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Optimality Conditions of First and Second Order in Vector Optimization Problems on Metric Spaces

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Abstract—For mappings defined on metric spaces with values in Banach spaces, the notions of derivative vectors of first and second order are introduced. These notions are used to establish necessary conditions and sufficient conditions of first and second order for points of local $\prec$-minimum of such mappings, where $\prec$ is a strict preorder relation defined on the space of values of the mapping that is minimized. Minimality conditions are obtained as corollaries for the case when the mapping is defined on a subset of a normed space.
Keywords: vector optimization, metric spaces, conical local approximations of sets, derivatives of mappings.

1. INTRODUCTION

In this paper, we study necessary conditions and sufficient conditions for the optimality in a vector optimization problem on a metric space; this problem is formulated below.

Let Q be a metric space with the distance function $d_Q(\cdot,\cdot) : Q \times Q \to \mathbb{R}$, and let $F : Q \to Y$ be a mapping from Q to a Banach space Y ordered by a strict preorder relation $\prec$ such that

$$y_1 \prec y_2 \iff y_2 - y_1 \in P,$$

where $P \subset Y$ is an asymmetric convex cone with nonempty interior.

Recall [20, 25] that a strict preorder relation is a binary relation that is asymmetric and transitive. In the sequel, we will sometimes use the symbol $(Y, \prec)$ meaning that the Banach space Y is ordered by the relation $\prec$. The symbol $\not\prec$ denotes the negation of the relation $\prec$; i.e., $y_1 \not\prec y_2 \iff y_2 - y_1 \notin P$.

A point $x^0 \in Q$ is said to be a *point of local $\prec$-minimum* of the mapping $F : Q \to (Y, \prec)$ if, for some real number $\varepsilon > 0$, the relation $F(x) \not\prec F(x^0)$ holds for all $x \in B_Q(x^0, \varepsilon)$, where $B_Q(x^0, \varepsilon) := \{x \in Q \mid d_Q(x, x^0) \le \varepsilon\}$ is the ball (in the metric space Q) of radius ε centered at the point x^0.

In addition to the relation $\prec\!\!\prec$, we introduce the derivative relations $\prec\!\!\prec$ and $\preceq$ on the Banach space Y defining them as follows:

$$y_1 \prec\!\!\prec y_2 \iff y_2 - y_1 \in \mathrm{int} P$$

and

$$y_1 \preceq y_2 \iff y_2 - y_1 \in \mathrm{cl} P.$$

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Here, int$P$ and cl$P$ are the interior and the closure of the cone P, respectively.

Note that the relation $\prec\!\!\prec$ is also a strict preorder relation on Y, while the relation $\preceq$ is reflexive and transitive and, hence, is a preorder relation.

A point $x^0 \in Q$ is said to be a *point of local weak $\prec$-minimum* of the mapping $F : Q \to (Y, \prec)$ if there exists a real number $\varepsilon > 0$ such that the relation $F(x) \not\prec\!\!\prec F(x^0)$ holds for all $x \in B_Q(x^0, \varepsilon)$.

We say that a point $x^0 \in Q$ is a *point of local strong $\prec$-minimum* of the mapping $F : Q \to (Y, \prec)$ if there exists a real number $\varepsilon > 0$ such that $F(x) \not\preceq F(x^0)$ for all $x \in B_Q(x^0, \varepsilon)$, $F(x) \neq F(x^0)$.

In these definitions, the symbols $\not\prec\!\!\prec$ and $\not\preceq$ denote the negations of the relations $\prec\!\!\prec$ and $\preceq$, respectively.

Note that, in the case when $P = \mathrm{int}P$, the notions of local $\prec$-minimum and local weak $\prec$-minimum introduced above coincide. If $P = \mathrm{cl}\, P \setminus \{0\}$, then the notions of local $\prec$-minimum and local strong $\prec$-minimum coincide. In particular, if $Y = \mathbb{R}$ and P is the set of positive real numbers, then all three notions are reduced to the notion of local minimum of a real-valued function on a set. Thus, the notions of local weak $\prec$-minimum and local strong $\prec$-minimum can be considered as some kind of a regularization of the notion of local $\prec$-minimum; any point of local strong $\prec$-minimum is a point of local $\prec$-minimum and any point of local $\prec$-minimum is a point of local strong $\prec$-minimum. The converse implications do not hold in the general case.

Problems of minimizing (or maximizing) mappings taking values in ordered vector spaces are usually called vector optimization problems.

The main goal of this paper is to obtain necessary conditions and sufficient conditions of first and second order for points of local (weak, strong) $\prec$-minimum in the vector optimization problem formulated above. According to the method originally suggested in [2], we first "scalarize" the original vector optimization problem (see Theorem 1), replacing the set-theoretic definitions of points of weak and strong $\prec$-minimum by equivalent scalar functional inequalities, from which, using variational analysis, we obtain local minimum conditions.

The scalarization of the vector optimization problem is based on the fact that, for any convex cone $P \subset Y$ with nonempty interior, one can specify a bounded sublinear function $\sigma : Y \to \mathbb{R}$ satisfying the equalities

$$\mathrm{int}P = \{y \in Y \mid \sigma(-y) < 0\} \quad \text{and} \quad \mathrm{cl}P = \{y \in Y \mid \sigma(-y) \le 0\}. \tag{1}$$

A sublinear function $\sigma : Y \to \mathbb{R}$ is called bounded if $|\sigma(y)| \le L\|y\|$ for any $y \in Y$, where L is a fixed positive real number.

As a function $\sigma : Y \to \mathbb{R}$, we can take, for instance, the symmetrized distance to the cone $-P$ [2]

$$\sigma : y \to \inf_{z \in -P} \|y - z\| - \inf_{z \in Y \setminus (-P)} \|y - z\|$$

or the function [11, 18]

$$\sigma : y \to \inf\{t \in \mathbb{R} \mid y \in ty^0 - P\},$$

where y^0 is an arbitrary fixed vector from int$P$.

Let Y^* be the space topologically conjugate to the space Y, and let $P^+ := \{y^* \in Y^* \mid y^*(y) \ge 0 \ \forall\, y \in P\}$ be the cone conjugate to the cone P.

Any bounded sublinear function $\sigma : Y \to \mathbb{R}$ satisfying equalities (1) can be presented as follows:

$$\sigma(y) = \max_{b^* \in B^+} b^*(y),$$

where B^+ is a w^*-weakly compact base of the cone P^+ (the existence of such a base B^+ follows from the condition $\mathrm{int}P \neq \varnothing$ [20]).

Conversely, any w^*-weakly compact base B^+ of the cone P^+ coincides with the subdifferential $\partial\sigma := \{y^* \in Y^* \,|\, y^*(y) \leq \sigma(y)\ \forall\, y \in Y\}$ of some bounded sublinear function σ satisfying equalities (1).

Recall [20] that a convex subset D of a convex cone K from a normed space Z is called a *base* of K if $0 \notin \mathrm{cl}D$ and, for any $z \in K$, $z \neq 0$, there exists a uniquely defined vector $d \in D$ and a real number $\lambda > 0$ such that $z = \lambda d$.

In what follows, without special mention, we will denote by $\sigma : Y \to \mathbb{R}$ some bounded sublinear function satisfying equalities (1). Note here that all the conditions of $\prec$-minimality obtained below (see Theorems 1–6 and their corollaries) do not depend on the choice of the function $\sigma : Y \to \mathbb{R}$.

Theorem 1 (on scalarization)**.** (a) *A point $x^0 \in Q$ is a point of local weak $\prec$-minimum of the mapping $F : Q \to Y$ if and only if there exists a positive real number $\varepsilon > 0$ such that*

$$\sigma(F(x) - F(x^0)) \geq 0 \quad \textit{for any} \quad x \in B_Q(x^0, \varepsilon). \tag{2}$$

(b) *A point $x^0 \in Q$ is a point of local strong $\prec$-minimum of the mapping $F : Q \to Y$ if and only if there exists a positive real number $\varepsilon > 0$ such that*

$$\sigma(F(x) - F(x^0)) > 0 \quad \textit{for any} \quad x \in B_Q(x^0, \varepsilon), \quad F(x) \neq F(x^0). \tag{3}$$

It is clear that condition (2) is a necessary condition and condition (3) is a sufficient condition for the points of local $\prec$-minimum of the mapping $F : Q \to Y$.

However, these conditions are not very far from the definitions. Hence, they do not simplify substantially the verification of whether a point is a point of local weak or strong minimum. As noted above, the genuine meaning of conditions (2) and (3) is that they reformulate the original set-theoretic definitions in terms of functional inequalities, which are easier to analyze by variational methods.

2. LOCAL APPROXIMATIONS OF SETS AND MAPPINGS IN NORMED SPACES

The minimized mapping in the vector optimization problem formulated above is defined on a metric space; hence, local minimum conditions cannot be derived by means of the traditional methods of classical and modern variational analysis [1–6, 9, 12–17, 19, 21–23, 26], which are appropriate for studying mappings defined on normed or topological vector spaces. This is why we introduce in this paper the new notions of derivative vectors of first and second order for mappings defined on metric spaces. These notions are in turn based on the notions of conical local approximations of sets of first and second order. Let us recall briefly (see [9, 12, 26] for more details) the definitions of conical local approximations of sets from normed spaces, the definitions of directional derivatives of mappings acting between normed spaces, and basic properties of these notions.

Let Z be a real normed space, and let M be a subset of Z.

Denote by $S(0)$ the family of sequences of positive real numbers (t_n), $t_n > 0$, $n = 1, 2, \ldots$, converging to 0. Denote by U_M the family of all sequences (z_n) such that $z_n \in M$, $n = 1, 2, \ldots$, and let $U_M(z)$ be the subfamily of U_M consisting of sequences (z_n) converging (in sense of the norm defined on Z) to z.

A vector $z \in Z$ is said to be a *first-order tangent vector* to the set M at a point $z^0 \in \mathrm{cl}M$ if there exist subsequences $(t_n) \in S(0)$ and $(z_n) \in U_Z(z)$ such that $z^0 + t_n z_n \in M$ for all $n = 1, 2, \ldots$.

The collection of all vectors from Z that are (first-order) tangent to the set M at a point $z^0 \in \mathrm{cl}M$ forms a closed cone, which will be denoted by $T(z^0 \,|\, M)$.

The cone $T(z^0 \,|\, M)$ is known by various names in the literature: the *cone of directions consistent with constraints* [7], the *cone of directions admissible in the broad sense* [8], the *cone of possible directions* [9], and the *contingent cone* [12,16,23]. According to [1,26], we will call the cone $T(z^0 \,|\, M)$ the *first-order tangent cone* to the set M at the point $z^0 \in \mathrm{cl}M$.

The most natural way to extend the ideas underlying the notion of first-order tangent vectors to the case of second-order tangent vectors is presented in the following definition.

Definition 1 (see, for instance, [1, 3, 6, 12, 15, 16, 26]). A vector $w \in Z$ is called a *second-order true tangent vector* to the set $M \subset Z$ at a point $z^0 \in \mathrm{cl}M$ in a direction $z \in Z$ (or, simply, at $(z^0,\, z) \in \mathrm{cl}M \times Z$) if there exist sequences $(t_n) \in S(0)$ and $(w_n) \in U_Z(w)$ such that $z^0 + t_n z + t_n^2 w_n \in M$ for all $n = 1, 2, \ldots$.

The set of all second-order true tangent vectors to the set $M \subset Z$ at a point $z^0 \in \mathrm{cl}M$ in a direction $z \in Z$ will be denoted by $M^2(z^0, z)$.

It is known [3, 6, 12, 16] that $M^2(z^0,\, z)$ is a closed (possibly empty) subset of Z satisfying the following properties.

(*i*) $M^2(z^0,\, 0) = T(z^0 \,|\, M)$.

(*ii*) If $M^2(z^0,\, z) \neq \varnothing$, then $z \in T(z^0 \,|\, M)$.

(*iii*) $M^2(z^0,\, \alpha z) = \alpha^2 M^2(z^0,\, z)$ for all $\alpha > 0$.

(*iv*) If $w \in M^2(z^0,\, z)$, then $w + \beta z \in M^2(z^0,\, z)$ for all $\beta \in \mathbb{R}$.

Example 1. Assume that $M = z^0 + K$, where K is a closed cone from Z. Then, $T(z^0 \,|\, M) = K$ and $M^2(z^0, z) = T(z \,|\, K)$ for all $z \in K$.

Example 2. Assume that $Z = \mathbb{R}^2$ and $M := \{(z_1,\, z_2) \in \mathbb{R}^2 \,|\, z_2 = z_1^2\,\}$. For the point $z^0 = (0,\, 0)$ contained in this set, $T(0 \,|\, M) = \{(z_1,\, z_2) \,|\, z_2 = 0\,\}$ and $M^2(0,\, z) = \{(w_1,\, w_2) \in \mathbb{R}^2 \,|\, w_2 = z_1^2\,\}$ for all $z \in T(0 \,|\, M)$, $z \neq 0$.

Example 3. Assume that $Z = \mathbb{R}^2$, $M := \{(z_1,\, z_2) \in \mathbb{R}^2 \,|\, z_1 \geq 0,\, z_1^2 = z_2^3\,\}$, and $z^0 = (0,\, 0)$. It can be obtained directly from the definitions of tangent vectors of first and second order that $T(0 \,|\, M) = \{(z_1,\, z_2) \in \mathbb{R}^2 \,|\, z_1 = 0,\, z_2 \geq 0\,\}$ and $M^2(0,\, z) = \varnothing$ for any $z \in T(0 \,|\, M)$, $z \neq 0$.

Example 3 shows that the cone of second-order true tangent vectors may be empty for some sets even in finite-dimensional spaces and, hence, provide no additional information on the local structure of such sets in a neighborhood of the studied point. Moreover, even in the cases when the cone of second-order true tangent vectors is nonempty, it may contain no information on some sequences converging to the studied point and, hence, it may characterize the local structure of the set incompletely. This circumstance stimulates one to introduce, as a local second-order approximation of sets, a somewhat wider set of vectors than the cone of second-order true tangent vectors.

Definition 2 [6,17]. A vector $w \in X$ is called a *second-order tangent vector* to the set $M \subset Z$ at a point $z^0 \in \mathrm{cl}M$ in a direction $z \in Z$ if there exist sequences $(t_n),\, (\tau_n) \in S(0)$ and $(w_n) \in U_X(w)$ such that $z^0 + t_n z + t_n \tau_n w_n \in M$ for all $n = 1, 2, \ldots$.

The set of all second-order tangent vectors to the set $M \subset Z$ at a point $z^0 \in \mathrm{cl}M$ in a direction $z \in Z$ is a closed cone, which will be denoted by $T^2(z^0,\, z \,|\, M)$ and called the *second-order tangent cone* to the set M at the point $z^0 \in \mathrm{cl}M$ in the direction z.

Note that the notion of second-order projective tangent cone introduced in [24] is close to this definition.

It is easy to see that $T^2(z^0, 0\,|\,M) = T(z^0\,|\,M)$ and $M^2(z^0, z) \subset T^2(z^0, z\,|\,M)$ $\forall\, z^0 \in \mathrm{cl}M$, $z \in Z$. In addition,

$$T^2(z^0, z\,|\,M) \neq \varnothing \Longrightarrow z \in T(z^0\,|\,M).$$

In finite-dimensional spaces, the converse implication is also true; i.e., if $\dim Z < \infty$, then $T^2(z^0, z\,|\,M) \neq \varnothing$ for any $z^0 \in \mathrm{cl}M$ and any $z \in T(z^0\,|\,M)$.

It is also verified directly that, if $w \in T^2(z^0, z\,|\,M)$, then $w + \mu z \in T^2(z^0, z\,|\,M)$ for all $\mu \in \mathbb{R}$.

It was shown in Example 3 that the set of true tangent vectors $M^2(0, z)$ for the set $M := \{(z_1, z_2) \in \mathbb{R}^2\,|\, z_1 \geq 0,\ z_1^2 = z_2^3\}$, the point $z^0 = (0, 0) \in M$, and any nonzero vector $z \in T(0\,|\,M)$ is empty, whereas $T^2(0, z\,|\,M) = \{(w_1, w_2) \in \mathbb{R}^2\,|\, w_1 \geq 0\}$.

Let us now consider the notions of directional derivatives of mappings.

A mapping $F : X \to Y$ from a normed space X to another normed space Y is called [9] *differentiable at a point* $x^0 \in X$ *in a direction* $h \in X$ if there exists the limit

$$F'(x^0\,|\,h) := \lim_{t\to 0,\, t>0} \frac{F(x^0 + th) - F(x^0)}{t}.$$

If $F'(x^0\,|\,h)$ exists for any $h \in X$, then the mapping $F'(x^0\,|\cdot) : h \to F'(x^0\,|h)$ is called the *directional derivative of the mapping* F *at the point* x^0.

The mapping $F : X \to Y$ is said to be *directionally H-differentiable at a point* $x^0 \in X$ if $F'(x^0\,|\,h)$ exists for all $h \in X$ and the following relation holds:

$$F'(x^0\,|\,h) = \lim_{\substack{z\to h,\\ t\to 0,\, t>0}} \frac{F(x^0 + tz) - F(x^0)}{t}.$$

If the mapping $F : X \to Y$ is directionally H-differentiable at a point $x^0 \in X$, then the directional derivative $F'(x^0\,|\cdot) : X \to Y$ is continuous on the whole space X.

According to [13, 14] (see also [2, 16]), the mapping $F : X \to Y$ is called *twice* (*parabolically*) *differentiable at a point* x^0 *in a direction* $h \in X$ *with divergence* $w \in X$ if F is differentiable at the point x^0 in the direction h and, moreover, there exists the limit

$$F''(x^0\,|\,h, w) := \lim_{t\to 0,\, t>0} \frac{F(x^0 + th + 1/2\, t^2 w) - F(x^0) - tF'(x^0\,|\,h)}{1/2\, t^2}.$$

If the limit $F''(x^0\,|\,h, w)$ exists and, moreover,

$$F''(x^0\,|\,h, w) = \lim_{\substack{z\to w,\\ t\to 0,\, t>0}} \frac{F(x^0 + th + \frac{1}{2}t^2 z) - F(x^0) - tF'(x^0\,|\,h)}{1/2\, t^2}, \tag{4}$$

then the mapping F is said to be *twice* (*parabolically*) *H-differentiable at the point* x^0 in the direction h with divergence w.

In the case when $F''(x^0\,|\,h, w)$ exists for all $h \in X$ and $w \in X$, the mapping $F''(x^0\,|\cdot, \cdot) : (h, w) \to F''(x^0\,|\,h, w)$ is called the *second parabolic derivative* (or, if (4) holds, the *second parabolic H-derivative*) of the mapping F at the point x^0.

It is easy to check that the restriction of the second parabolic derivative $F''(x^0\,|\cdot, \cdot) : X \times X \to Y$ to $\{0\} \times X$ coincides with the first directional derivative; i.e., the equality $F''(x^0\,|0, w) = F'(x^0\,|w)$ holds for all $w \in X$.

The restriction of the second parabolic derivative $F''(x^0 \,|\, \cdot, \cdot) : X \times X \to Y$ to $X \times \{0\}$ is called the *second* (*usual*) *directional derivative* of the mapping F at the point x^0 in the direction $h \in X$ and is denoted by $F''(x^0 \,|\, \cdot) : X \to Y$. (Note that the second (usual) directional derivative was introduced (see, for instance, [8]) long before the introduction of the second parabolic derivative.) Evidently, in the general case, the existence of $F''(x^0 \,|\, h)$ does not imply the existence of $F''(x^0 \,|\, h, w))$ for $w \neq 0$. At the same time, if the mapping $F : X \to Y$ is strictly differentiable [10] at the point x^0 and, in addition, $F''(x^0 \,|\, h)$ exists, then $F''(x^0 \,|\, h, w)$ also exists for all $w \in X$ and, moreover,

$$F''(x^0 \,|\, h, w) = F'(x^0)h + F''(x^0 \,|\, h),$$

where $F'(x^0)$ is the strict derivative of the mapping F at the point x^0.

It is also easy to check that the second parabolic derivative $F''(x^0 \,|\, \cdot, \cdot) : X \times X \to Y$ satisfies the following condition of positive homogeneity:

$$F''(x^0 \,|\, \tau h, \tau^2 w) = \tau^2 F''(x^0 \,|\, h, w) \quad \text{for all} \quad h, w \in X \quad \text{and all} \quad \tau > 0.$$

3. FIRST-ORDER CONDITIONS FOR POINTS OF LOCAL $\prec$-MINIMUM OF A VECTOR MAPPING

As above, we denote by $S(0)$ the family of sequences of positive real numbers (t_n), $t_n > 0$, $n = 1, 2, \ldots$, converging to zero. We denote by U_Q the family of all sequences (x_n) such that $x_n \in Q$, $n = 1, 2, \ldots$; $U_Q(x)$ is the subfamily from U_Q consisting of all sequences (x_n) that converge (in the sense of the metric defined on Q) to x.

Definition 3. A vector $y \in Y$ is called a (first-order) *derivative vector* of a mapping $F : Q \to Y$ at a point $x^0 \in Q$ if there exist subsequences $(x_n) \in U_Q(x^0)$ and $(t_n) \in S(0)$ such that $y = \lim\limits_{n\to\infty} t_n^{-1}(F(x_n) - F(x^0))$.

The set of all derivative vectors of a mapping $F : Q \to Y$ at a point $x^0 \in Q$ will be denoted by $DF(x^0)$.

It is easy to verify that $DF(x^0)$ is a cone. In addition, since the equalities $y = \lim\limits_{n\to\infty} t_n^{-1}(F(x_n) - F(x^0))$ and $\lim\limits_{n\to\infty} t_n = 0$ imply $\lim\limits_{n\to\infty} F(x_n) = F(x^0)$, it follows that $DF(x^0) \subset T(F(x^0) \,|\, F(Q))$, where $T(F(x^0) \,|\, F(Q))$ is the tangent (contingent) cone (see Definition 1) to the set $F(Q)$ at the point $F(x^0)$.

Theorem 2 (a necessary condition of first-order minimality). *Suppose that $F : Q \to Y$ is a mapping defined on the metric space Q and taking values in the ordered Banach space $(Y, \prec)$. If a point $x^0 \in Q$ is a point of local $\prec$-minimum of the mapping F, then*

$$\sigma(y) \geq 0 \quad \textit{for all} \quad y \in DF(x^0). \tag{5}$$

Proof. Suppose that $y \in DF(x^0)$ and sequences $(x_n) \in U_Q(x^0)$ and $(t_n) \in S(0)$ are such that $y = \lim\limits_{n\to\infty} t_n^{-1}(F(x_n) - F(x^0))$. If the point $x^0 \in Q$ is a point of local weak $\prec$-minimum of the mapping F (on Q), then, by statement (a) of Theorem 1, $\sigma(F(x_n) - F(x^0)) \geq 0$ for sufficiently large numbers n. Multiplying this inequality by t_n^{-1} and taking n to infinity, we obtain, by the continuity of the function $\sigma(\cdot)$, the inequality $\sigma(y) \geq 0$. The theorem is proved.

Corollary 1 [2,3]**.** *Suppose that X is a normed space, Q is a subset of X, and $F : X \to (Y, \prec)$ is a mapping from X to the ordered Banach space $(Y, \prec)$. If F is directionally H-differentiable at* *a point x^0, then the point $x^0 \in Q$ is a point of local weak $\prec$-minimum of the mapping F on the set Q if and only if the inequality*

$$\sigma(F'(x^0 \,|\, h)) \geq 0 \quad \text{for all} \quad h \in T(x^0 \,|\, Q) \tag{6}$$

holds, where $T(x^0 \,|\, Q)$ is the tangent cone of first order to the set Q at the point x^0.

To validate inequality (6), we note that, if Q is a subset of the normed space X and the mapping F is defined in some open domain of the space X containing Q and is directionally H-differentiable at the point x^0, then $F'(x^0 \,|\, h) \in DF(x^0)$ for any $h \in T(x^0 \,|\, Q)$, where $F'(x^0 \,|\, \cdot) : X \to Y$ is the directional H-derivative of the mapping F at the point x^0 and $T(x^0 \,|\, Q)$ is the tangent (contingent) cone of first order to the set Q at the point x^0.

Theorem 3 (a sufficient condition of first-order minimality). *Suppose that $F : Q \to Y$ is a mapping defined on the metric space Q and taking values in the ordered Banach space $(Y, \prec)$. If the space Y is finite-dimensional and the mapping $F : Q \to Y$ is continuous at the point $x^0 \in Q$, then the condition*

$$\sigma(y) > 0 \quad \text{for all} \quad y \in DF(x^0), \quad y \neq 0, \tag{7}$$

is sufficient for the point $x^0 \in Q$ to be a point of local strong $\prec$-minimum of the mapping F on the metric space Q.

Proof is by contradiction. Assume that condition (7) holds but the point $x^0 \in Q$ is not a point of local strong $\prec$-minimum of the mapping $F : Q \to X$. Then, there exists a sequence $(x_n) \in U_Q(x^0)$, $F(x_n) \neq F(x^0)$, such that $\sigma(F(x_n) - F(x^0)) \leq 0$ for all n. Define $t_n := \|F(x_n) - F(x^0)\|$. Since $F(x_n) \neq F(x^0)$ and F is continuous at the point x^0, we have $t_n > 0$ and $t_n \to 0$. Without loss of generality, we can assume that the sequence $y_n := t_n^{-1}(F(x_n) - F(x^0))$ converges to some vector $y \in Y$, $\|y\| = 1$. It follows from the definition of $DF(x^0)$ that $y \in DF(x^0)$, $y \neq 0$. In addition, $\sigma(y) = \lim\limits_{n\to\infty} t_n^{-1}\sigma(F(x_n) - F(x^0)) \leq 0$. This contradicts equality (7) and, thus, the theorem is proved.

4. SECOND-ORDER CONDITIONS FOR POINTS OF LOCAL $\prec$-MINIMUM OF A VECTOR MAPPING

Definition 4. A vector $z \in Y$ is called a *second-order derivative vector* of a mapping $F : Q \to Y$ at a point $x^0 \in Q$ in a direction $y \in Y$ if there exist sequences $(x_n) \in U_Q(x^0)$ and $(t_n) \in S(0)$, $(\tau_n) \in S(0)$ such that $z = \lim\limits_{n\to\infty} \dfrac{F(x_n) - F(x^0) - t_n y}{1/2\, t_n \tau_n}$.

The set of all second-order derivative vectors of the mapping $F : Q \to Y$ at the point $x^0 \in Q$ in the direction $y \in Y$ will be denoted by $D^2F(x^0, y)$.

If $D^2F(x^0, y) \neq \varnothing$, then $y \in DF(x^0)$. It is also easy to see that $D^2F(x^0, y)$ is a cone in Y and $D^2F(x^0, y) \subset T^2(y^0, y \,|\, F(Q))$, where $T^2(y^0, y \,|\, F(Q))$ is the second-order tangent cone to the set $F(Q)$ at the point y^0 in the direction y [6].

Theorem 4 (a necessary condition of second-order minimality). *Suppose that $F : Q \to Y$ is a mapping defined on the metric space Q and F takes values in the ordered Banach space $(Y, \prec)$. If $x^0 \in Q$ is a point of local weak $\prec$-minimum of the mapping F, then, for any $y \in DF(x^0)$ such that $\sigma(y) = 0$, the following inequality holds:*

$$\max_{y^* \in \partial\sigma(y)} y^*(z) \geq 0 \quad z \in D^2F(x^0, y), \tag{8}$$

where $\partial\sigma(y) := \{y^* \in \partial\sigma \mid y^*(y) = 0\}$ is the subdifferential of the function σ at the point y.

Proof. Let $y \in DF(x^0)$ satisfy the condition $\sigma(y) = 0$, and let $z \in D^2F(x^0, y)$. Then, there exist sequences $(x_n) \in U_Q(x^0)$ and $(t_n) \in S(0)$, $(\tau_n) \in S(0)$ such that $z = \lim\limits_{n\to\infty} \dfrac{F(x_n) - F(x^0) - t_n y}{1/2\, t_n \tau_n}$. Since $x^0 \in Q$ is a point of local weak $\prec$-minimum of the mapping F on the set Q, we have $\sigma(F(x_n) - F(x^0)) \geq 0$ for sufficiently large n. Assume that $z_n := \dfrac{F(x_n) - F(x^0) - t_n y}{1/2\, t_n \tau_n}$. Then, $F(x_n) - F(x^0) = t_n y + 1/2\, t_n \tau_n z_n$. Hence, $\sigma(y + 1/2\, \tau_n z_n) \geq 0$ for sufficiently large n. Dividing this inequality by $1/2\, \tau_n$ and taking n to $+\infty$, we obtain $\sigma'(y \mid z) \geq 0$, where $\sigma'(y \mid \cdot) : Y \to \mathbb{R}$ is the directional derivative of the function σ at the point y. Using the inequality $\sigma'(y \mid z) = \max\limits_{y^* \in \partial\sigma(y)} y^*(z)$, we get inequality (8). Theorem is proved.

Corollary 2 [3]**.** *Suppose that X is a normed space, Q is a subset of X, and $F : X \to (Y, \prec)$ is a mapping from X to the ordered Banach space $(Y, \prec)$. Suppose also that the mapping $F : X \to Y$ is twice parabolically H-differentiable at the point $x^0 \in Q$. If $x^0 \in Q$ is a point of local weak minimum of the mapping F on the set Q, then the inequality*

$$\max_{y^* \in \partial\sigma(F'(x^0 \mid h))} y^*(F''(x^0 \mid h,\, w)) \geq 0 \quad \textit{for all} \quad w \in Q^2(x^0,\, h)$$

holds for any $h \in T(x^0 \mid Q)$ such that $\sigma(F'(x^0 \mid h)) = 0$. Here, $Q^2(x^0,\, h)$ is the set of all second-order true tangent vectors to the set Q at the point x^0 in the direction h.

If Q is a subset of a normed space X, a mapping F is defined in some open domain of the space X containing Q, and F is twice parabolically H-differentiable at a point $x^0 \in Q$, then, for any $h \in T(x^0 \mid Q)$,

$$F''(x^0 \mid h,\, w) \in D^2F(x^0,\, F'(x^0 \mid h)) \quad \text{for all} \quad w \in Q^2(x^0,\, h),$$

where $Q^2(x^0,\, h)$ is the set of second-order true tangent vectors to the set Q at the point $x^0 \in Q$ in the direction $h \in Y$.

This remark proves Corollary 2.

Definition 5. A vector $y \in Y$ will be called a *radial derivative vector* of the mapping $F : Q \to X$ at a point $x^0 \in Q$ if there exist sequences $(x_n) \in U_Q(x^0)$ and $(t_n) \in S(0)$ such that $F(x_n) = F(x^0) + t_n y$ for all n.

The set of all radial derivative vectors of the mapping $F : Q \to X$ at the point $x^0 \in Q$ is denoted by $RF(x^0)$. If is clear that $RF(x^0) \subset DF(x^0)$.

Theorem 5 (a sufficient condition of second-order minimality)**.** *Let $F : Q \to Y$ be a mapping defined on the metric space Q and taking values in the ordered Banach space $(Y, \prec)$. Suppose that the space Y is finite-dimensional and the mapping $F : Q \to Y$ is continuous at a point $x^0 \in Q$. Suppose also that condition* (5) *holds and, for any nonzero vector $y \in DF(x^0)$ satisfying the equality $\sigma(y) = 0$, the inclusion $y \in DF(x^0) \setminus RF(x^0)$ holds and the inequality*

$$\max_{y^* \in \partial\sigma(y)} y^*(z) > 0 \quad \textit{for all} \quad z \in D^2F(x^0,\, y) \cap L_y^{\perp}, \quad z \neq 0, \tag{9}$$

holds, where $L_y^{\perp}$ is an arbitrary direct complement to the one-dimensional space $L_y := \{\beta y \mid \beta \in \mathbb{R}\}$. Then, $x^0 \in Q$ is a point of local strong $\prec$-minimum of the mapping $F : Q \to Y$.

Proof is by contradiction. Assume that condition (9) holds but x^0 is not a point of local strong $\prec$-minimum of the mapping F. Then, there exists a sequence $(x_n) \in U_Q(x^0)$, $F(x_n) \neq F(x^0)$, such that $\sigma(F(x_n) - F(x^0)) \leq 0$ for all n and the corresponding sequence $y_n = \dfrac{F(x_n) - F(x^0)}{\|F(x_n) - F(x^0)\|}$ converges to some vector y, $\|y\| = 1$. Since $\sigma(y_n) \leq 0$, we have $\sigma(y) \leq 0$. On the other hand, since $y \in DF(x^0)$, we have $\sigma(y) \geq 0$. Hence, $\sigma(y) = 0$ and $y \in DF(x^0)$.

If $y_n = y$ for infinitely many numbers n, then the sequence (y_n) has a subsequence (y_{n_k}) such that $y_{n_k} = y$ for all k. Without loss of generality, we can assume that the original sequence (x_n) is such that $y_n = y$ for all n. In this case, $F(x_n) = F(x^0) + t_n y$, where $t_n = \|F(x_n) - F(x^0)\|$, for all n and, hence, $y \in RF(x^0)$, which is impossible by the conditions of the theorem.

Thus, without loss of generality, we can assume that $y_n \neq y$ for all n. Consider the sequence $z_n := 2\dfrac{y_n - y}{\|y_n - y\|}$ and assume that $z_n \to z$ as $n \to \infty$ (if necessary, take a subsequence for which this condition holds). It follows from the definitions of the vectors y_n and z_n that $F(x_n) = F(x^0) + t_n y + 1/2\, t_n \tau_n z_n$, where $t_n := \|F(x_n) - F(x^0)\|$ and $\tau_n := \|y_n - y\|$. Hence, $z \in D^2F(x^0, y)$ and $z \neq 0$. Since the space Y is finite-dimensional, we can assume without loss of generality that a scalar product $\langle \cdot, \cdot \rangle$: $Y \times Y \to \mathbb{R}$ is defined on Y and $\|y\| = (\langle y, y \rangle)^{\frac{1}{2}}$. For any n, we have $1 = \langle y_n, y_n \rangle = \|y_n - y\|^2 + 2\langle y_n - y, y \rangle + \|y\|^2$ and, since $1 = \langle y_n, y_n \rangle = \|y_n - y\|^2 + 2\langle y_n - y, y \rangle + \|y\|^2$, we also have $\|y_n - y\| + \Big\langle \|y_n - y\|^{-1}(y_n - y), y \Big\rangle = 0$. Taking $n \to \infty$, we get $\langle z, y \rangle = 0$. Hence, z belongs to the orthogonal complement to L_y.

Since the sequence (x_n) satisfies the inequality $\sigma(F(x_n) - F(x^0)) \leq 0$ for all n, we have $\sigma(y + 1/2\, \tau_n z_n)) \leq 0$, $n = 1, 2, \ldots$. In addition, $\sigma(y) = 0$; hence,

$$\frac{\sigma(y + 1/2\, \tau_n z_n)) - \sigma(y)}{1/2\, \tau_n} \leq 0, \quad n = 1, 2, \ldots.$$

Therefore, $\sigma'(y \,|\, z) = \max\limits_{y^* \in \partial\sigma(y)} y^*(z) \leq 0$. We have come to a contradiction, which proves the theorem.

Definition 6. A subset Q of a normed space X is said [15, 16] to satisfy the *second-order regularity condition* at a point $x^0 \in Q$ if, for any sequence $(x_n) \in Q$ of the form $x_n = x^0 + t_n h + \frac{1}{2} t_n^2 w_n$, where $h \in X$, $(t_n) \in S(0)$, and $(w_n) \in U_X$ are such that $t_n w_n \to 0$, the following inequality holds:

$$\lim_{n \to \infty} \operatorname{dist}\big(w_n, Q^2(x^0, h)\big) = 0,$$

where $\operatorname{dist}\big(w, Q^2(x^0, h)\big) := \inf\limits_{u \in Q^2(x^0, h)} \|w - u\|$ is the distance from the point w to the set $Q^2(x^0, h)$.

Theorem 6 (a sufficient condition of second-order minimality in the regular case)**.** *Let Q be a subset of a normed space X satisfying the second-order regularity condition at a point $x^0 \in Q$, and let a mapping $F : X \to Y$ be twice Fréchet differentiable at the point x^0. Suppose that the normed spaces X and Y are finite-dimensional, condition* (6) *holds, and, for any $h \in T(x^0 \,|\, Q)$ such that $\sigma(F'(x^0)h) = 0$, the inequality*

$$\inf_{w \in Q^2(x^0, h)} \max_{y^* \in \partial\sigma(F'(x^0)h)} y^*\big(F'(x^0)w + F''(x^0)[h, h]\big) > 0 \tag{10}$$

holds, where $Q^2(x^0, h)$ is the set of second-order true tangent vectors to the set Q at the point x^0 in the direction h and $F'(x^0) : X \to Y$ and $F''(x^0)[\cdot, \cdot] : X \times X \to Y$ are the first and second

Fréchet derivatives of the mapping F at the point x^0, respectively. Then, $x^0 \in Q$ is a point of local strong $\prec$-minimum of the mapping $F : X \to Y$ on the set Q.

Proof is by contradiction, as in the previous theorem. Assume that the condition of the theorem is satisfied but $x^0 \in Q$ is not a point of local strong $\prec$-minimum of the mapping $F : X \to Y$. Then, there exists a sequence $(x_n) \in U_Q(x^0)$, $F(x_n) \neq F(x^0)$, such that $\sigma(F(x_n) - F(x^0)) \leq 0$ for all n all the corresponding sequence $h_n = \dfrac{x_n - x^0}{\|x_n - x^0\|}$ converges to some vector h, $\|h\| = 1$. From the Fréchet differentiability of the mapping F at the point x^0 and from the condition $\dfrac{\sigma(F(x_n) - F(x^0))}{\|x_n - x^0\|} \leq 0$, we get $\sigma(F'(x^0)h) \leq 0$. On the other hand, since $h \in T(x^0 \,|\, Q)$, we have $\sigma(F'(x^0)h) \geq 0$ by condition (6). Thus, $\sigma(F'(x^0)h) = 0$ and, hence, $-F'(x^0)h \in \mathrm{bd}P$, where $\mathrm{bd}P$ is the boundary of the cone P.

Define $w_n = 2(h_n - h)/t_n$, where $t_n = \|x_n - x^0\|$. Then, $x_n = x^0 + t_n h + 1/2\, t_n^2 w_n$, where $(t_n) \in S(0)$, $(w_n) \in U_X$, and $t_n w_n = 2(h_n - h) \to 0$. Since the set Q satisfies the second-order regularity condition, there exists a sequence $(u_n) \in Q^2(x^0, h)$ such that $w_n - u_n \to 0$ as $n \to \infty$. Choose from the set $\partial\sigma(F'(x^0)h) := \{y^* \in \partial\sigma \,|\, y^*(F'(x^0)h) = 0\}$ a sequence (y_n^*) such that

$$\max_{y^* \in \partial\sigma(F'(x^0)h)} y^*(F'(x^0)u_n + F''(x^0)[h,\, h]) = y_n^*\big(F'(x^0)u_n + F''(x^0)[h,\, h]\big).$$

The existence of such y_n^* follows from the compactness of the set $\partial\sigma(F'(x^0)h)$ and from the continuity of the linear function $y^* \to y^*(F'(x^0)u_n + F''(x^0)[h,\, h])$. Note that, by condition (10), the inequality

$$y_n^*\big(F'(x^0)u_n + F''(x^0)[h,\, h]\big) \geq \gamma \tag{11}$$

holds for all n, where $\gamma := \inf\limits_{w \in Q^2(x^0,\, h)} \max\limits_{y^* \in \partial\sigma(F'(x^0)h)} y^*\big(F'(x^0)w + F''(x^0)[h,\, h]\big) > 0$.

Since the set $\partial\sigma(F'(x^0)h)$ is compact, the sequence (y_n^*) has a subsequence converging to some $y^* \in \partial\sigma(F'(x^0)h)$. We assume without loss of generality that the sequence (y_n^*) itself converges to $y^* \in \partial\sigma(F'(x^0)h)$. Further, we use the fact that the mapping $F : X \to Y$ is twice Fréchet differentiable at the point x^0. By the Taylor formula, we get

$$\begin{aligned} F(x_n) - F(x^0) &= t_n F'(x^0)h + \frac{1}{2}t_n^2\big(F'(x^0)w_n + F''(x^0)[h,\, h]\big) + o(t_n^2) \\ &= t_n F'(x^0)h + \frac{1}{2}t_n^2\big(F'(x^0)u_n + F''(x^0)[h,\, h]\big) + \frac{1}{2}t_n^2 F'(x^0)(w_n - u_n) + o(t_n^2). \end{aligned} \tag{12}$$

Note that the inclusion $y_n^* \in \partial\sigma(F'(x^0)h)$ implies that $y_n^*(F'(x^0)h) = 0$. Therefore, we find from equality (12) and condition (11) that

$$\begin{aligned} y_n^*(F(x_n) - F(x^0)) &= \frac{1}{2}t_n^2 y_n^*\big(F'(x^0)u_n + F''(x^0)[h,\, h]\big) + y_n^*\left(\frac{1}{2}t_n^2 F'(x^0)(w_n - u_n) + o(t_n^2)\right) \\ &\geq \frac{1}{2}\|x_n - x^0\|^2 \left(\gamma + y_n^*(F'(x^0)(w_n - u_n)) + \frac{y_n^*(o(\|x_n - x^0\|^2))}{\|x_n - x^0\|^2}\right), \end{aligned}$$

which implies that $y_n^*(F(x_n) - F(x^0)) > 0$ for sufficiently large n.

On the other hand, since $-(F(x_n) - F(x^0)) \in \mathrm{cl}P$ and $y_n^* \in \partial\sigma$, we have $y_n^*(F(x_n) - F(x^0)) \leq 0$ for all n.

The obtained contradiction proves the theorem.

In the case of the scalar optimization problem, i.e., when the space Y coincides with the real line $\mathbb{R}$ with the natural order, Theorem 6 takes the following form.

Corollary 3 [27]**.** *Let Q be a subset of a normed space X satisfying the second-order regularity condition at a point $x^0 \in Q$. Suppose that a real-valued function $f : X \to \mathbb{R}$ is twice Fréchet differentiable at the point x^0. Suppose also that the normed space X is finite-dimensional and the condition*

$$f'(x^0)h \geq 0 \quad \textit{holds for any} \quad h \in T(x^0 \,|\, Q)$$

and the inequality

$$\inf_{w \in Q^2(x^0,\, h)} \big(f'(x^0)w + f''(x^0)[h,\, h]\big) > 0 \tag{13}$$

holds for any $h \in T(x^0 \,|\, Q)$ such that $f'(x^0)h = 0$. Then, $x^0 \in Q$ is a point of local minimum of the function $f : X \to Y$ on the set Q.

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