \partial_\alpha \psi_{\beta\nu}, \quad (52)$$

$$\begin{aligned} \partial_{[\mu} \psi_{\nu\alpha\beta]} &\equiv \partial_\mu \psi_{\nu\alpha\beta} - \partial_\nu \psi_{\mu\alpha\beta} + \\ &\partial_\alpha \psi_{\mu\nu\beta} - \partial_\beta \psi_{\mu\nu\alpha}. \end{aligned} \quad (53)$$

After introduction into the system (51) the dual conjugates tensors $\tilde{\psi}_{\mu\nu}$, $\tilde{\psi}_\mu$ and pseudoscalar $\tilde{\psi}_0 = \frac{1}{4!} \varepsilon_{\mu\nu\alpha\beta} \psi_{\mu\nu\alpha\beta}$ instead of the tensors $\psi_{\mu\nu}$, $\psi_{\nu\alpha\beta}$, $\psi_{\mu\nu\alpha\beta}$, it is conveniently rewritten as

$$\alpha \partial_\mu \tilde{\psi}_\mu + a \tilde{\psi}_0 = 0, \quad (54a)$$

$$\beta^* \partial_\nu \tilde{\psi}_{\mu\nu} + \alpha^* \partial_\mu \tilde{\psi}_0 + b \tilde{\psi}_\mu = 0, \quad (54b)$$

$$\beta \left(-\partial_\mu \tilde{\psi}_\nu + \partial_\nu \tilde{\psi}_\mu\right) + c \tilde{\psi}_{\mu\nu} = 0. \quad (54c)$$

As seen from the comparison between (54) and (5), these systems are dual – one can be derived from the other by the substitutions

$$\psi_0 \leftrightarrow \tilde{\psi}_0, \quad \psi_\mu \leftrightarrow \tilde{\psi}_\mu, \quad \psi_{\mu\nu} \leftrightarrow \tilde{\psi}_{\mu\nu}. \quad (55)$$

Clearly, the use of the system (54) gives the same results as the use of (5).

3. Simultaneous Description of Massless Fields

Returning to the set of representations (4) and the system (5), we consider the case

$$b = 0; \quad (56)$$

then we get

$$\alpha \partial_\mu \psi_\mu + a \psi_0 = 0, \quad (57a)$$

$$\beta^* \partial_\nu \psi_{\mu\nu} + \alpha^* \partial_\mu \psi_0 = 0, \quad (57b)$$

$$\beta \left(-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu\right) + c \psi_{\mu\nu} = 0; \quad (57c)$$

it is associated with the matrix γ_0 of the form

$$\gamma_0 = \begin{pmatrix} a & & \\ & 0_4 & \\ & & cI_4 \end{pmatrix}. \quad (58)$$

From the system (57) we obtain the d'Alembert equation (15) for the scalar function ψ_0 and the second-order equation

$$\square \psi_\mu - \left(1 - \frac{c|\alpha|^2}{a|\beta|^2}\right) \partial_\mu \partial_\nu \psi_\nu = 0 \quad (59)$$

for the vector ψ_μ . This means that we deal with a massless field. When considering the quantities ψ_0 and ψ_μ as potentials of this field, we treat the equation (57c) as the definition of the intensity $\psi_{\mu\nu}$ in terms of the potentials, (57a) is an additional constraint similar to the Feynman gauge. Then the equation (57b) plays the role of the equation of motion.

With this treatment, the system (57) and the equation (59) are invariant with respect to the gauge transformation

$$\psi_\mu \rightarrow \psi_\mu + \partial_\mu \Lambda, \quad (60)$$

where the choice of Λ is constrained by (18). The gauge transformations (60) and (18), in combination with an additional requirement

(57a), indicate that, among the four components of the potential ψ_μ , only two of them are independent. They describe the transverse components of the field under study. In turn, the longitudinal component of this field is described by the scalar function ψ_0 .

In this way the choice (56) in the system (5) leads to the theory of a massless field with three helicity values $\pm 1, 0$. This is one of the distinguishing features of the system (5) as opposed to the massless theory of Duffin–Kemmer for spin 1 with helicities ± 1 .

Also, note that the equation (59), regarding (57a), may be rewritten as

$$\square\psi_\mu + \left(1 - \frac{c|\alpha|^2}{a|\beta|^2}\right) \frac{a}{\alpha} \partial_\mu \psi_0 = 0, \quad (61)$$

from whence it follows that the gradient of the scalar component plays a role of an (internal) source of the transverse component of the massless field.

Next, we consider the case

$$a = 0, \quad b = 0, \quad (62)$$

which gives

$$\partial_\mu \psi_\mu = 0, \quad (63a)$$

$$\beta^* \partial_\nu \psi_{\mu\nu} + \alpha^* \partial_\mu \psi_0 = 0, \quad (63b)$$

$$\beta (-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu) + c \psi_{\mu\nu} = 0. \quad (63c)$$

This system differs from (57) by the potential gauge requirements (compare (57a) with (63a)). In this case, the matrix γ_0 is of the form

$$\gamma_0 = \begin{pmatrix} 0 & & \\ & 0_4 & \\ & & cI_6 \end{pmatrix}. \quad (64)$$

From (63) one can obtain the equation (15) for the function ψ_0 and the second-order equation

$$\square\psi_\mu - \frac{\alpha^* c}{|\beta|^2} \partial_\mu \psi_0 = 0 \quad (65)$$

for ψ_μ that, similar to the system (63), is invariant with respect to the gauge transformations (60), (18).

All this indicates that here we again deal with two interrelated massless fields: vector field with helicity ± 1 and a scalar field with helicity 0, the gradient of the scalar field acts as a source of the vector field.

The other two massless analogs of the system (5), when

$$a = 0, \quad c = 0 \quad (66)$$

and

$$b = 0, \quad c = 0, \quad (67)$$

are associated with the description of a massless field of zero helicity. We will omit further details.

Considering the possibility for simultaneous description of different massless fields, we next analyze a set of representations (33) and the first-order system (34).

First, we take the case

$$c = 0, \quad a = b; \quad (68)$$

then the system (34) reduces to

$$\alpha \partial_\nu \psi_{\mu\nu} + a \psi_\mu = 0, \quad (69a)$$

$$\beta \partial_\nu \tilde{\psi}_{\mu\nu} + a \tilde{\psi}_\mu = 0, \quad (69b)$$

$$\alpha^* (-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\psi}_\beta = 0, \quad (69c)$$

and the matrix γ_0 is

$$\gamma_0 = \begin{pmatrix} aI_8 & \\ & 0_6 \end{pmatrix}. \quad (70)$$

In (69) we take components of the tensor $\psi_{\mu\nu}$ as potentials, assuming the vector ψ_μ and the pseudovector $\tilde{\psi}_\mu$ are intensities. Then the equations (69a) and (69b) are the intensity definitions in terms of the potentials, and (69c) plays the role of the equation of motion.

From the system (69) we derive the second-order equation for the tensor-potential $\psi_{\mu\nu}$

$$\square\psi_{\mu\nu} = 0. \quad (71)$$

The equations (69) and (71) are invariant with respect to the gauge transformation

$$\psi_{\mu\nu} \rightarrow \psi_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \quad (72)$$

where the function Λ_μ is constrained by

$$\square \Lambda_\mu - \partial_\mu \partial_\nu \Lambda_\nu = 0. \quad (73)$$

The equation (71) and the gauge transformations (72) and (73) indicate that the choice (68) leads to the theory for a massless particle of zero helicity carrying spin 1.

By the present time, two approaches to the description of such a particle have been known: Ogievetsky and Polubarinov approach [17] in which intensity is represented by the vector (this particle was called the notoph) and Kalb-Ramond approach [13], where intensity is presented by the antisymmetric third-rank tensor (Kalb-Ramond field). The system (69) combines the description of both fields in the frame of one wave equation.

In turn, if we take the system (34) when setting

$$a = 0, \quad b = 0, \quad (74)$$

we will obtain the following system

$$\partial_\nu \psi_{\mu\nu} = 0, \quad (75a)$$

$$\partial_\nu \tilde{\psi}_{\mu\nu} = 0, \quad (75b)$$

$$\alpha^* (-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\psi}_\beta + c \psi_{\mu\nu} = 0 \quad (75c)$$

that is associated with the matrix γ_0 of the form

$$\gamma_0 = \begin{pmatrix} 0_8 & \\ & cI_6 \end{pmatrix}. \quad (76)$$

In the system (75), the components ψ_μ and $\tilde{\psi}_\mu$ are considered as potentials, and $\psi_{\mu\nu}$ is taken as intensity. Then the system is invariant with respect to the gauge transformations

$$\psi_\mu \rightarrow \psi_\mu + \Lambda_\mu, \quad \tilde{\psi}_\mu \rightarrow \tilde{\psi}_\mu + \tilde{\Lambda}_\mu, \quad (77)$$

where the gauge functions $\Lambda_\mu, \tilde{\Lambda}_\mu$ are constrained by

$$\alpha^* (-\partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\Lambda}_\beta = 0. \quad (78)$$

In other words, at $\alpha = \beta = 1$ the system (75) represents the well-known two-potential

formulation of electrodynamics (e.g., see [19]) for a massless spin 1 field with helicity ± 1 .

Thus, reciprocal complementarity of the theories based on the systems (69) and (75) is exhibited in interpretation of the field components $\psi_\mu, \tilde{\psi}_\mu, \psi_{\mu\nu}$ as well as in helicities of the particles under consideration.

Of particular interest is the case when in (34) we set

$$a = 0, \quad c = 0. \quad (79)$$

This results in the system

$$\partial_\nu \psi_{\mu\nu} = 0, \quad (80a)$$

$$\beta \partial_\nu \tilde{\psi}_{\mu\nu} + b \tilde{\psi}_\mu = 0, \quad (80b)$$

$$\alpha^* (-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu) + \beta^* \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\psi}_\beta = 0 \quad (80c)$$

and leads to the matrix

$$\gamma_0 = \begin{pmatrix} 0_4 & & \\ & bI_4 & \\ & & 0_6 \end{pmatrix}. \quad (81)$$

We rewrite (80) in the following form

$$\partial_\nu \psi_{\mu\nu} = 0, \quad (82a)$$

$$\beta (\partial_\mu \psi_{\nu\alpha} + \partial_\alpha \psi_{\mu\nu} + \partial_\nu \psi_{\alpha\mu}) + b \psi_{\mu\nu\alpha} = 0, \quad (82b)$$

$$\alpha^* (-\partial_\nu \psi_\alpha + \partial_\alpha \psi_\nu) + \beta^* \partial_\mu \psi_{\mu\nu\alpha} = 0, \quad (82c)$$

where $\psi_{\mu\nu\alpha}$ is an antisymmetric three-rank tensor dual to the pseudovector $\tilde{\psi}_\mu$.

According to the structure of the system (82), ψ_μ and $\psi_{\mu\nu}$ are potentials, $\psi_{\mu\nu\alpha}$ is the intensity. Then the equation (82b) is the definition of the intensity, and (82a) acts as an additional constraint imposed on the tensor-potential $\psi_{\mu\nu}$ and included originally in the system itself. This constraint leaves for tensor $\psi_{\mu\nu}$ satisfying the second-order equation

$$\square \psi_{\mu\nu} + \frac{|\alpha|^2}{|\beta|^2} \frac{b}{\alpha} (\partial_\mu \psi_\nu - \partial_\nu \psi_\mu) = 0 \quad (83)$$

two independent components. As this takes place, the system (82) is invariant with respect to the gauge transformations (72), (73). Due to the gauge freedom in Λ_μ we have only one

independent component for $\psi_{\mu\nu}$ that is associated with the state of a massless field with zero helicity.

To elucidate the meaning of the term $\partial_\mu\psi_\nu - \partial_\nu\psi_\mu$ in (83), we turn to the potential ψ_μ . Apart from the transformations (72) and (73), the system (82) is also invariant with respect to the gauge transformation

$$\psi_\mu \rightarrow \psi_\mu + \partial_\mu\Lambda. \quad (84)$$

From the equation (82c) for ψ_μ we derive the second-order equation

$$\square\psi_\mu - \partial_\mu\partial_\nu\psi_\nu = 0, \quad (85)$$

in combination with (73) indicating that the potential ψ_μ gives description for the transverse component (helicity ± 1) of the massless field. The relation

$$\partial_\mu\psi_\nu - \partial_\nu\psi_\mu \equiv F_{\mu\nu} \quad (86)$$

in the equations (82c) and (83) may be considered as definition of intensity associated with this transverse component. Then the equation (82c) rewritten with regard to the notation of (86) as

$$\beta^*\partial_\mu\psi_{\mu\nu\alpha} - \alpha^*F_{\nu\alpha} = 0, \quad (87)$$

plays the role of the equation of motion.

Thus, the choice (79) of mass parameters in the initial system (34) leads to the theory of a generalized massless field with helicities 0, ± 1 .

The choice of the parameters

$$b = 0, \quad c = 0. \quad (88)$$

in the system (34) also results in a theory of the generalized massless field with helicities 0, ± 1 featuring a dual conjugate of that obtainable in the case of (79). Details are beyond the scope of this paper.

4. Mass generation and RWE theory

In 1974 [13, 14], a mechanism of mass generation was proposed differing from the well-known Higgs mechanism. Later this mechanism

has been identified as a gauge-invariant field mixing. Let us consider it.

Two massless systems of equations are considered together as initial systems:

$$\partial_\nu\psi_{\mu\nu} = 0, \quad (89a)$$

$$-\partial_\mu\varphi_\nu + \partial_\nu\varphi_\mu + \psi_{\mu\nu} = 0, \quad (89b)$$

and

$$\partial_\mu\psi_{\mu\nu\alpha} = 0, \quad (90a)$$

$$-\partial_\mu\varphi_{\nu\rho} - \partial_\nu\varphi_{\rho\mu} - \partial_\rho\varphi_{\mu\nu} + \psi_{\mu\nu\rho} = 0. \quad (90b)$$

The first system describes an electromagnetic field and the second one describes a field of Kalb-Ramond. In (90a), (90b) tensor $\psi_{\mu\nu\alpha}$ is considered to be intensity. Then the additional term

$$L_{int} = m\varphi_\mu\partial_\nu\varphi_{\mu\nu} \quad (91)$$

is included into the Lagrangian of this system without violation of the gauge-invariance for the initial Lagrangian L_0 . This term may be formally treated as an interaction of the fields under study (so-called topological interaction). Varying the Lagrangian $L = L_0 + L_{int}$ and introducing the pseudovector $\tilde{\psi}_\mu = \frac{1}{3!}\varepsilon_{\mu\nu\alpha\beta}\psi_{\nu\alpha\beta}$, we have the system

$$\partial_\nu\psi_{\mu\nu} + m\tilde{\psi}_\mu = 0, \quad (92a)$$

$$-\partial_\mu\tilde{\psi}_\nu + \partial_\nu\tilde{\psi}_\mu + m\psi_{\mu\nu} = 0, \quad (92b)$$

$$\partial_\nu\tilde{\varphi}_{\mu\nu} + \tilde{\psi}_\mu = 0, \quad (92c)$$

$$-\partial_\mu\varphi_\nu + \partial_\nu\varphi_\mu + \psi_{\mu\nu} = 0, \quad (92d)$$

where

$$\tilde{\varphi}_{\mu\nu} = \frac{1}{2!}\varepsilon_{\mu\nu\alpha\beta}\varphi_{\alpha\beta}. \quad (93)$$

Now in the system (92), we replace φ_μ and $\tilde{\varphi}_{\mu\nu}$ by the quantities \tilde{G}_μ and $G_{\mu\nu}$ using the formulae

$$\tilde{G}_\mu = \varphi_\mu - \frac{1}{m}\tilde{\psi}_\mu, \quad (94a)$$

$$G_{\mu\nu} = \tilde{\varphi}_{\mu\nu} - \frac{1}{m}\psi_{\mu\nu}. \quad (94b)$$

Finally, the system (92) is reduced to the following form

$$\partial_\nu \psi_{\mu\nu} + m\tilde{\psi}_\mu = 0, \quad (95a)$$

$$-\partial_\mu \tilde{\psi}_\nu + \partial_\nu \tilde{\psi}_\mu + m\psi_{\mu\nu} = 0, \quad (95b)$$

$$\partial_\nu G_{\mu\nu} = 0, \quad (95c)$$

$$-\partial_\mu \tilde{G}_\nu + \partial_\nu \tilde{G}_\mu = 0. \quad (95d)$$

As seen, the system (95) is reducible with respect to the Lorentz group into subsystems (95a), (95b) and (95c), (95d). The first of them describing a massive spin 1 particle is interpreted in [13] as an interaction transporter between open strings. The subsystem (95c), (95d) gives no description for a physical field, as it is associated with zero energy density. However, its presence is necessary to impart to the latter status of a gauge-invariant theory.

Using the formalism of the first order wave equations, all the above may be interpreted as follows. Let us consider a set of representations

$$\left(\frac{1}{2}, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, \frac{1}{2}\right)' \oplus 2(1, 0) \oplus 2(0, 1), \quad (96)$$

associated with the tensor system (89), (90).

It is obvious that on the basis of (96) one can derive an equation of the type (1) with the matrices

$$\gamma_\mu = \begin{pmatrix} \gamma_\mu^{DK} & \\ & \gamma_\mu^{DK} \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0_4 & & \\ & I_6 & \\ & & 0_6 \\ & & & I_4 \end{pmatrix}, \quad (97)$$

where γ_μ^{DK} are 10-dimensional Duffin-Kemmer matrices. Addition to the Lagrangian of the topological term (91), leads to varying the matrices γ_μ leaving the matrix γ_0 unaltered. The substitutions (94) are equivalent to the unitary transformation restoring the form of γ_μ matrix given in (97). As this is the case, the matrix γ_0 takes the form

$$\gamma_0 = \begin{pmatrix} mI_{10} & \\ & 0_{10} \end{pmatrix}. \quad (98)$$

In this way we actually arrive at RWE reducible to the ordinary Duffin-Kemmer

equation for a massive spin 1 particle and at the massless fermionic limit of this equation. Nontrivial nature of the mass generation method consists in the fact that on passage from the initial massless field(s) to the massive one neither the form of γ_μ matrices nor the rank of singular γ_0 matrix is affected, the procedure being reduced to permutation of zero and unity blocks of this matrix only. In the process, the number of degrees of freedom (that is equal to three) for a field system is invariable. It seems as if the notoph passes its degree of freedom to the photon, that automatically leads to a massive spin 1 particle.

5. Discussion and conclusions

Based on the examples considered, important conclusions can be made.

Conclusion 1. The generalized equation (1) with the singular matrix γ_0 can describe not only massless but also massive fields. Featuring the gauge invariance, these equations just form the class of massive gauge-invariant theories.

As demonstrated in Sec. 2 using the equations (37) and (45) as an example, a theory of generalized wave equations suggests also a variant of the generalized description for massive and massless fields based on equations irreducible with respect to the Lorentz group. Thus, we arrive at the following conclusion.

Conclusion 2. The equation of the form (1) with the singular matrix γ_0 can describe the fields involving both massive and massless components. In this case it is more correct to refer to massive-massless gauge-invariant theories rather than to the massive ones.

As demonstrated in Sec. 3, on adequate selection of the Lorentz group representations in a space of the wave function ψ and interpretation of its components, one can give the description of a massless field not only with helicity ± 1 but also with helicity 0 as well as simultaneous description of the indicated fields. Generalizing this result for the case of arbitrary spin S , we can conclude the

following.

Conclusion 3. *A theory of the generalized wave equations with the singular matrix γ_0 makes it possible to describe not only massless fields with maximal (for the given set of representations) helicity $\pm S$, but also fields with intermediate helicity values as well as to offer simultaneous description of these fields.*

It is clear that the character of the field described by equation (1) with the singular matrix γ_0 depends on the form of this matrix. To find when the singular matrix γ_0 leads to massless theories and when it results in massive or massive-massless gauge-invariant theories, we examine the Lorentz structure of the “massive” term $\gamma_0\psi$ in the foregoing cases.

It is observed that in the case of (6), (13), (14) associated with a massive gauge-invariant spin 1 theory, the matrix γ_0 (14) affecting the wave function ψ (6) in the expression $\gamma_0\psi$ retains (without reducing to zero) the Lorentz covariants $\psi_\mu, \psi_{\mu\nu}$, on the basis of which an ordinary (of the form (2)) massive spin 1 theory can be framed.

But in the case of a massless theory given by (6), (57), (58), the matrix γ_0 in the expression $\gamma_0\psi$ retains the covariant $\psi_{\mu\nu}$, on the basis of which it is impossible to construct an equation of the form (2) for a massive particle. A similar pattern is characteristic one for the remaining cases: in all the massive (massive-massless) gauge-invariant theories the matrix γ_0 affecting the wave function ψ retains its covariant components necessary for framing of an ordinary massive spin 1 or 0 theory;

provided the expression $\gamma_0\psi$ does not involve such a necessary set of covariants, massless theories can be framed only. This leads us to the fourth conclusion.

Conclusion 4. *Should the generalized RWE (1) with the singular matrix γ_0 in the product $\gamma_0\psi$ retain a set of the Lorentz covariants sufficient to frame an ordinary (with $\det \gamma_0 \neq 0$) theory of a massive spin S particle, this equation may be associated with a massive gauge-invariant spin S theory. Otherwise, when this requirement is not fulfilled for any S , the equation (1) can describe a massless field only.*

Proceeding from all the afore-said, we arrive at the following important though obvious conclusion.

Conclusion 5. *To frame both massive (massive-massless) gauge-invariant spin S theory and massless theory with intermediate helicity values from $+S$ to $-S$ we need an extended, in comparison with a minimally necessary for the description of this spin (helicity), set of the irreducible Lorentz group representations in a space of the wave function ψ .*

In the present work, when considering spin 1, the above-mentioned extension has been accomplished by the introduction of the scalar representation $(0, 0)$ into a set of representations given by (4) and of pseudoscalar representation $(\frac{1}{2}, \frac{1}{2})'$ – into a set given by (33).

More possibilities are offered by the use of multiple (recurrent) Lorentz group representations.

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