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Existence Conditions for a Classical Solution of the Cauchy Problem for the Diffusion-Wave Equation with a Partial Caputo Derivative

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Consider the Cauchy problem

$$({}^cD^{\alpha}_{0+,\,t}u)(x,t) = \lambda^2\Delta_x u(x,t), \quad x \in \mathbf{R}^m, \quad t>0, \quad \alpha>0, \quad \lambda>0, \tag{1}$$

$$\frac{\partial^k u}{\partial t^k}(x, 0+) = f_k(x), \quad k = 0, 1, \ldots, n-1, \quad n = -[-\alpha], \quad x \in \mathbf{R}^m. \tag{2}$$

Here, $({}^cD^{\alpha}_{0+,\,t}u)(x,t)$ is the αth-order Caputo derivative of $u(x, t)$ with respect to the second argument ($\alpha > 0$) [1, Section 2.4.1],

$$({}^cD^{\alpha}_{0+,\,t}u)(x,t) = \left(\frac{\partial}{\partial x}\right)^n \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u(x,\tau) - \sum\limits_{k=0}^{n-1}\frac{\tau^k}{k!}\frac{\partial^k u}{\partial \tau^k}(x,0)}{(t-\tau)^{\alpha-n+1}}d\tau, \quad n = -[-\alpha], \tag{3}$$

and Δ_x is the Laplacian with respect to the first argument $x = (x_1, x_2, \ldots, x_m) \in \mathbf{R}^m$: $\Delta_x = \sum\limits_{j=1}^{m}\frac{\partial^2 u}{\partial x_j^2}$. For $\alpha = 1$ and $\alpha = 2$, Eq. (1) coincides with the heat (diffusion) equation and the wave equation, respectively. That is why Eq. (1) is known as the diffusion-wave equation [1, p. 146].

A function $u(x, t)$ ($x \in \mathbf{R}^m$, $t > 0$) is a classical solution to Cauchy problem (1), (2) if (i) $u(x, t)$ is twice continuously differentiable with respect to x for every $t > 0$; (ii) for every $x \in \mathbf{R}^m$, $u(x, t)$ is a continuous function of t and has a continuous partial derivative of order α with respect to t; and (iii) Eqs. (1) and (2) hold true.

We examine the existence conditions for a classical solution to problem (1), (2) when $0 < \alpha < 2$. Specifically, our study is based on the scheme proposed in [2] for analyzing the Cauchy problem with a regularized fractional derivative of order $0 < \alpha < 1$.

It was shown in [3] that problem (1), (2) with $0 < \alpha \le 1$ and $1 < \alpha < 2$ has the explicit solutions

$$u(x,t) = \int_{\mathbf{R}^m} {}^cG_1^{\alpha}(x-\tau, t) f_0(\tau)d\tau, \quad 0<\alpha\le 1, \tag{4}$$

$$u(x,t) = \int_{\mathbf{R}^m} [{}^cG_1^{\alpha}(x-\tau,t)f_0(\tau) + {}^cG_2^{\alpha}(x-\tau,t)f_1(\tau)]d\tau, \quad 1<\alpha<2, \tag{5}$$

respectively. Here,

$${}^cG_k^{\alpha}(x,t) = \frac{2^{-m}|x|^{1-m/2}}{\lambda^{1+m/2}\pi^{(m-1)/2}}t^{k-1-\alpha(m+2)/4} \times H^{2,0}_{2,2}\left[\frac{|x|}{\lambda}t^{-\alpha/2}\left|\begin{matrix}(m/4, 1/2), (k-\alpha(m+2)/4, \alpha/2)\\(m/2-1, 1), (1/2-m/4, 1/2)\end{matrix}\right.\right], \quad k = 1, 2, \tag{6}$$

and $H^{m,n}_{p,q}(z)$ is the so-called H-function (see, e.g., [4, Section 1.1]).

For integers m, n, p, and q such that $0 \le m \le q$ and $0 \le n \le p$; for $a_i, b_j \in C$; and for $\alpha_i, \beta_j \in R_+ = (0, \infty)$ ($i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$), the H-function $H^{m,n}_{p,q}(z)$ is defined by the Mellin–Barns integral

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$$H_{p,q}^{m,n}(z) \equiv H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)} z^{-s}ds. \quad (7)$$

Here,

$$z^{-s} = \exp[-s\{\ln|z| + i\arg z\}], \quad z \neq 0;$$

the poles

$$b_{jl} = \frac{-b_j - l}{\beta_j}, \quad i = 1, 2, \dots, m; \quad l = 0, 1, 2, \dots \quad (8)$$

of the gamma functions $\Gamma(b_j + \beta_j s)$ do not coincide with the poles

$$a_{ik} = \frac{1 - a_i + k}{\alpha_i}, \quad i = 1, 2, \dots, n; \quad k = 0, 1, 2, \dots \quad (9)$$

of the gamma functions $\Gamma(1 - a_i - \alpha_i k)$:

$$\alpha_i(b_j + l) \neq \beta_j(a_i - k - 1), \quad i = 1, 2, \dots, n;$$
$$j = 1, 2, \dots, m; \quad k, l = 0, 1, 2, \dots;$$

L in (7) is a special infinite contour such that all b_{jl} and a_{ik} lie on the left and right of it, respectively.

By using the properties of H-functions [4, Section 2.1], ${}^cG_1^\alpha(x, t)$ and ${}^cG_2^\alpha(x, t)$ can be simplified to

$$ {}^cG_k^\alpha(x, t)$$
$$= |x|^{-m}\pi^{-m/2}t^{k-1}H_{1,2}^{2,0}\left[\frac{|x|^2}{4\lambda^2 t^\alpha}\left|\begin{matrix}(k,\alpha)\\(m/2,1),(1,1)\end{matrix}\right.\right], \quad (10)$$
$$k = 1, 2.$$

To analyze the asymptotics of solutions (4) and (5) at large x, we estimate ${}^cG_k^\alpha(x, t)$ $(k = 1, 2)$ and their derivatives with respect to x.

Below is an asymptotic estimate for the $H_{1,2}^{2,0}$-functions in (10).

Lemma 1. *Let* $m \in N$, $0 < \alpha < 2$, $k = 1$ *for* $0 < \alpha \le 1$; *and* $k = 1, 2$ *for* $1 < \alpha < 2$.

Then, for any fixed $t > 0$, *we have the asymptotic estimates*

$$H_{1,2}^{2,0}\left[\frac{|x|^2}{4\lambda^2 t^\alpha}\left|\begin{matrix}(k,\alpha)\\(m/2,1),(1,1)\end{matrix}\right.\right]$$

$$= A_k\exp\left[-(2-\alpha)\alpha^{\alpha/(2-\alpha)}\left(\frac{|x|^2}{4\lambda^2 t^\alpha}\right)^{1/(2-\alpha)}\right]$$
$$\times\left(\frac{|x|^2}{4\lambda^2 t^\alpha}\right)^{(m/2+1-k)/(2-\alpha)}\left[1 + O\left(\left(\frac{t^\alpha}{|x|^2}\right)^{1/(2-\alpha)}\right)\right], \quad (11)$$
$$|x| \to \infty, \quad k = 1, 2,$$

where A_k *are constants.*

Proof. Formula (11) follows from the exponential behavior at infinity of the more general $\mathrm{H}_{p,q}^{q,0}$-function as shown in Theorem 1.10 in [4, p. 17].

Formula (11) yields the following asymptotic estimates for ${}^cG_1^\alpha(x, t)$ and ${}^cG_2^\alpha(x, t)$ in (4) and (5) for fixed $t > 0$ as $|x| \to \infty$:

$${}^cG_k^\alpha(x, t)$$
$$= B_k\exp\left[-(2-\alpha)\alpha^{\alpha/(2-\alpha)}\left(\frac{|x|^2}{4\lambda^2 t^\alpha}\right)^{1/(2-\alpha)}\right]|x|^{-m}t^{k-1}$$
$$\times\left(\frac{|x|^2}{4\lambda^2 t^\alpha}\right)^{(m/2+1-k)/(2-\alpha)}\left[1 + O\left(\left(\frac{t^\alpha}{|x|^2}\right)^{1/(2-\alpha)}\right)\right], \quad (12)$$
$$|x| \to \infty,$$

where

$$B_k = \pi^{-m/2}A_k; \quad (13)$$

$k = 1$ for $0 < \alpha \le 1$; and $k = 1, 2$ for $1 < \alpha < 2$.

Since $0 < \alpha < 2$, it follows from (12) that, for any fixed $t > 0$, the functions ${}^cG_1^\alpha(x, t)$ and ${}^cG_2^\alpha(x, t)$ tend to zero as $|x| \to 0$:

$$\lim_{|x|\to\infty} {}^cG_1^\alpha(x, t) = \lim_{|x|\to\infty} {}^cG_2^\alpha(x, t) = 0. \quad (14)$$

Let us estimate ${}^cG_k^\alpha(x, t)$ $(k = 1, 2)$ for $|x|^2 t^{-\alpha} < 1$, $x \neq 0$.

According to Theorem 1.12 in [4, p. 20], the asymptotic expansion of the H-function

$$H_{1,2}^{2,0}\left[z\left|\begin{matrix}(k,\alpha)\\(m/2,1),(1,1)\end{matrix}\right.\right]$$

at zero is given by the formula

$$H_{p,q}^{m,n}(z) = \sum_j{}'[h_j^* z^{b_j/\beta_j} + o(z^{b_j/\beta_j})]$$

$$+ \sum_j{}''[H_j^* z^{b_j/\beta_j}[\ln z]^{N_j^*-1} + o(z^{b_j/\beta_j}[\ln z]^{N_j^*-1})], \quad z \to 0, \quad (15)$$

where the sums in $\sum'$ and $\sum''$ are taken over j ($j = 1, 2, \dots, m$) such that the gamma functions $\Gamma(b_j + \beta_j s)$ in

(7) have simple poles and poles of order N_j^*, respectively, at the points $b_j = b_{j0}$ defined by (8).

Using formula (15), we derive the following estimates at $z \to 0$:

$$H_{1,2}^{2,0}\left[z\left|\begin{matrix}(k,\alpha)\\(m/2,1),(1,1)\end{matrix}\right.\right] = h_1^* z^{m/2} + o(z^{m/2}) + h_2^* z + o(z) \quad (16)$$

for odd m;

$$H_{1,2}^{2,0}\left[z\left|\begin{matrix}(k,\alpha)\\(m/2,1),(1,1)\end{matrix}\right.\right] = H_1^* z\ln z + o(z\ln z) \quad (17)$$

for $m = 2$; and

$$H_{1,2}^{2,0}\left[z\left|\begin{matrix}(k,\alpha)\\(m/2,1),(1,1)\end{matrix}\right.\right] = H_1^* z^{m/2}\ln z + o(z^{m/2}\ln z) + h_2^* z + o(z) \quad (18)$$

for even $m > 2$, where h_1^*, h_2^*, and H_1^* are constants.

Formulas (16)–(18) yield estimates of ${}^cG_k^\alpha(x, t)$ for $|x|^2t^{-\alpha} < 1$, $x \neq 0$:

$$\left|{}^cG_k^\alpha(x,t)\right| \le Ct^{k-1-\alpha}|x|^{-m+2}, \quad k = 1, 2, \quad x \in \mathbf{R}^m, \quad m > 2, \quad (19)$$

$$\left|{}^cG_k^\alpha(x,t)\right| \le Ct^{k-1-\alpha}(\left|\ln(|x|^2t^{-\alpha})\right| + 1), \quad k = 1, 2, \quad x \in \mathbf{R}^2, \quad (20)$$

$$\left|{}^cG_k^\alpha(x,t)\right| \le Ct^{k-1-\alpha/2}, \quad k = 1, 2, \quad x \in \mathbf{R}. \quad (21)$$

Inequalities (19)–(21) imply that $|G_1^\alpha(x, t)|$ and $|G_2^\alpha(x, t)|$ are integrable with respect to $x \in \mathbf{R}^m$ at zero for any $t > 0$.

Taking into account asymptotics (12) of these functions at infinity, we infer the following assertion.

Lemma 2. *Let* $m \in N$, $0 < \alpha < 2$; $k = 1$ *for* $0 < \alpha \le 1$; *and* $k = 1, 2$ *for* $1 < \alpha < 2$.

Then ${}^cG_k^\alpha(x, t) \in L_1(\mathbf{R}^m)$ *for any* $t_0 > 0$.

The result below follows from Lemmas 1 and 2.

Theorem 1. *Let* $0 < \alpha \le 2$, $m \in N$, *and* $\lambda > 0$.

(a) *If* $0 < \alpha \le 1$ *and* $f_0(x)$ *is bounded and absolutely integrable on* $\mathbf{R}^m$, *then solution* (4) *to Cauchy problem* (1), (2) *exists and tends to zero as* $|x| \to \infty$ *for any fixed* $t > 0$.

(b) *If* $1 < \alpha < 2$ *and if* $f_0(x)$ *and* $f_1(x)$ *are bounded and absolutely integrable on* $\mathbf{R}^m$, *then solution* (5) *to Cauchy problem* (1), (2) *exists and tends to zero as* $|x| \to \infty$ *for any fixed* $t > 0$.

Proof. Since ${}^cG_1^\alpha(x, t)$ is absolutely integrable with respect to $x \in \mathbf{R}^m$ for every $t > 0$ and $f_0(x)$ is bounded on $\mathbf{R}^m$, the assertion in [5, p. 532] implies that convolution (4) exists. The rest of the proof is based on direct estimates.

An asymptotic expansion of ${}^cG_k^\alpha(x, t)$ as $|x|^2t^{-\alpha} \to \infty$ and $|x|^2t^{-\alpha} \to 0$ is given by formulas (12) and (19)–(21), respectively. The derivatives of these functions can be found as follows. Introducing $\rho = |x|$ and using formula (2.2.2) from [4] for the derivatives of H-functions, we obtain

$$D_\rho^n({}^cG_k^\alpha(x,t)) = D_\rho^n\left(\rho^{-m}\pi^{-m/2}t^{k-1}H_{1,2}^{2,0}\left[\frac{\rho^2}{4\lambda^2t^\alpha}\left|\begin{matrix}(k,\alpha)\\(m/2,1),(1,1)\end{matrix}\right.\right]\right) = (-1)^n\rho^{-m-n}\pi^{-m/2}t^{k-1}$$

$$\times H_{2,3}^{3,0}\left[\frac{\rho^2}{4\lambda^2t^\alpha}\left|\begin{matrix}(k,\alpha),(m,2)\\(m+n,2),(m/2,1),(1,1)\end{matrix}\right.\right]; \quad (22)$$

$$\rho > 0, \quad t > 0, \quad n = 1, 2, 3.$$

According to Theorem 1.10 in [4, p. 17] on the exponential behavior of an H-function at infinity, it follows from (22) that

$$D_x^n({}^cG_k^\alpha(x,t)) = B_k\exp\left[-(2-\alpha)\alpha^{\alpha/(2-\alpha)}\left(\frac{\rho^2}{4\lambda^2t^\alpha}\right)^{1/(2-\alpha)}\right] \times \rho^{-m}t^{k-1}\left(\frac{\rho^2}{4\lambda^2t^\alpha}\right)^{(m/2+n-k+1)/(2-\alpha)} \times\left[1 + O\left(\left(\frac{t^\alpha}{\rho^2}\right)^{1/(2-\alpha)}\right)\right], \quad \rho^2t^{-\alpha} \to \infty. \quad (23)$$

The asymptotics of $D_\rho^n({}^cG_k^\alpha(\rho, t))$ as $\rho \to 0$ can be found as follows. According to (15) and (22), for $\rho^2t^{-\alpha} < 1$, $\rho > 0$, $t > 0$, and $n \ge 1$, we have

$$\left|D_x^nG_k^\alpha(x,t)\right| \le Ct^{-k}|x|^{-m-n+2}, \quad k = 1, 2, \quad m \ge 2, \quad C > 0. \quad (24)$$

To derive an estimate at $m = 1$, we use the properties of H-functions to transform ${}^cG_k^\alpha(x, t)$ into

$${}^cG_k^\alpha(x,t) = \frac{1}{2\lambda}t^{k-1-\alpha/2}H_{1,1}^{1,0}\left[\frac{|x|}{\lambda t^{\alpha/2}}\left|\begin{matrix}(k-\alpha/2,\alpha/2)\\(0,1)\end{matrix}\right.\right].$$

According to formula (2.2.2) in [4],

$$D_x^n({}^cG_k^\alpha(x,t)) = \frac{(-1)^n}{2\lambda}t^{k-1-\alpha/2}|x|^{-n}$$

$$\times H_{2,2}^{2,0}\left[\frac{|x|}{\lambda t^{\alpha/2}}\middle|\begin{matrix}(k-\alpha/2,\alpha/2),(0,1)\\(n,1),(0,1)\end{matrix}\right]$$

$$= \frac{(-1)^n}{2\lambda}t^{k-1-\alpha/2}|x|^{-n}H_{1,1}^{1,0}\left[\frac{|x|}{\lambda t^{\alpha/2}}\middle|\begin{matrix}(k-\alpha/2,\alpha/2)\\(n,1)\end{matrix}\right].$$

Using (15) for $|x|^2t^{-\alpha} < 1$, $x \neq 0$, $n \geq 1$, we obtain

$$\left|D_x^n({}^cG_k^\alpha(x,t))\right| \leq Ct^{k-1-(n+1)\alpha/2}. \quad (25)$$

Combining estimates (12), (19)–(21), and (23)–(25) yields the following result.

Theorem 2. (a) *Suppose that* $f_0(x)$ *is continuous on* $\mathbf{R}^m$; *increases exponentially at infinity*,

$$|f_0(x)| \leq C\exp(h|x|^\mu),\ \ C>0,\ \ h>0,\ \ \mu<\frac{2}{2-\alpha}; \quad (26)$$

and, additionally, is locally Hölder for $m > 1$. *Let* ${}^cG_1^\alpha(x,t)$ *be defined by formula* (10) *with* $k = 1$.

Then function (4) *is a classical solution to problem* (1), (2) *with* $0 < \alpha \leq 1$; *moreover*, $u(x,t) \in C[0,\infty)$ *for any* $x \in \mathbf{R}^m$.

(b) *Let* $f_0(x)$ *and* $f_1(x)$ *be continuous on* $\mathbf{R}^m$, *satisfy estimate* (26), *and be locally Hölder for* $m > 1$; *and let* ${}^cG_1^\alpha(x,t)$ *and* ${}^cG_2^\alpha(x,t)$ *be defined by formula* (10) *with* $k = 1$ *and* $k = 2$, *respectively.*

Then function (5) *is a classical solution to problem* (1), (2) *with* $1 < \alpha < 2$ *and* $u(x,t) \in C[0,\infty)$ *for any* $x \in \mathbf{R}^m$.

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