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PIECEWISE AFFINE FUNCTIONS AND POLYHEDRAL SETS*

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In this paper we present a number of characterizations of piecewise affine and piecewise linear functions defined on finite dimensional normed vector spaces. In particular we prove that a real-valued function is piecewise affine [resp., piecewise linear] if both its epigraph and its hypograph are (nonconvex) polyhedral sets [resp., polyhedral cones]. Also, we show that the collection of all piecewise affine [resp., piecewise linear] functions coincides with the smallest vector lattice containing the vector space of affine [resp., linear] functions. Furthermore, we prove that a function is piecewise affine [resp., piecewise linear] if it can be represented as a difference of two convex [resp., sublinear] polyhedral functions.

KEY WORDS: Polyhedral sets and functions, piecewise affine and piecewise linear functions, approximation.

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1. INTRODUCTION

Piecewise affine functions and polyhedral sets are mainly studied in piecewise linear topology [17, 20, 23]. We are interested in piecewise affine functions and mappings due to their possible applications in nonsmooth analysis (see, e.g., [1, 5, 8, 16] and the references therein) and, in particular, in the quasidifferentiation theory [6–11, 18, 19]. In our preceding paper [11] we have already used piecewise linear functions as local approximations for defining the notion of polyhedral quasidifferentiability. In that paper we have also announced some results concerning properties of piecewise linear functions. Here we shall present different qualitative and analytical characterizations of piecewise affine and piecewise linear functions in detail.

In Section 1 of this paper, we give the definitions of piecewise affine and piecewise linear functions and prove that both the collection of all piecewise affine functions and the collection of all piecewise linear functions form vector lattices, the second of which is a sublattice of the first one. Section 2 deals with polyhedral sets. We say that a set of a finite dimensional normed space is polyhedral if it is the result of unions and intersections of finitely many closed halfspaces. A convex set turns out to be polyhedral in the above sense if and only if it is the intersection of a finite family of closed

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halfspaces. Thus the above definition extends the notion of a convex polyhedral set, which is well-known in convex analysis (see, e.g., [3, 14, 21, 22]) to the nonconvex case. Some properties of polyhedral sets are also studied in Section 2. In Section 3 we prove the main results of this paper. So a real-valued function is proved to be piecewise affine (respectively, piecewise linear) if and only if both its epigraph and its hypograph are polyhedral sets (respectively, polyhedral cones). Using this criterion we obtain a number of analytical representations characterizing piecewise affine and piecewise linear functions. In particular, we show that the vector sublattice of piecewise affine (respectively, piecewise linear) functions consists exactly of those functions which are represented as a difference of two convex (respectively, sublinear) polyhedral functions. Thus, piecewise affine (respectively, piecewise linear) functions form a sublattice in the lattice of difference convex (respectively, difference sublinear) functions. Section 4 is devoted to approximating arbitrary compact sets by polyhedral sets and continuous functions by piecewise affine functions. Also we prove that the sublattice of piecewise linear functions is dense in the Banach lattice of all positively homogeneous continuous functions.

2. PIECEWISE AFFINE FUNCTIONS

Let X be a real normed vector space and let X^* be its dual vector space consisting of all linear continuous real-valued functions defined on X. The canonical bilinear form on $X \times X^*$ will be denoted by $\langle \cdot, \cdot \rangle$. Throughout this paper we shall assume, without any special mention, that X is finite dimensional. So we could identify X and X^* with $\mathbb{R}^n$, where $n = \dim X$, and $\langle \cdot, \cdot \rangle$ with the inner product on $\mathbb{R}^n$. However, we find it more convenient to use different notations for the primal space X and its dual X^*.

We recall (see, e.g., [3, 14, 22]) that a convex set $Q \subset X$ is said to be *polyhedral* if it can be represented as the intersection of a finite family of closed halfspaces, i.e., if $Q = \bigcap_{i=1}^{k} H_{\leqslant}(a_i^*, \alpha_i)$, where $H_{\leqslant}(a_i^*, \alpha_i) := \{x \in X \mid \langle x, a_i^* \rangle \leqslant \alpha_i\}$, $a_i^* \in X^*$, $a_i^* \neq 0$, $\alpha_i \in \mathbb{R}$, $i = 1, \dots, k$ ($\mathbb{R}$ is the set of real numbers).

A finite family $\sigma := \{Q_1, Q_2, \dots, Q_k\}$ of convex polyhedral sets is called a *polyhedral partition* of X if

(i) $\operatorname{int} Q_i \neq \phi$, $i = 1, 2, \dots, k$;
(ii) $\operatorname{int} Q_i \cap \operatorname{int} Q_j = \phi, i, j = 1, 2, \dots, k, i \neq j$;
(iii) $X = \bigcup_{i=1}^{k} Q_i$.

If, in addition, all subsets $Q_1, Q_2, \dots, Q_k$ of the polyhedral partition σ are convex polyhedral cones we say that σ is a *conically polyhedral partition* of X.

A function $f: X \to \mathbb{R}$ is said to be *piecewise affine* (respectively, *piecewise linear*) on X [13, 15, 23] if one can associate with f a polyhedral (respectively, conically polyhedral) partition $\sigma_f := \{Q_1, Q_2, \dots, Q_k\}$ of X such that the restriction of f to each set Q_i, $i = 1, 2, \dots, k$, is affine (respectively, linear).

Proposition 1.1: *The collection $PA(X)$ of all piecewise affine functions defined on X is a vector lattice with respect to the standard algebraic operations and pointwise maximum and pointwise minimum as lattice operations.*

For the proof we need the following

Lemma 1.1: *For any polyhedral partitions* $\sigma_1 := \{Q_1, Q_2, \dots, Q_k\}$ *and* $\sigma_2 := \{S_1, S_2, \dots, S_m\}$ *of* X *their intersection defined by* $\sigma_1 \cap \sigma_2 := \{Q_i \cap S_j \mid \operatorname{int}(Q_i \cap S_j) \neq \phi, i = 1, \dots, k, j = 1, \dots, m\}$ *is also a polyhedral partition of* X.

Proof: It follows from properties of σ_1 and σ_2 that $X = \bigcup_{i=1}^{k} \bigcup_{j=1}^{m} (Q_i \cap S_j)$, with all $Q_i \cap S_j$, $i = 1, \dots, k, j = 1, \dots, m$, being closed because they are convex polyhedral sets. In as much as the union of a finite family of nowhere dense sets is also nowhere dense (see, e.g., [4]), at least one of the sets $Q_i \cap S_j$ has a nonempty interior. Hence the family $\sigma_1 \cap \sigma_2$ is nonvoid. Now we have to show that $\sigma_1 \cap \sigma_2$ satisfies the conditions (i)–(iii) from the definition of a polyhedral partition. Condition (i) is true due to the definition of $\sigma_1 \cap \sigma_2$. As for condition (ii) the family $\sigma_1 \cap \sigma_2$ satisfies it because both σ_1 and σ_2 satisfy this condition. To prove condition (iii) we consider the set $G := X \backslash \cup \{Q_i \cap S_j \mid Q_i \cap S_j \in \sigma_1 \cap \sigma_2\}$. We note that G is open and $cl\, G \subset \cup \{Q_i \cap S_j \mid Q_i \cap S_j \notin \sigma_1 \cap \sigma_2\}$. Since $\operatorname{int}(Q_i \cap S_j) = \phi$ for any $Q_i \cap S_j \notin \sigma_1 \cap \sigma_2$ we have $\operatorname{int} G = \phi$ and, consequently, $G \neq \phi$. Hence, $X = \cup \{Q_i \cap S_j \mid Q_i \cap S_j \in \sigma_1 \cap \sigma_2\}$. This completes the proof of Lemma 1.1. ■

Remark 1.1: If σ_1 and σ_2 are conically polyhedral partitions of X, then their intersection $\sigma_1 \cap \sigma_2$ is also a conically polyhedral partition.

Proof of Proposition 1.1: First of all we note that if f is a piecewise affine function then αf is also a piecewise affine function for all real α. Now we consider two arbitrary piecewise affine functions f and g. Let σ_f and σ_g be polyhedral partitions associated with f and g respectively. The functions f and g are evidently affine on every subset of the intersection $\sigma_f \cap \sigma_g := \{Q_1, Q_2, \dots, Q_r\}$ and therefore the function $f + g$ is also affine on every Q_i, $i = 1, \dots, r$. Thus $PA(X)$ is a vector space.

To prove that the functions $\max(f, g)$ and $\min(f, g)$ are piecewise affine we construct the refinement of the family $\sigma_f \cap \sigma_g := \{Q_1, \dots, Q_r\}$ in the following way. Together with every subset $Q_i \in \sigma_f \cap \sigma_g$, $i = 1, 2, \dots, r$, we consider affine functions a_i and b_i such that $f(x) = a_i(x)$ and $g(x) = b_i(x)$ for all $x \in Q_i$. If the hyperplane $H_i = \{x \in X \mid a_i(x) = b_i(x)\}$ does not intersect $\operatorname{int} Q_i$ then the set Q_i belongs to one of the closed halfspaces generated by the hyperplane H_i. Let $Q_i \subset \{x \in X \mid a_i(x) \leqslant b_i(x)\}$. Then $\max(f(x), g(x)) = b_i(x)$, $\min(f(x), g(x)) = a_i(x)$ for all $x \in Q_i$. Therefore in the case when $H_i \cap \operatorname{int} Q_i = \phi$ we retain the set Q_i without changing. In the case when $H_i \cap \operatorname{int} Q \neq \phi$, we replace the set Q_i by two sets $Q_i^- := \{x \in Q_i \mid a_i(x) \leqslant b_i(x)\}$ and $Q_i^+ := \{x \in Q_i \mid a_i(x) \geqslant b_i(x)\}$, that are polyhedral. It is easily to see that the functions $\max(f, g)$ and $\min(f, g)$ are affine on every subset Q_i^- and Q_i^+. In addition we have i) $\operatorname{int} Q_i^- \neq \phi$, $\operatorname{int} Q_i^+ \neq \phi$; ii) $\operatorname{int} Q_i^- \cap \operatorname{int} Q_i^+ = \phi$; iii) $Q_i = Q_i^- \cup Q_i^+$, and therefore the family that is obtained from $\sigma_f \cap \sigma_g$ by replacing the set Q_i with the subsets Q_i^- and Q_i^+ is also a polyhedral partition of X. Using this way for refining the family $\sigma_f \cap \sigma_g$ we construct a polyhedral partition of X such that the functions $\max(f, g)$ and $\min(f, g)$ are affine on its subsets. Thus the functions $\max(f, g)$ and $\min(f, g)$ are piecewise affine and we have proved the proposition. ■

Proposition 1.2: *The collection* $PL(X)$ *of all piecewise linear functions is a vector sublattice of the lattice of piecewise affine functions* $PA(X)$.

The proof is similar to the proof of Proposition 1.1.

3. POLYHEDRAL SETS

Let X be a finite dimensional normed space and let $\mathfrak{M}(X)$ be the Boolean lattice [2] of all subsets of X with the set-theoretic operations of union and intersection as lattice operations. By the symbol $M(X)$ we shall denote the smallest sublattice in $\mathfrak{M}(X)$ that contains all closed halfspaces $H_{\leqslant}(a^*, \alpha) := \{x \in X \mid \langle x, a^* \rangle \leqslant \alpha\}$, $a^* \in X^*$, $a^* \neq 0$, $\alpha \in \mathbb{R}$.

Any subset of X that belongs to the sublattice $M(X)$ will be called *a polyhedral set.*

Note that any affine manifold (in particular, X, ϕ and any singleton) is a polyhedral set. Besides any convex set that is polyhedral in the sense of convex analysis (see the preceding section) is also polyhedral in the above sense. Below in Proposition 2.1 we shall show that the converse is also true.

By the definition any polyhedral set Q is of the form

$$Q = p(H_{\leqslant}(a_1^*, \alpha_1), H_{\leqslant}(a_2^*, \alpha_2), \ldots, H_{\leqslant}(a_m^*, \alpha_m)),$$

where $p(Q_1, Q_2, \ldots, Q_m)$ is some lattice polynomial [2] defined over the Boolean lattice $\mathfrak{M}(X)$. Since $\mathfrak{M}(X)$ is distributive then (see, e.g., [2]) any lattice polynomial over $\mathfrak{M}(X)$ is equivalent to an union of intersections or an intersection of unions. Thus, we have

Proposition 2.1: *Any polyhedral set $Q \in M(X)$ can be represented in the form*

$$Q = \bigcup_{i=1}^{m} \bigcap_{j=1}^{k(i)} H_{\leqslant}(a_{ij}^*, \alpha_{ij}), \tag{2.1}$$

as well as in the form

$$Q = \bigcap_{\nu=1}^{r} \bigcup_{\mu=1}^{s(\nu)} H_{\leqslant}(b_{\nu\mu}^*, \beta_{\nu\mu}), \tag{2.2}$$

where $a_{ij}^*, b_{\nu\mu}^* \in X^*$, α_{ij}, $\beta_{\nu\mu} \in \mathbb{R}$, $j = 1, \ldots, k(i)$, $i = 1, \ldots, m$, $\mu = 1, \ldots, s(\nu)$, $\nu = 1, \ldots, r$.

From the equalities (2.1) and (2.2) it follows that any polyhedral set is closed.

Proposition 2.2: *A polyhedral set $Q \in M(X)$ is convex if and only if it can be represted as the intersection of a finite family of closed halfspaces:*

$$Q = \bigcap_{i=1}^{k} H_{\leqslant}(a_i^*, \alpha_i), \tag{2.3}$$

where $a_i^* \in X^*$, $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, k$.

Proof: The sufficient part is evident. For proving the necessity let us consider a convex polyhedral set Q. From equality (2.2) it follows that $Q = \bigcup_{i=1}^{m} Q_i$, where $Q_i = \bigcap_{j=1}^{k(i)} H_{\leqslant}(a_{ij}^*, \alpha_{ij})$, $i = 1, 2, \ldots, m$. Since Q is closed and convex then $Q = cl(\text{conv}(Q_1 \cup Q_2 \cup \cdots \cup Q_m))$. Now using Theorem 19.6 from [22] we conclude that Q can be represented in the form (2.3). This completes the proof. ■

Proposition 2.3: *A set $Q \subset X$ is polyhedral if and only if it can be represented as a union of a finite family of convex polyhedral sets.*

The validity of the proposition follows from representation (2.1) and Proposition 2.2.

It is not difficult to verify that the sublattice $M(X)$ of polyhedral sets is closed with respect to the main algebraic operations. In particular, the following statements are true.

A. Let X and Y be finite dimensional normed spaces and let Q and P be polyhedral sets in X and Y, respectively. Then

(a) the direct product $Q \times P$ is also a polyhedral set in the space $X \times Y$;
(b) for any linear operator $A\colon X \to Y$ the sets

$$AQ := \{y \in Y \mid y = Ax, x \in Q\} \qquad A^{-1}P := \{x \in X \mid Ax \in P\}$$

are also polyhedral sets.

B. If Q_1 and Q_2 are polyhedral sets in X, then

$$Q_1 + Q_2 := \{x_1 + x_2 \mid x_i \in Q_i,\ i = 1, 2\}$$

is also a polyhedral set.

C. If Q is a polyhedral set in X, then $\lambda Q := \{\lambda x \mid x \in Q\}$ is also a polyhedral set for all $\lambda \in \mathbb{R}$.

D. If Q is a polyhedral set in X, then the closure of its complement $cl(X \setminus Q)$ is a polyhedral set in X and, consequently, the boundary of Q is a polyhedral set.

Any maximal (in the sense of inclusion) convex subset of Q will be referred to as *a convex component* of the set Q.

The existence of convex components for any nonempty set Q follows from Zorn's Lemma [12]. Indeed, the collection $\omega(Q)$ of all convex subsets of Q is nonvoid since $\omega(Q)$ contains singletons. Besides, if $\{Q_i, i \in I\}$ is an increasing chain in $\omega(Q)$ then $\bigcup_{i \in I} Q_i \in \omega(Q)$. Using Zorn's Lemma we conclude that for any convex subset of Q there exists a maximal convex subset (a convex component) of Q containing this subset. Thus the collection $\bar{\omega}(Q) := \{S \mid S$ is a convex component of $Q\}$ is nonvoid and $Q = \cup\{S \mid S \in \bar{\omega}(Q)\}$.

Preposition 2.4: *Any convex component of a polyhedral set is also a polyhedral set.*

Proof: Let Q be a polyhedral set in X. It follows from Proposition 2.1 (see equality (2.2)) that Q can be represented as $Q = \bigcap_{v=1}^{r} Q_v$ where $Q_v := \bigcup_{\mu=1}^{s(v)} H_{\leqslant}(b_{v\mu}^*, \beta_{v\mu})$, $v = 1, 2, \dots, r$. We note that the complements $X \setminus Q_v = \bigcap_{\mu=1}^{s(v)} (X \setminus H_{\leqslant}(b_{v\mu}^*, \beta_{v\mu}))$, $v = 1, 2, \dots, r$, are open convex sets.

Now let us consider an arbitrary convex subset S of Q. It is evident that $S \cap (X \setminus Q_v) = \phi$ for all $v = 1, \dots, r$. The Separation Theorem (see, e.g., [22]) point out for every $v = 1, \dots, r$ a closed halfspace $H_{\leqslant}(a_v^*, \alpha_v)$, such that $S \subset H_{\leqslant}(a_v^*, \alpha_v)$, and $X \setminus Q_v \subset X \setminus H_{\leqslant}(a_v^*, \alpha_v)$. Since the last inclusion is equivalent to $H_{\leqslant}(a_v^*, \alpha_v) \subset Q_v$ we have $S \subset \bigcap_{v=1}^{r} H_{\leqslant}(a_v^*, \alpha_v) \subset \bigcap_{v=1}^{r} Q_v = Q$. In the case when S is a convex component of Q we conclude from these inclusions that $S = \bigcap_{v=1}^{r} H_{\leqslant}(a_v^*, \alpha_v)$. Due to Proposition 2.3 this completes the proof. ■

Remark 2.1: There are simple examples showing that the collection $\omega(Q)$ of convex components of a polyhedral set Q is generally infinite. However, it follows from the representation (2.1) that any polyhedral set can be represented as a union of finitely many of its convex components. We remark that, in general, minimal (in the sense of inclusion) finite subcollections of $\omega(Q)$ covering Q may be different both by the number and by constituents.

Before closing this section we shall discuss some facts concerning polyhedral cones. First of all it easily follows from Proposition 2.2 that a polyhedral cone $K \subset X$ is convex if and only if it can be represented in the form

$$K=\bigcap_{i=1}^{m} H_{\leqslant}(a_i^*,0)\; a_i^*\in X^*,\quad a_i^*\neq 0, \tag{2.4}$$

i.e., if and only if K is an intersection of finitely many positively homogeneous closed halfspaces.

In the general case we have

Proposition 2.5: *A set $K\subset X$ is a polyhedral cone if and only if it can be represented in the form*

$$K=\bigcup_{i=1}^{m}\bigcap_{j=1}^{k(i)} H_{\leqslant}(a_{ij}^*,0),\quad (a_{ij}^*\in X^*) \tag{2.5}$$

or, equivalently, in the form

$$K=\bigcap_{\nu=1}^{r}\bigcup_{\mu=1}^{s(\nu)} H_{\leqslant}(b_{\nu\mu}^*,0),\quad (b_{\nu\mu}^*\in X^*). \tag{2.6}$$

Proof: It is easily shown that any set $K\subset X$ which can be represented in the form (2.5) or (2.6) is a cone and polyhedral (the last property follows from Proposition 1.1). For proving the converse assertion let us consider a polyhedral cone $K\subset X$. By Remark 2.1 we can represent K in the form $K=\bigcup_{i=1}^{m}C_i$, where $\{C_1,\dots,C_m\}$ is a finite family of convex components of X. Since K is a cone it follows from the maximality of C_i and Proposition 2.4 that every C_i, $i=1,\dots,m$, is a convex polyhedral cone. Representing every C_i, $i=1,\dots,m$, in the form (2.4) we get equality (2.5).

Let $M_0(X)$ consist of all polyhedral cones in X. It is straight-forward verified that $M_0(X)$ is a sublattice in $M(X)$ (and in $\mathfrak{M}(X)$). Besides, from representation (2.5) we see that $M_0(X)$ is the smallest sublattice in $M(X)$ (in $\mathfrak{M}(X)$) that contains all positively homogeneous closed halfspaces. Using now arguments based on the equivalence of lattice polynomials we prove the validity of the representation (2.6). This completes the proof of the Proposition 2.5. ■

It should be noted that all above assertions concerning the lattice $M(X)$ of polyhedral sets are true with obvious changing for the sublattice $M_0(X)$ of polyhedral cones.

4. POLYHEDRAL FUNCTIONS, THEIR ANALYTICAL REPRESENTATIONS AND EQUIVALENCE TO PIECEWISE AFFINE FUNCTIONS

As before, we consider a finite dimensional normed space X.

We shall say that a function $f\colon X\to\mathbb{R}\cup\{+\infty\}$ is *lower polyhedral* if its epigraph $\operatorname{epi} f:=\{(x,\alpha)\in X\times\mathbb{R}\mid f(x)\leqslant\alpha\}$ is a polyhedral set in $X\times\mathbb{R}$.

Similarly, we shall say that a function $f\colon X\to\mathbb{R}\cup\{-\infty\}$ is *upper polyhedral* if its hypograph $\operatorname{hyp} f:=\{(x,\alpha)\in X\times\mathbb{R}\mid f(x)\geqslant\alpha\}$ is a polyhedral set in $X\times\mathbb{R}$.

Evidently, a function f is lower polyhedral if and only if $-f$ is upper polyhedral and vice versa.

We shall say that a function $f: X \to \mathbb{R}$ is *polyhedral* if it is both lower polyhedral and upper polyhedral.

Since polyhedral sets are closed we have that a lower (respectively, upper) polyhedral function is lower (respectively, upper) semicontinuous on X and, consequently, a polyhedral function is continuous. It is easy verified that a lower (respectively, upper) polyhedral function $f: X \to \mathbb{R}$ is polyhedral if and only if it is upper (respectively, lower) semicontinuous on X.

Affine functions $a(x) := \langle x, a^* \rangle + \alpha$ $(a^* \in X^*, \alpha \in \mathbb{R})$ are evidently the simplest examples of polyhedral functions. Really, both the epigraph and the hypograph of any affine function are "nonvertical" closed halfspaces in $X \times \mathbb{R}$. The simplest example of a lower polyhedral function that is not polyhedral gives the indicator function

$$\delta(x|Q) := \begin{cases} 0, & x \in Q \\ +\infty, & x \notin Q \end{cases}$$

of an arbitrary polyhedral set $Q \subset X$, $Q \neq X$. Below we shall show that any (lower, upper) polyhedral function can be obtained from a finite family of affine and indicator functions by means of pointwise maximum or minimum operations and algebraic operations of addition or subtraction.

We begin with reformulating the following statement well-known in convex analysis (see, e.g., [22]).

Proposition 3.1: *A function $f: X \to \mathbb{R} \cup \{+\infty\}$ is lower polyhedral and convex if and only if there exist affine functions $a_i: X \to \mathbb{R}$, $i = 1, 2, \ldots, m$, and a convex polyhedral set C in X such that*

$$f(x) = \max_{1 \leqslant i \leqslant m} a_i(x) + \delta(x|C), \quad x \in X. \tag{3.1}$$

If, in addition, the function f is finite on the whole space X then, in fact, $C = X$ and, consequently, in (3.1) $\delta(x|C) \equiv 0$ *for all* $x \in X$.

Generalizing this proposition we prove the following

Proposition 3.2: *A function $f: X \to \mathbb{R} \cup \{+\infty\}$ is lower polyhedral if and only if there exist affine functions $a_{ij}: X \to \mathbb{R}$, $j = 1, 2, \ldots, m_i$, $i = 1, 2, \ldots, k$, and convex polyhedral subsets $C_1, \ldots, C_k$ of X such that*

$$f(x) = \min_{1 \leqslant i \leqslant k} \left[\max_{1 \leqslant j \leqslant m_i} a_{ij}(x) + \delta(x|C_i) \right], \quad x \in X. \tag{3.2}$$

Proof: Since the epigraph of a lower polyhedral function $f: X \to \mathbb{R} \cup \{+\infty\}$ is a polyhedral set then by Remark 2.1 there exists a finite subfamily $\{F_1, \ldots, F_k\}$ of convex components of epi f such that epi $f = \bigcup_{i=1}^{k} F_i$. We note that $F_i \subset \tilde{F}_i := \{(x, \alpha + t) \in X \times \mathbb{R} | (x, \alpha) \in F_i, t \geqslant 0\} \subset \operatorname{epi} f$. In as much as $\tilde{F}_i$ is convex and F_i is a maximal convex subset of epi f we have $F_i = \tilde{F}_i$ and we see that every subset F_i is the epigraph of some lower polyhedral convex function $f_i: X \to \mathbb{R} \cup \{+\infty\}$. It follows from the equality $\operatorname{epi} f = \bigcup_{i=1}^{k} \operatorname{epi} f_i$ that

$$f(x) = \min_{1 \leqslant i \leqslant k} f_i(x), \; x \in X.$$

Representing the functions f_i, $i = 1, \dots, k$, in the form (3.1) we get (3.2) for f. To prove the converse statement we have to argue in reverse way. ■

Proposition 3.3: *A function $f: X \to \mathbb{R} \cup \{-\infty\}$ is upper polyhedral if and only if there exist affine functions $a_{ij}: X \to \mathbb{R}$, $j = 1, 2, \dots, m_i$, $i = 1, 2, \dots, k$, and convex polyhedral subsets $C_1, \dots, C_k$ of X such that*

$$f(x) = \max_{1 \leqslant i \leqslant k} \left[\min_{1 \leqslant j \leqslant m_i} a_{ij}(x) - \delta(x \mid C_i) \right], \quad x \in X. \tag{3.3}$$

We omit the easy proof.

Now we shall show that in the case when f is a polyhedral function we can put $C_1 = C_2 = \cdots = C_k = X$ or equivalently, $\delta(x \mid C_i) \equiv 0$ for all $i = 1, 2, \dots, k$, in (3.2) and (3.3). To show this we prove first the following important property of polyhedral functions.

Proposition 3.4: *Any polyhedral function $f: X \to \mathbb{R}$ is Lipschitzian on the whole space X, i.e., there exists a constant $L > 0$ such that*

$$|f(x) - f(y)| \leqslant L \|x - y\| \quad \textit{for all} \quad x, y \in X.$$

Proof: Represent the function f in the form(3.2) and consider the function $\tilde{f}_i(x) := \max_{1 \leqslant j \leqslant m_i} a_{ij}(x)$ where the functions a_{ij} are the same as in (3.2). It follows from Proposition 3.1 that every function $\tilde{f}_i$ is convex and lower polyhedral. Besides, it is not hard to see, that $\tilde{f}_i$ is Lipschitzian on the whole space X with the Lipschitz constant $L_i = \max_{1 \leqslant j \leqslant m_i} \|a_{ij}\|$. Now we shall prove that the function f is also Lipschitzian on the whole space X with a Lipschitz constant $L = \max_{1 \leqslant j \leqslant k} L_i$. To this end we introduce the sets $\tilde{Q}_i = \{x \in X \mid f(x) = \tilde{f}_i(x)\}$, $i = 1, \dots, k$, and note that $\tilde{Q} = pr_X(\operatorname{epi} \tilde{f}_i \cap \operatorname{hyp} f)$, $i = 1, 2, \dots, k$, where $pr_X: X \times \mathbb{R} \to X$ is the operator of projection on X. Since the function $\tilde{f}_i$ is lower polyhedral and the function f is polyhedral, the sets $\operatorname{epi} \tilde{f}_i$ and $\operatorname{hyp} f$ are polyhedral and, hence, the intersection $\operatorname{epi} \tilde{f}_i \cap \operatorname{hyp} f$ is also a polyhedral set. Consequently, every set $\tilde{Q}_i, i = 1, 2, \dots, k$, is the projection on X of the polyhedral set $\operatorname{epi} \tilde{f}_i \cap \operatorname{hyp} f$ and therefore $\tilde{Q}_i$ is polyhedral.

Now we represent every set $\tilde{Q}_i$ as the union of finitely many convex polyhedral sets Q_{is}, $S = 1, 2, \dots, r_i$. Since $|f(x)| < +\infty$ for all $x \in X$, we have $X = \bigcup_{i=1}^{k} \tilde{Q}_i$ and, hence, $X = \bigcup_{i=1}^{k} \bigcup_{s=1}^{r(i)} Q_{is}$. Thus the family $\sigma := \{Q_{is}, s = 1, 2, \dots, r_i, i = 1, 2, \dots, k\}$ consisting of convex polyhedral sets is a finite covering of the space X. It follows from the finitness of σ and the convexity of its subsets that for any points $x, y \in X$ one can choose finitely many points $x_\nu = x + t_\nu(y - x)$, $\nu = 1, 2, \dots, p$ $(0 = t_0 < t_1 < t_2 < \cdots < t_p = 1)$, on the interval $[x, y]$ in such a way that every subinterval $[x_{\nu-1}, x_\nu]$, $\nu = 1, 2, \dots, p$, will wholly belong to some (possibly not only) subset Q_{is}. Since the function f is Lipschitzian on every subset Q_{is} with the Lipschitz constant L, we have

$$|f(x)-f(y)| = \left|\sum_{\nu=1}^{p}(f(x_{\nu-1})-f(x_\nu))\right|$$

$$\leqslant \sum_{\nu=1}^{p}|f(x_{\nu-1})-f(x_\nu)| \leqslant \sum_{\nu=1}^{p} L\|x_{\nu-1}-x_\nu\|$$

$$= L\|x-y\|\sum_{\nu=1}^{p}(t_\nu - t_{\nu-1}) = L\|x-y\|.$$

This does prove that the function f is Lipschitzian on the whole space X. ■

Now we are in the position to prove our main

Theorem 3.1: *For any function $f\colon X \to \mathbb{R}$ the following statements are equivalent:*
(i) *f is piecewise affine;*
(ii) *f is polyhedral;*
(iii) *f can be represented in the form*

$$f(x) = \min_{1\leqslant i\leqslant k}\ \max_{1\leqslant j\leqslant m_i} a_{ij}(x), \quad x\in X, \tag{3.4}$$

where $a_{ij}\colon X\to\mathbb{R}$, $j=1,2,\ldots,m_i$, $i=1,2,\ldots,k$, are affine functions;
(iv) *f can be represented in the form*

$$f(x) = \max_{1\leqslant i\leqslant k}\ \min_{1\leqslant j\leqslant m_i} b_{ij}(x), \quad x\in X, \tag{3.5}$$

where $b_{ij}\colon X\to\mathbb{R}$, $j=1,2,\ldots,m_i$, $i=1,2,\ldots,k$, are affine functions;
(v) *f can be represented as a difference of two convex polyhedral functions:*

$$\mathrm{f}(x) = \max_{1\leqslant i\leqslant k} a_i(x) - \max_{1\leqslant j\leqslant m} b_j(x), \quad x\in X, \tag{3.6}$$

where $a_i\colon X\to\mathbb{R}$, $i=1,2,\ldots,k$, and $b_j\colon X\to\mathbb{R}$, $j=1,2,\ldots,m$, are affine functions.

Proof: We shall prove the following chain of implications:

(i) ⇒ (ii) ⇒ (iii) ⇒ (v) ⇒ (iv) ⇒ (i)

(i) ⇒ (ii) Let $f\colon X\to\mathbb{R}$ be a piecewise affine function on X. We associate with f both a polyhedral partition $\sigma_f := \{Q_1,\ldots,Q_r\}$ of the space X and a family $\{a_1,\ldots,a_r\}$ of affine functions such that $f(x)=a_i(x)$ for all $x\in Q_i$, $i=1,2,\ldots,r$. It is not difficult to verify that $\operatorname{epi} f = \bigcup_{i=1}^{r}((Q_i\times\mathbb{R})\cap\operatorname{epi} a_i)$ and $\operatorname{hyp} f = \bigcup_{i=1}^{r}((Q_i\times\mathbb{R})\cap\operatorname{hyp} a_i)$. From these equalities we can see that the epigraph and the hypograph of the function f are polyhedral sets in $X\times\mathbb{R}$. Thus the function f is polyhedral on X.

(ii) ⇒ (iii) Argueing as in the proof of Proposition 3.2 we can represent the function f in the form

$$f(x) = \min_{1\leqslant i\leqslant k} f_i(x), \quad x\in X, \tag{3.7}$$

where $f_i\colon X\to\mathbb{R}\cup\{+\infty\}$, $i=1,\ldots,k$, are lower polyhedral convex functions, with the epigraphs $\operatorname{epi} f_i$, $i=1,2,\ldots,k$, being convex components of the epigraph of f. Since the function f is polyhedral, it follows from Proposition 3.4 that f is Lipschitzian on X with some Lipschitz constant $L > 0$. This implies that the recession cone of any convex component of epi f contains the epigraph of the function $\omega_L: x \to L\|x\|$. Hence, the epigraph of any function f_i, $i = 1,2,\dots,k$, satisfies the inclusion $\operatorname{epi} f_i + \operatorname{epi} \omega_L \subset \operatorname{epi} f_i$, $i = 1,2,\dots,k$. This inclusion implies that $|f_i(x) - f_i(y)| \leqslant L\|x - y\|$ for all $x, y \in X$ and, consequently, every function f_i, $i = 1,2,\dots,k$, is finite on X. By the second part of Proposition 3.1 we conclude that every f_i can be represented in the form

$$f_i(x) = \max_{1 \leqslant j \leqslant m_i} a_{ij}(x), \tag{3.8}$$

where $a_{ij}: X \to \mathbb{R}$ ($j = 1,2,\dots,m_i$, $i = 1,2,\dots,k$, are affine functions. Now it follows from (3.7) and (3.8) that f can be represented in the form (3.4).

(iii)$\to$(v) Let

$$f(x) = \min_{1 \leqslant i \leqslant k} \max_{1 \leqslant j \leqslant m_i} a_{ij}(x).$$

Letting $f_i(x) = \max_{1 \leqslant j \leqslant m_i} a_{ij}(x), i = 1,2,\dots,k$, we can represent the function f in the following form:

$$f(x) = \min_{1 \leqslant i \leqslant k} f_i(x) = -\max_{1 \leqslant i \leqslant k}\left(\sum_{s=1,\, s \neq i}^{k} f_s(x) - \sum_{s=1}^{k} f_s(x)\right)$$

$$= \sum_{s=1}^{k} f_s(x) - \max_{1 \leqslant i \leqslant k} \sum_{s=1,\, s \neq i}^{k} f_s(x)$$

Since the functions f_s, $s = 1,\dots,k$, are convex and polyhedral then the functions $g: x \to \sum_{s=1}^{k} f_s(x)$ and $h: x \to \max_{1 \leqslant i \leqslant k} \sum_{s=1, s \neq i}^{k} f_s(x)$ are also convex and polyhedral. By Proposition 3.1 there exist affine functions $a_i: X \to \mathbb{R}, i = 1,2,\dots,k$, $b_j: X \to \mathbb{R}$, $j = 1,2,\dots,m$, such that $g(x) = \max_{1 \leqslant i \leqslant k} a_i(x), h(x) = \max_{1 \leqslant j \leqslant m} b_j(x)$, $x \in X$. Thus the function f can be represented as (3.6).

(v)$\Rightarrow$(iv) Let f be defined by (3.6). Then for the family of affine functions $a_{ij}(x) := a_i(x) - b_j(x), i = 1,\dots,k, j = 1,2,\dots,m$, one has (3.5).

(iv)$\to$(i) By Proposition 1.1 the collection $PA(X)$ of all piecewise affine functions is a vector lattice containing affine functions. It implies that any function f taking the form (3.5) is piecewise affine.

Thus we have proved the equivalence of all statements of Theorem 3.1. ■

Now we shall formulate a theorem characterizing piecewise linear functions. The proof is essentially the same as the preceding one and, therefore, it is omitted.

Theorem 3.2: *For any function $f: X \to \mathbb{R}$ the following statements are equivalent:*

(i) *f is piecewise linear;*

(ii) *f is positively homogeneous (i.e., $f(\lambda x) = \lambda f(x)$ for all $x \in X$ u $\lambda > 0$) and polyhedral;*

(iii) *f can be written in the form*

$$f(x) = \min_{1 \leqslant i \leqslant k} \max_{1 \leqslant j \leqslant m_i} \langle x, a_{ij}^* \rangle, \quad x \in X,$$

where $a_{ij}^* \in X^*, j = 1,2,\dots,m_i, i = 1,2,\dots,k$;

(iv) *f can be written in the form*

$$f(x) = \max_{1 \leqslant i \leqslant k} \min_{1 \leqslant j \leqslant m_i} \langle x, b_{ij}^* \rangle, \quad x \in X,$$

where $b_{ij}^* \in X^*$, $j = 1, 2, \ldots, m_i$, $i = 1, 2, \ldots, k$;
(v) *f can be written as a difference of two sublinear polyhedral functions:*

$$f(x) = \max_{1 \leqslant i \leqslant k} \langle x, a_i^* \rangle - \max_{1 \leqslant j \leqslant m} \langle x, b_j^* \rangle, \quad x \in X,$$

where $a_i^* \in X^*$, $i = 1, 2, \ldots, k$; $b_j^* \in X^*$, $j = 1, 2, \ldots, m$.

Remark 3.2: The equivalence of the statements (i) and (v) of Theorem 3.2 has been first proved by Meltzer D. [15]. The proof given here is quite different from his one.

In [13] Kripfganz A. and Schulze R. have proposed two techniques for representing a piecewise affine function as a difference of two convex piecewise affine functions.

5. POLYHEDRAL AND PIECEWISE AFFINE APPROXIMATIONS

In this section we prove theorems characterizing approximate properties of polyhedral sets and piecewise affine functions.

Theorem 4.1: *For any compact set $C \subset X$ and any real $\varepsilon > 0$ there exist compact polyhedral sets $Q_1, Q_2 \in M(X)$, such that*

(i) $Q_1 \subset C \subset Q_2$;
(ii) $h(Q_1, C) \leqslant \varepsilon, h(Q_2, C) \leqslant \varepsilon$,

where $h(Q_i, C) := \max\{\sup_{y \in C} \inf_{x \in Q_i} \| y - x \|, \sup_{x \in Q_i} \inf_{y \in C} \| y - x \|\}$ *is the Hausdorff distance between the sets Q_i and C.*

Proof: Since C is a compact set, for any $\varepsilon > 0$ there exists a finite ε-net, i.e., a finite subset $Q_1 \subset C$ such that $d(x, Q_1) := \inf_{y \in Q_i} \| x - y \| < \varepsilon$ for all $x \in C$. It is not hard to see that Q_1 is a compact polyhedral set and $h(Q_1, C) \leqslant \varepsilon$.

For constructing the set Q_2 we consider the family $\omega(C)$ consisting of all convex components of the set C and form the family $\omega_{\varepsilon/2}(C) := \{B^0(S, \varepsilon/2) | S \in \omega(C)\}$ with $B^0(S, \varepsilon/2) = \{x \in X | \exists\, y \in S : d(x, y) < \varepsilon/2\}$. It is evidently that $\omega_{\varepsilon/2}(C)$ is an open covering of C. Since C is compact we can find a finite subcovering $\{B^0(S_1, \varepsilon/2), \ldots, B^0(S_m, \varepsilon/2)\}$ of the covering $\omega_{\varepsilon/2}(C)$. Thus we have $C \subset \bigcup_{i=1}^m B^0(S_i, \varepsilon/2) \subset \bigcup_{i=1}^m B(S_i, \varepsilon/2)$, where $B(S_i, \varepsilon/2) = cl\, B^0(S_i, \varepsilon/2) = \{x \in X | \exists\, y \in S : d(x, y) \leqslant \varepsilon/2\}$. Every $B(S_i, \varepsilon/2)$ is a convex compact set and therefore there exists (see, e.g., [14, 22]) a convex polyhedral set $Q_{2i} \subset X$ such that $B(S_i, \varepsilon/2) \subset Q_{2i}$ and $h(Q_{2i}, B(S_i, \varepsilon/2)) < \varepsilon/2$. Thus we have $C \subset \bigcup_{i=1}^m B(S_i, \varepsilon/2) \subset \bigcup_{i=1}^m Q_{2i} =: Q_2$. Since Q_2 is the union of finitely many convex polyhedral sets, it is compact and by the triangle inequality $h(Q_2, C) \leqslant h(C, \bigcup_{i=1}^m B(S_i, \varepsilon/2)) + h(\bigcup_{i=1}^m B(S_i, \varepsilon/2), Q_2) \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon$.

The proof is complete. ■

Theorem 4.2: *Let ε be an arbitrary positive real. For any continuous function $f: X \to \mathbb{R}$ and any compact set $Q \subset X$ there exists a piecewise affine function $w: X \to \mathbb{R}$ such that*

$$\max_{x \in Q} |f(x) - w(x)| < \varepsilon.$$

Proof: Let $C(Q)$ be the Banach lattice of continuous (on Q) functions $f: Q \to \mathbb{R}$ with the norm $\|f\| = \max_{x \in Q} |f(x)|$. It is not hard to verify that the collection $PA(Q)$ of restrictions of piecewise affine (on X) functions to Q is a normed sublattice of $C(Q)$, which contains constants and disjoint points on Q (since $PA(Q)$ contains restrictions of linear functionals). Consequently, by the Stone – Weierstrass theorem (see, e.g., [4]) the normed sublattice $PA(Q)$ is dense in $C(Q)$. This completes the proof. ■

Theorem 4.3: *Let ε be an arbitrary positive real. For any positively homogeneous* continuous *function $p: X \to \mathbb{R}$* there exists a piecewise linear function $v: X \to \mathbb{R}$ *such that*

$$|p(x) - v(x)| < \varepsilon \|x\| \quad \textit{for all} \quad x \in X.$$

Proof: Since any two norms defined on a finite dimensional space are equivalent, the assertion of Theorem 4.3 does not depend on the norm given on X. Therefore without loss generality we can consider that the norm given on X is piecewise linear, for instance, $\|x\| = \max_{1 \leqslant i \leqslant k} |\langle x, e_i^* \rangle|$, where $e_i^*, \ldots, e_n^*$ is a vector basis in X^*. We note that the restriction of the norm $x \to \|x\|$ to the unit sphere $S = \{x \in X \mid \|x\| = 1\}$ is an identical unit. Arguing further as in the proof of the preceding theorem we get by the Stone–Weierstrass theorem that the collection of restrictions of piecewise-linear functions to S is dense in $C(S)$. The proof is complete. ■

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