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PARTIAL DIFFERENTIAL EQUATIONS

Goursat Problem for Two-Dimensional Second-Order Hyperbolic Operator-Differential Equations with Variable Domains

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Abstract—We develop a modification of the energy inequality method and use it to prove the well-posedness of the Goursat problem for linear second-order hyperbolic differential equations with operator coefficients whose domains depend on the two-dimensional time. An energy inequality for strong solutions of this Goursat problem is derived with the help of abstract smoothing operators, and we prove that the range of the problem is dense by using properties of a regularizing Cauchy problem whose inverse operator is a family of smoothing operators of a new type. We give an example of a well-posed boundary value problem for a two-dimensional complete second-order hyperbolic partial differential equation with Goursat conditions and with a boundary condition depending on the two-dimensional time.

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1. INTRODUCTION. STATEMENT OF THE PROBLEM

The two-dimensional Goursat problem for hyperbolic partial differential equations with stationary (time-independent) boundary conditions and for hyperbolic operator-differential equations with time-independent domains of operator coefficients was studied in [1–3]. Here we study this problem in the case of nonstationary boundary conditions and domains. In this situation, the method in [1–3] can no longer be used to analyze well-posedness, and so the present paper develops new techniques for deriving an a priori estimate with the use of abstract smoothing operators in [4] and for proving that the range of the Goursat problem is dense with the use of a regularizing Cauchy problem. This regularizing Cauchy problem is an analog of the Cauchy problem whose solutions are used in the proof of the uniqueness of weak solutions in [5]. The operators providing the solutions of these problems are smoothing operators of a new type. The existence and uniqueness of classical solutions of a second-order hyperbolic partial differential equation with degenerating mixed derivative under Goursat conditions and without boundary conditions were studied in [6]. A nonlocal boundary value problem was solved in [7] for hyperbolic equations whose coefficients are square matrices of scalar functions.

Let H be a Hilbert space with inner product $(\cdot\,,\cdot)$ and norm $|\cdot|$. In the bounded rectangle $\mathcal{T} =]0,T_1[\,\times\,]0,T_2[$ of the plane $\mathbb{R}^2$ of the variables $t = \{t_1, t_2\}$, consider the hyperbolic operator-differential equation

$$\mathcal{L}(t)u \equiv \frac{\partial^2 u(t)}{\partial t_2 \partial t_1} + A_1(t)\frac{\partial u(t)}{\partial t_1} + A_2(t)\frac{\partial u(t)}{\partial t_2} + A(t)u(t) = f(t), \qquad t \in \mathcal{T}, \tag{1}$$

with the Goursat conditions

$$l_1 u \equiv u(t_1, 0) = \varphi_1(t_1), \qquad l_2 u \equiv u(0, t_2) = \varphi_2(t_2), \tag{2}$$

where $\varphi_1(t_1)$, $t_1 \in [0, T_1]$, and $\varphi_2(t_2)$, $t_2 \in [0, T_2]$, are functions ranging in H and satisfying the coordination condition $\varphi_1(0) = \varphi_2(0)$. The unknown function $u(t)$ and the given right-hand side

$f(t)$ depend on the two-dimensional variable t and range in H. For each t, the coefficient $A(t)$ is a positive self-adjoint operator on H, and the $A_i(t)$, $i = 1, 2$, are linear unbounded operators on H; their domains $D(A(t))$ and $D(A_i(t))$ depend on t.

Remark 1. The well-posedness of problem (1), (2) was studied in [1] (respectively, [2]) by the Fourier method with the use of the representation of the solution via the Riemann function (respectively, by the energy inequality method) for the case in which $A(t)$ is a stationary (respectively, nonstationary) elliptic partial differential operator in the space variables with stationary normal boundary conditions and the $A_i(t)$, $i = 1, 2$, are operators of multiplication by scalar functions. In [2], an a priori estimate was established by inner multiplication by a Leray separating operator with subsequent integration by parts, and the fact that the range is dense was derived from a dual Gårding type inequality, which was proved with the use of Friedrichs averaging integral operators. For strong solutions of the abstract problem (1), (2), an a priori estimate was also obtained in a traditional way in [3], where the domain of the operator $A(t)$ is stationary, and the operators $A_i(t)$, $i = 1, 2$, are bounded in H; but the fact that the range is dense was derived there from an a priori estimate for the formally adjoint equation on the basis of the representation of its solution via the operator Riemann function.

2. DEFINITIONS AND STATEMENT OF THE MAIN RESULTS

The Goursat problem (1), (2) defines a linear unbounded operator $L \equiv \{\mathcal{L}(t), l_1, l_2\} : E \supset D(L) \to F$ from the Banach space E obtained as the closure of the set

$$D(L) = \Big\{u \in \mathcal{H} = L_2(\mathcal{T}, H) :\ u \in D(A(t)),\ \partial u/\partial t_i \in D(A_i(t)),\ t \in \overline{\mathcal{T}} = [0, T_1] \times [0, T_2];$$
$$\partial u/\partial t_i, \partial^2 u/\partial t_2 \partial t_1, A_i(t)\partial u/\partial t_i, A(t)u \in \mathcal{H},\ i = 1, 2\Big\}$$

in the norm

$$\|u\|_E = \left[\int\limits_{\mathcal{T}} \sum_{i=1}^{2} \left(\left|\frac{\partial u}{\partial t_i}\right|^2 + |A^{1/2}(t)u|^2\right) dt\right]^{1/2}$$

to the space $F = \mathcal{H}_\gamma \times H_1 \times H_2$ of functions $\Phi = \{f, \varphi_1, \varphi_2\}$ such that $\varphi_1(0) = \varphi_2(0)$ with the norm

$$\|\Phi\|_F = (\|f\|_\gamma^2 + \|\varphi_1\|_1^2 + \|\varphi_2\|_2^2)^{1/2}.$$

Here $\mathcal{H}_\gamma = L_{2,\gamma(t)}(\mathcal{T}, H)$ is the Hilbert space with Hermitian norm $\|f\|_\gamma = \int_{\mathcal{T}} \gamma(t)|f(t)|^2\, dt$ and weight $\gamma(t) = T_2 - t_2 + T_1 - t_1$, and the Hilbert spaces H_i are the closures of the sets of traces of functions $u \in D(L)$ at $t_j = 0$ in the norms

$$\|u\|_i = \left(\int\limits_0^{T_i} \left(\left|\frac{\partial u}{\partial t_i}\right|^2 + |A^{1/2}(t)u|^2\right)\Bigg|_{t_j=0} dt_i\right)^{1/2}, \qquad j \neq i, \qquad i = 1, 2.$$

In a standard way, one can show that if the set

$$D^*(L) = \Big\{v \in D(L) :\ v(t) \in D(A_i^*(t)),\ t \in \overline{\mathcal{T}},\ A_i^*(t)v \in \mathcal{H},\ i = 1, 2\Big\}$$

is dense in $\mathcal{H}$, where the $A_i^*(t)$ are the adjoint operators of $A_i(t)$ in H, then the operator L admits a strong closure $\overline{L} :\ E \supset D(\overline{L}) \to F$.

Definition 1. The solutions $u \in D(\overline{L})$ of the operator equation $\overline{L}u = \Phi$, $\Phi \in F$, are referred to as *strong solutions* of the Goursat problem (1), (2).

Let us state the main results of the present paper. First, by using abstract smoothing operators $A_\varepsilon^{-1}(t)$, we derive an energy inequality for strong solutions.

Theorem 1. *Let the set $D^*(L)$ be dense in $\mathcal{H}$, and let the following conditions be satisfied.*

1. *For all $t \in \mathcal{T}$, the operators $A_i(t)$ are subordinate to the square roots $A^{1/2}(t)$ [with domains $D(A^{1/2}(t))$] of the operators $A(t)$ [i.e., $|A_i(t)w| \le c_i|A^{1/2}(t)w|$ for all $w \in D(A^{1/2}(t))$, $c_i > 0$, $i = 1, 2$] and satisfy the estimate*

$$-\operatorname{Re}(A_1(t)v_1 + A_2(t)v_2, v_1 + v_2) \le c_3(|v_1|^2 + |v_2|^2) \quad \forall v_i \in D(A_i(t)), \quad i = 1, 2, \quad c_3 \ge 0. \tag{3}$$

2. *In H, for all $t \in \mathcal{T}$, there exist bounded inverse operators*

$$A^{-1}(t) \in \mathcal{B}(\mathcal{T}, \mathfrak{L}(H)),$$

which are strongly continuous with respect to t and have the strong partial derivatives [8, *pp.* 216–218]

$$\partial A^{-1}(t)/\partial t_i \in \mathcal{B}(\mathcal{T}, \mathfrak{L}(H)), \qquad i = 1, 2,$$

such that

$$|((\partial A^{-1}(t)/\partial t_i)g, g)| \le c_4(A^{-1}(t)g, g) \qquad \forall g \in H, \qquad c_4 \ge 0, \qquad i = 1, 2. \tag{4}$$

Then one has the energy inequality

$$\|u\|_E^2 \le c_5\|\overline{L}u\|_F^2 \quad \forall u \in D(\overline{L}), \quad c_5 = \max\{1, T_1 + T_2\}\exp\{(T_1 + T_2)\max\{c_4, 2 + 2c_3\}\}. \tag{5}$$

This inequality readily implies the following assertion.

Corollary 1. *The relation $R(\overline{L}) = \overline{R(L)}$ holds under the assumptions of Theorem* 1, *where $R(\overline{L})$ is the range of the operator $\overline{L}$ and $\overline{R(L)}$ is the closure of the range $R(L)$ of the operator L in F.*

Then we use this corollary to prove the existence theorem for strong solutions with the use of a regularizing Cauchy problem.

Theorem 2. *Let the assumptions of Theorem* 1 *hold, and let the following conditions be satisfied.*

3. *The operators $\partial A^{-1}(t)/\partial t_i$, $i = 1, 2$, satisfy the inequalities*

$$|((\partial A^{-1}(t)/\partial t_i)g, h)| \le c_6|g||A^{-1/2}(t)h| \qquad \forall g, h \in H, \qquad 2 > c_6 \ge 0, \qquad i = 1, 2, \tag{6}$$

for all $t \in \mathcal{T}$.

4. *There exist constants $c_7, c_8 \ge 0$ such that the estimates*

$$-\operatorname{Re}(A_1(t)v_1 + A_2(t)v_2, A(t)(v_1 + v_2)) \le c_7[|A^{1/2}(t)v_1|^2 + |A^{1/2}(t)v_2|^2] \quad \forall v_1, v_2 \in D(A(t)), \tag{7}$$

$$|(A_i(t)(\partial A^{-1}(t)/\partial t_i)g, v)| \le c_8|g|\,|A^{-1/2}(t)v| \qquad \forall g, v \in H, \qquad i = 1, 2, \tag{8}$$

hold for all $t \in \mathcal{T}$.

5. *For almost all $t \in \mathcal{T}$, there exists a bounded strong mixed derivative $\partial^2 A^{-1}(t)/\partial t_2 \partial t_1 \in L_\infty(\mathcal{T}, \mathfrak{L}(H))$ satisfying the inequality*

$$|((\partial^2 A^{-1}(t)/\partial t_2 \partial t_1)g, v)| \le c_9|g|\,|A^{-1/2}(t)v| \qquad \forall g, v \in H, \qquad c_9 \ge 0. \tag{9}$$

Then for all $\Phi = \{f, \varphi_1, \varphi_2\} \in F$, there exists a strong solution $u \in E$ of the Goursat problem (1), (2).

In Theorems 1 and 2, the constants c_i, $i = 1, \dots, 9$, are independent of w, v_1, v_2, g, h, v, and t. The stability of strong solutions of problem (1), (2) with respect to f, φ_1, and φ_2 follows from inequality (5).

3. REGULARIZING CAUCHY PROBLEM

Consider the Cauchy problem

$$\frac{\partial v(t)}{\partial t_1} + \frac{\partial v(t)}{\partial t_2} = J_\delta^{-1}(t) e^{-\theta(t)} w(t), \qquad t \in \mathcal{T}, \qquad \delta > 0, \tag{10}$$

$$v(t_1, 0) = 0, \qquad t_1 \in [0, T_1], \qquad v(0, t_2) = 0, \qquad t_2 \in [0, T_2], \tag{11}$$

where $\theta(t) = c\gamma(t)$ for all $c \geq 0$, and the smoothing integral operators $J_\delta^{-1}(t) = (I + \delta J(t))^{-1}$ are the inverses of the operators $J_\delta(t) = I + \delta J(t)$, $\delta > 0$, where the differential operator $J(t)h = \partial^2 h/\partial t_2 \partial t_1$ has the domain

$$D(J) = \Big\{ h \in \mathcal{H} : \ \partial h/\partial t_i, \partial^2 h/\partial t_2 \partial t_1 \in \mathcal{H}, \ i = 1, 2; \\ h(t_1, 0) = 0, \ t_1 \in [0, T_1]; \ h(0, t_2) = 0, \ t_2 \in [0, T_2] \Big\}.$$

Set $\hat{C}^{(m)}(\overline{\mathcal{T}}, H) = \{ v \in C(\overline{\mathcal{T}}, H) : \ v \in C^{(m)}(\mathcal{T}_k, H), \ k = 1, 2 \}$, $m = 1, 2, \dots$ Obviously, the following assertion holds.

Theorem 3. *All smooth solutions* $v = v_\delta(t) \in C^{(3)}(\overline{\mathcal{T}}, H)$ *of the Cauchy problem* (10), (11) *satisfy the identity*

$$\int_{\mathcal{T}} \gamma(t) |v|^2 \, dt + \delta \sum_{i=1}^{2} \int_{\mathcal{T}} (T_i - t_i) \left| \frac{\partial v}{\partial t_i} \right|^2 dt \\ = 2 \operatorname{Re} \int_{\mathcal{T}} \gamma(t) (J_\delta v, v) \, dt - 2 \operatorname{Re} \int_{\mathcal{T}} (T_1 - t_1)(T_2 - t_2) \left(J_\delta \left(\frac{\partial v}{\partial t_1} + \frac{\partial v}{\partial t_2} \right), v \right) dt \quad \forall \delta > 0. \tag{12}$$

Corollary 2. *If* $w \in \mathcal{H}_\gamma$ *in Eq.* (10), *then for the solutions* $v = v_\delta(t)$ *of the Cauchy problem* (10), (11), *there exists a constant* $c_{10} > 0$, *independent of* v *and* δ, *such that*

$$\frac{1}{4} \int_{\mathcal{T}} \gamma(t) |v|^2 \, dt + \delta \sum_{i=1}^{2} \int_{\mathcal{T}} (T_i - t_i) \left| \frac{\partial v}{\partial t_i} \right|^2 dt \leq c_{10} \int_{\mathcal{T}} \gamma(t) |w|^2 \, dt. \tag{13}$$

Proof. It is known that the solution of the Cauchy problem

$$\frac{\partial \widetilde{v}(t)}{\partial t_1} + \frac{\partial \widetilde{v}(t)}{\partial t_2} = g(t), \quad t \in \mathcal{T}; \quad \widetilde{v}(t_1, 0) = 0, \quad t_1 \in [0, T_1], \quad \widetilde{v}(0, t_2) = 0, \quad t_2 \in [0, T_2],$$

is given by the formula

$$\widetilde{v}(t) = \begin{cases} \displaystyle\int_{t_2 - t_1}^{t_2} g(s + t_1 - t_2, s) \, ds & \text{for} \quad t \in \mathcal{T}_1 = \{ t \in \mathcal{T} : \ 0 \leq t_1 \leq t_2 \leq T_2 \} \\ \displaystyle\int_{t_1 - t_2}^{t_1} g(s, s - t_1 + t_2) \, ds & \text{for} \quad t \in \mathcal{T}_2 = \{ t \in \mathcal{T} : \ 0 \leq t_2 < t_1 \leq T_1 \}. \end{cases} \tag{14}$$

A straightforward verification shows that this formula gives unique solutions $\widetilde{v} \in \mathcal{D}$ for all right-hand sides

$$g \in \mathcal{D} = \{g \in \mathcal{H}: \ \partial g/\partial t_i \in \mathcal{H}, \ \partial^2 g/\partial t_2\, \partial t_1 \in \mathcal{H}_k = L_2(\mathcal{T}_k, H), \ k, i = 1, 2\}.$$

First, by passage to the limit, one can generalize identity (12) from smooth solutions $v \in C^{(3)}(\overline{\mathcal{T}}, H)$ of the Cauchy problem (10), (11) to all solutions $v \in \mathcal{D}$, because, by Eq. (10),

$$J_\delta(\partial v/\partial t_1 + \partial v/\partial t_2) \in \mathcal{H}_\gamma \qquad \forall \delta > 0.$$

Then, by applying the Cauchy–Schwarz inequality and elementary estimates to the right-hand side of identity (12), we obtain the inequality

$$\begin{aligned} &\frac{1}{4} \int\limits_{\mathcal{T}} \gamma(t)|v|^2\, dt + \delta \sum_{i=1}^{2} \int\limits_{\mathcal{T}} (T_i - t_i) \left| \frac{\partial v}{\partial t_i} \right|^2 dt \\ &\qquad \le 2 \int\limits_{\mathcal{T}} \gamma(t)|J_\delta v|^2\, dt + \int\limits_{\mathcal{T}} \gamma(t)(T_1 - t_1)(T_2 - t_2) \left| J_\delta \left(\frac{\partial v}{\partial t_1} + \frac{\partial v}{\partial t_2} \right) \right|^2 dt. \end{aligned} \tag{15}$$

Lemma 1. *The values of the operators $J_\delta(t)$ on the solutions $v = v_\delta(t)$, $\delta > 0$, of the Cauchy problem* (10), (11) *have the form*

$$J_\delta v = \begin{cases} \displaystyle\int\limits_{t_2 - t_1}^{t_2} e^{-\theta(s + t_1 - t_2, s)} w(s + t_1 - t_2, s)\, ds & \text{for} \quad t \in \mathcal{T}_1 \\ \displaystyle\int\limits_{t_1 - t_2}^{t_1} e^{-\theta(s, s - t_1 + t_2)} w(s, s - t_1 + t_2)\, ds & \text{for} \quad t \in \mathcal{T}_2 \end{cases} \tag{16}$$

almost everywhere in $\mathcal{T}$.

Proof. Formula (14) with $g(t) = J_\delta^{-1}(t) e^{-\theta(t)} w(t)$ implies that

$$J_\delta v = \int\limits_{t_2 - t_1}^{t_2} (J_\delta^{-1} e^{-\theta} w)(s + t_1 - t_2, s)\, ds + \delta \frac{\partial^2}{\partial t_2 \partial t_1} \int\limits_{t_2 - t_1}^{t_2} (J_\delta^{-1} e^{-\theta} w)(s + t_1 - t_2, s)\, ds, \qquad t \in \mathcal{T}_1.$$

Here in the second integral, we make the change of variables $\nu = s - t_2$, differentiate with respect to t_1 and t_2 by the formula for the derivative of a composite function, use the homogeneous Goursat conditions for the operator $J(t)$, and find that the mixed derivative of this integral is equal to

$$\int\limits_{-t_1}^{0} \frac{\partial^2 (J_\delta^{-1} e^{-\theta} w)}{\partial \eta\, \partial \mu} (\mu, \eta) \Bigg|_{\eta = \nu + t_2}^{\mu = \nu + t_1} d\nu.$$

In the last integral, we make the inverse change of variables $s = \nu + t_2$ and obtain relation (16) in the domain $\mathcal{T}_1$. Relation (16) in the domain $\mathcal{T}_2$ can be proved in a similar way. The proof of Lemma 1 is complete.

The existence of a constant $c_{10} > 0$ independent of v and δ in the estimate (13) follows from inequality (15) by virtue of relation (16) and Eq. (10). The proof of Corollary 2 is complete.

Corollary 3. *If $w \in \mathcal{H}_\gamma$, then the integral*

$$\int\limits_{\mathcal{T}} \gamma(t) |J_\delta^{-1}(t) h|^2\, dt < +\infty \qquad \forall h \in \mathcal{H}_\gamma \tag{17}$$

is uniformly bounded for all $\delta > 0$.

Proof. By setting $J_\delta v = h$, from (13) we obtain the estimate (17) for functions h of the form (16). For all functions $w \in \mathcal{H}_\gamma$, the set of functions $h = J_\delta v$, where $v = v_\delta(t)$ are the solutions of the Cauchy problem (10), (11), is dense in $\mathcal{H}_\gamma$. Indeed, by formulas (14) and (16), for all infinitely differentiable functions $w \in \hat{C}^\infty(\overline{\mathcal{T}}, H)$, the solutions of the Cauchy problem

$$\partial h/\partial t_1 + \partial h/\partial t_2 = e^{-\theta(t)} w, \quad t \in \mathcal{T}, \quad h(t_1, 0) = 0, \quad t_1 \in [0, T_1], \quad h(0, t_2) = 0, \quad t_2 \in [0, T_2],$$

are given by functions h in the space

$$\hat{C}^\infty_{[0]}(\overline{\mathcal{T}}, H) = \left\{ h \in \hat{C}^\infty(\overline{\mathcal{T}}, H) : \ h(t_1, 0) = 0, \ t_1 \in [0, T_1], \ h(0, t_2) = 0, \ t_2 \in [0, T_2] \right\},$$

which are brought by the equation of this Cauchy problem into functions $w \in \hat{C}^\infty(\overline{\mathcal{T}}, H)$. Since the set $\hat{C}^\infty_{[0]}(\overline{\mathcal{T}}, H)$ dense in $\mathcal{H}_\gamma$ is a set of second Baire category, it follows from the uniform boundedness principle (Corollary in [9, p. 108 of the Russian translation]) that the estimate (17) is indeed satisfied for all $h \in \mathcal{H}_\gamma$. The proof of Corollary 3 is complete.

By virtue of the estimate (17) and the Banach–Steinhaus theorem, we have the following assertion.

Corollary 4. *For each function $w \in \mathcal{H}_\gamma$, one has $\delta \gamma^{1/2}(t) J(t)(\partial v/\partial t_1 + \partial v/\partial t_2) \to 0$ in $\mathcal{H}$ as $\delta \to 0$.*

Remark 2. The integral smoothing operators $J_\delta^{-1}(t)$ differ from the integral smoothing operators in [10, 11] in that the operator $J(t) = \partial^2/\partial t_2 \partial t_1$ defined on $D(J)$ is not semibounded below or above in $\mathcal{H}$; consequently, one cannot used well-known proofs of the property that $J_\delta^{-1}(t) g \to g$ for all $g \in \mathcal{H}$ in $\mathcal{H}$ as $\delta \to 0$. There is a more important difference: the integral smoothing operators $J_\delta^{-1}(t)$ and the abstract smoothing operators $A_\delta^{-1}(t)$ converge to the identity operator in $\mathcal{H}_\gamma$ and $\mathcal{H}$, respectively, as $\delta \to 0$, while the Cauchy problem (10), (11) has smoothed right-hand side and converges to a Cauchy problem with nonsmooth right-hand side in $\mathcal{H}_\gamma$ as $\delta \to 0$. The assertion of Lemma 1 describes a typical property of the regularizing Cauchy problem (10), (11) as a smoothing operator of a new type: for each $\delta > 0$, the function $h = J_\delta v_\delta$ is a solution of the corresponding limit Cauchy problem in $\mathcal{H}$ as $\delta \to 0$. One can show that, for all $w \in \mathcal{H}_\gamma$, the Cauchy problem (10), (11) has smoother solutions $v_\delta \in C^{(1)}(\overline{\mathcal{T}}, H)$, $\partial^2 v_\delta/\partial t_2\, \partial t_1 \in \mathcal{H}$, by virtue of properties of the smoothing operators $J_\delta^{-1}(t)$ on the right-hand side in Eq. (10) and the homogeneous conditions

$$h(t_1, 0) = 0, \qquad t_1 \in [0, T_1]; \qquad h(0, t_2) = 0, \qquad t_2 \in [0, T_2],$$

for the functions $h \in D(J)$. In what follows, this fact is used in the proof of Lemma 2.

4. PROOF OF THEOREM 1

We take the inner product of Eq. (1) by $e^{\theta(t)} \gamma(t) \times (\partial u/\partial t_1 + \partial u/\partial t_2)$ in $\mathcal{H}$ and obtain the identity

$$\int_{\mathcal{T}} e^{\theta(t)} \gamma(t) \left(\mathcal{L}(t) u, \left(\frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2} \right) \right) dt = \sum_{i=1}^{2} \int_{\mathcal{T}} e^{\theta(t)} \gamma(t) \left(\frac{\partial^2 u}{\partial t_2 \partial t_1}, \frac{\partial u}{\partial t_i} \right) dt$$
$$+ \int_{\mathcal{T}} e^{\theta(t)} \gamma(t) \left(A_1(t) \frac{\partial u}{\partial t_1} + A_2(t) \frac{\partial u}{\partial t_2}, \frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2} \right) dt + \sum_{i=1}^{2} \int_{\mathcal{T}} e^{\theta(t)} \gamma(t) \left(A(t) u, \frac{\partial u}{\partial t_i} \right) dt \qquad (18)$$

for all $u \in D(L)$.

We integrate by parts with respect to t_j, $j \neq i$, in the first sum on the right-hand side in relation (18):

$$\int\limits_{\mathcal{T}} e^{\theta(t)} \left|\frac{\partial u}{\partial t_i}\right|^2 dt = 2\,\mathrm{Re} \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t) \left(\frac{\partial^2 u}{\partial t_2 \partial t_1}, \frac{\partial u}{\partial t_i}\right) dt$$
$$- \int\limits_0^{T_i} e^{\theta(t)}\gamma(t) \left|\frac{\partial u}{\partial t_i}\right|^2 \Bigg|_{t_j=0}^{t_j=T_j} dt_i - c\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t) \left|\frac{\partial u}{\partial t_i}\right|^2 dt, \qquad j \neq i, \quad i = 1,2. \tag{19}$$

By virtue of the estimate (3), the second integral on the right-hand side in relation (18) (we denote it by I_1) satisfies the inequality

$$-2c_3 \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t) \left(\left|\frac{\partial u}{\partial t_1}\right|^2 + \left|\frac{\partial u}{\partial t_2}\right|^2 \right) dt \leq 2\,\mathrm{Re}\, I_1. \tag{20}$$

In the second sum on the right-hand side in relation (18), we use the smoothing operators $A_\varepsilon^{-1}(t) = (I + \varepsilon A(t))^{-1}$, $\varepsilon > 0$, ranging in $D(A(t))$ and possessing the following properties [4].

1. $|A_\varepsilon^{-1}(t)g - g| \to 0$ uniformly with respect to t for all $g \in H$ as $\varepsilon \to 0$.
2. In the Hilbert space H, for all $t \in \mathcal{T}$, there exist strong derivatives

$$\partial A_\varepsilon^{-1}(t)/\partial t_i \in \mathcal{B}(\mathcal{T}, \mathcal{L}(H)),$$

and the relations

$$\frac{\partial (A(t)A_\varepsilon^{-1}(t))}{\partial t_i} = -A(t)A_\varepsilon^{-1}(t)\frac{\partial A^{-1}(t)}{\partial t_i}A(t)A_\varepsilon^{-1}(t), \qquad i = 1,2,$$

are satisfied.

By integrating once with respect to t_j, by letting ε tend to zero, and by taking into account the properties of the operators $A_\varepsilon^{-1}(t)$, we obtain the relation

$$\int\limits_{\mathcal{T}} e^{\theta(t)}(A(t)u,u)\,dt = 2\,\mathrm{Re} \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t) \left(A(t)u, \frac{\partial u}{\partial t_j}\right) dt - \int\limits_0^{T_i} e^{\theta(t)}\gamma(t)(A(t)u,u)\Big|_{t_j=0}^{t_j=T_j} dt_i$$
$$+ \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t) \left[-\left(\frac{\partial A^{-1}(t)}{\partial t_i}A(t)u, A(t)u\right) - c(A(t)u,u) \right] dt, \qquad j \neq i, \qquad j = 1,2.$$

This, together with the estimate (4), implies the inequalities

$$\int\limits_{\mathcal{T}} e^{\theta(t)}|A^{1/2}(t)u|^2\,dt \leq 2\,\mathrm{Re} \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t) \left(A(t)u, \frac{\partial u}{\partial t_j}\right) dt + (T_1+T_2)\int\limits_0^{T_i} e^{T_1+T_2-t_i}|A^{1/2}(t)u|^2\Big|_{t_j=0} dt_i$$
$$+ (c_4 - c)\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)|A^{1/2}(t)u|^2\,dt, \qquad j \neq i, \qquad j = 1,2. \tag{21}$$

We add relations (19)–(21) term by term, use identity (18), the Cauchy–Schwartz–Bunyakovskii inequality, and elementary transformations and estimates, and obtain inequality (5) for smooth solutions $u \in D(L)$ of the Goursat problem (1), (2). Then we generalize this inequality by passage to the limit to all strong solutions $u \in D(\overline{L})$. The proof of Theorem 1 is complete.

5. PROOF OF THEOREM 2

By Corollary 1, to prove the existence of strong solutions of the Goursat problem (1), (2) for arbitrary $\Phi \in F$, it suffices to show that the set $R(L)$ is dense in F. Thus, let some element $\Phi = \{w, \psi_1, \psi_2\} \neq 0$ of the dual space $F' = cH_\gamma \times H_1 \times H_2$ be orthogonal to the set $R(L)$; i.e., let

$$\int\limits_{\mathcal{T}} \gamma(t)(\mathcal{L}(t)u, w)dt + (l_1 u, \psi_1)_1 + (l_2 u, \psi_2)_2 = 0 \qquad \forall u \in D(L), \tag{22}$$

where $(\cdot\,,\cdot)_i$ is the inner products in H_i, $i = 1, 2$. By setting

$$u \in D_0(L) = \{u \in D(L) :\ l_1 u = l_2 u = 0\}$$

in this relation, we obtain

$$\int\limits_{\mathcal{T}} \gamma(t)(\mathcal{L}(t)u, w)\, dt = 0 \qquad \forall u \in D_0(L). \tag{23}$$

Lemma 2. *Let the assumptions of Theorem 2 be satisfied. If identity* (23) *holds for some function $w \in \mathcal{H}_\gamma$, then $w = 0$.*

Proof. In (23), we set $u = A^{-1}(t)v$, where $v = v_\delta(t)$ is the solution in $\mathcal{H}$ of the Cauchy problem (10), (11) with the function w defined in Lemma 2, take the double real part, and obtain the relation

$$\begin{aligned} 2\operatorname{Re}\Bigg[&\sum_{i=1}^{2} \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left(\frac{\partial^2 A^{-1}(t)}{\partial t_2 \partial t_1}v, \frac{\partial v}{\partial t_i}\right) dt + \sum_{i=1}^{2} \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left(A^{-1}(t)\frac{\partial^2 v}{\partial t_2 \partial t_1}, \frac{\partial v}{\partial t_i}\right) dt \\ &+ \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left(\frac{\partial A^{-1}(t)}{\partial t_1}\frac{\partial v}{\partial t_2} + \frac{\partial A^{-1}(t)}{\partial t_2}\frac{\partial v}{\partial t_1}, \frac{\partial v}{\partial t_1} + \frac{\partial v}{\partial t_2}\right) dt \\ &+ \sum_{i=1}^{2} \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left[\left(A_i(t)\frac{\partial A^{-1}(t)}{\partial t_i}v, \frac{\partial v}{\partial t_1} + \frac{\partial v}{\partial t_2}\right) + \left(A_i(t)A^{-1}(t)\frac{\partial v}{\partial t_i}, \frac{\partial v}{\partial t_1} + \frac{\partial v}{\partial t_2}\right)\right] dt \\ &+ \sum_{i=1}^{2} \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left(v, \frac{\partial v}{\partial t_i}\right) dt + \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\Phi_1(v, v)\, dt\Bigg] = 0, \end{aligned} \tag{24}$$

where $\Phi_1(v, v)$ is the sesquilinear form given by the relation

$$\begin{aligned} \Phi_1(v, v) = \Bigg(&\frac{\partial^2 A^{-1}(t)}{\partial t_2 \partial t_1}v + A^{-1}(t)\frac{\partial^2 v}{\partial t_2 \partial t_1} + \frac{\partial A^{-1}(t)}{\partial t_1}\frac{\partial v}{\partial t_2} + \frac{\partial A^{-1}(t)}{\partial t_2}\frac{\partial v}{\partial t_1} + A_1(t)\frac{\partial A^{-1}(t)}{\partial t_1}v \\ &+ A_2(t)\frac{\partial A^{-1}(t)}{\partial t_2}v + A_1(t)A^{-1}(t)\frac{\partial v}{\partial t_1} + A_2(t)A^{-1}(t)\frac{\partial v}{\partial t_2} + v, \delta J\left(\frac{\partial v}{\partial t_1} + \frac{\partial v}{\partial t_2}\right)\Bigg). \end{aligned}$$

By virtue of inequality (9), we have the estimates

$$-2\operatorname{Re}\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left(\frac{\partial^2 A^{-1}(t)}{\partial t_2 \partial t_1}v, \frac{\partial v}{\partial t_i}\right) dt \le c_9 \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left[\left|A^{-1/2}(t)\frac{\partial v}{\partial t_i}\right|^2 + |v|^2\right] dt, \quad i = 1, 2. \tag{25}$$

By integrating by parts with respect to t_j, $j \neq i$, and by taking into account the homogeneous Goursat conditions, we obtain

$$-2\,\mathrm{Re}\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left(A^{-1}(t)\frac{\partial^2 v}{\partial t_2 \partial t_1}, \frac{\partial v}{\partial t_i}\right) dt \leq \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left(\frac{\partial A^{-1}(t)}{\partial t_i}\frac{\partial v}{\partial t_j}, \frac{\partial v}{\partial t_j}\right) dt - c\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left|A^{1/2}(t)\frac{\partial v}{\partial t_i}\right|^2 dt, \qquad j \neq i, \qquad i = 1, 2. \tag{26}$$

By virtue of the estimate (8), we have the inequalities

$$-2\,\mathrm{Re}\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left(A_i(t)\frac{\partial A^{-1}(t)}{\partial t_i}v, \frac{\partial v}{\partial t_1} + \frac{\partial v}{\partial t_2}\right) dt \leq c_8\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left[\sum_{i=1}^{2}\left|A^{-1/2}(t)\frac{\partial v}{\partial t_i}\right|^2 + 2|v|^2\right] dt, \qquad i = 1, 2. \tag{27}$$

By virtue of the estimate (7), we obtain the inequality

$$-2\,\mathrm{Re}\sum_{i=1}^{2}\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left(A_i(t)A^{-1}(t)\frac{\partial v}{\partial t_i}, \frac{\partial v}{\partial t_1} + \frac{\partial v}{\partial t_2}\right) dt \leq 2c_7\sum_{i=1}^{2}\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left|A^{-1/2}(t)\frac{\partial v}{\partial t_i}\right|^2 dt. \tag{28}$$

By integrating by parts with respect to t_i and by taking into account the homogeneous Goursat conditions, we arrive at the estimates

$$-2\,\mathrm{Re}\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left(v, \frac{\partial v}{\partial t_i}\right) dt \leq -c\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)|v|^2\, dt, \qquad i = 1, 2. \tag{29}$$

Relation (24), together with the estimates (25)–(29), implies the inequality

$$\sum_{i=1}^{2}\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left|A^{-1/2}(t)\frac{\partial v}{\partial t_i}\right|^2 dt \leq (1 + 2c_7 + 2c_8 + c_9 - c)\sum_{i=1}^{2}\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left|A^{-1/2}(t)\frac{\partial v}{\partial t_i}\right|^2 dt + (2c_9 + 4c_8 - 2c)\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)|v|^2\, dt - 2\,\mathrm{Re}\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\Phi_1(v,v)\, dt + \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\Phi_2(v,v)\, dt, \tag{30}$$

where, by virtue of the estimate (4), we have the relations

$$\begin{aligned}
\Phi_2(v,v) &= -\left(\frac{\partial A^{-1}(t)}{\partial t_1}\frac{\partial v}{\partial t_2}, \frac{\partial v}{\partial t_2}\right) - \left(\frac{\partial A^{-1}(t)}{\partial t_2}\frac{\partial v}{\partial t_1}, \frac{\partial v}{\partial t_1}\right) \\
&\quad - 2\,\mathrm{Re}\left(\frac{\partial A^{-1}(t)}{\partial t_1}\frac{\partial v}{\partial t_2}, \frac{\partial v}{\partial t_1}\right) - 2\,\mathrm{Re}\left(\frac{\partial A^{-1}(t)}{\partial t_2}\frac{\partial v}{\partial t_1}, \frac{\partial v}{\partial t_2}\right) \\
&= -\left(\frac{\partial A^{-1}(t)}{\partial t_1}\left(\frac{\partial v}{\partial t_1} + \frac{\partial v}{\partial t_2}\right) + \frac{\partial A^{-1}(t)}{\partial t_2}\left(\frac{\partial v}{\partial t_1} + \frac{\partial v}{\partial t_2}\right), \frac{\partial v}{\partial t_1} + \frac{\partial v}{\partial t_2}\right) \\
&\quad + \left(\frac{\partial A^{-1}(t)}{\partial t_1}\frac{\partial v}{\partial t_1}, \frac{\partial v}{\partial t_1}\right) + \left(\frac{\partial A^{-1}(t)}{\partial t_2}\frac{\partial v}{\partial t_2}, \frac{\partial v}{\partial t_2}\right) \leq 5c_4\sum_{i=1}^{2}\left|A^{-1/2}(t)\frac{\partial v}{\partial t_i}\right|^2.
\end{aligned} \tag{31}$$

By integrating by parts with respect to t_i, we find that all solutions $v \in C^{(3)}(\overline{\mathcal{T}})$ of the Cauchy problem (10), (11) satisfy the identities

$$\begin{aligned} &-2\operatorname{Re}\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left(A^{-1}(t)\frac{\partial^2 v}{\partial t_2\partial t_1}, J\frac{\partial v}{\partial t_i}\right) dt \\ &\quad= -\int\limits_0^{T_j} e^{\theta(t)}\gamma(t)\left|A^{-1/2}(t)\frac{\partial^2 v}{\partial t_2\partial t_1}\right|^2\Bigg|_{t_i=0}^{t_i=T_i} dt_j + \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left(\frac{\partial A^{-1}(t)}{\partial t_i}\frac{\partial^2 v}{\partial t_2\partial t_1}, \frac{\partial^2 v}{\partial t_2\partial t_1}\right) dt \\ &\qquad -\int\limits_{\mathcal{T}} e^{\theta(t)}\left|A^{-1/2}(t)\frac{\partial^2 v}{\partial t_2\partial t_1}\right|^2 dt - c\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left|A^{-1/2}(t)\frac{\partial^2 v}{\partial t_2\partial t_1}\right|^2 dt, \quad j\neq i, \quad i=1,2. \end{aligned} \tag{32}$$

By differentiating Eq. (10) with respect to t_j, we find the traces

$$(\partial^2 v/\partial t_2\partial t_1)|_{t_i=0} = (\partial(J_\delta^{-1}e^{-\theta(t)}w)/\partial t_j)|_{t_i=0} - (\partial^2 v/\partial t_j^2)|_{t_i=0} = 0, \qquad j\neq i, \quad i=1,2,$$

according to the homogeneous Goursat conditions for the operator J and the initial conditions (11). We extend the term-by-term sum of two identities (32) to all solutions of the Cauchy problem (10), (11) by passage to the limit; then on the left-hand side in the resulting relation, we obtain the corresponding term from the right-hand side of relation (30). By virtue of inequalities (4), the right-hand side of identity (32) does not exceed zero for $c > c_4$.

By using the Cauchy–Schwarz inequality and by taking into account assumption 2, we obtain the estimates

$$\begin{aligned} &-2\operatorname{Re}\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left(A_i(t)A^{-1}(t)\frac{\partial v}{\partial t_i}, \delta J\left(\frac{\partial v}{\partial t_1}+\frac{\partial v}{\partial t_2}\right)\right) dt \\ &\qquad \le c_i\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left|A^{-1/2}(t)\frac{\partial v}{\partial t_i}\right|^2 dt + \int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left|\delta J\left(\frac{\partial v}{\partial t_1}+\frac{\partial v}{\partial t_2}\right)\right|^2 dt, \qquad i=1,2. \end{aligned} \tag{33}$$

By using inequality (6) and elementary estimates, we obtain the relations

$$\begin{aligned} &\left(\frac{\partial A^{-1}(t)}{\partial t_1}\frac{\partial v}{\partial t_2} + \frac{\partial A^{-1}(t)}{\partial t_2}\frac{\partial v}{\partial t_1}, \delta J\left(\frac{\partial v}{\partial t_1}+\frac{\partial v}{\partial t_2}\right)\right) \\ &\qquad\qquad \le \frac{1}{4}\sum_{i=1}^{2}\left|A^{-1/2}\frac{\partial v}{\partial t_i}\right|^2 + 4c_6^2\left|\delta J\left(\frac{\partial v}{\partial t_1}+\frac{\partial v}{\partial t_2}\right)\right|^2. \end{aligned} \tag{34}$$

In inequality (30), we estimate the left-hand side from below and the right-hand side from above by using relations (31)–(34) and by omitting all nonpositive terms for

$$c = c_{11} = 1 + c_1 + c_2 + 5c_4 + 2c_7 + 2c_8 + c_9.$$

As a result, we obtain the inequality

$$\frac{1}{4}\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\left|A^{-1/2}(t)\left(\frac{\partial v}{\partial t_1}+\frac{\partial v}{\partial t_2}\right)\right|^2 dt \le 2\operatorname{Re}\int\limits_{\mathcal{T}} e^{\theta(t)}\gamma(t)\Psi(v,v)\,dt, \tag{35}$$

where

$$\Psi(v,v) = \left((1+4c_5^2)\delta J\left(\frac{\partial v}{\partial t_1}+\frac{\partial v}{\partial t_2}\right) - \frac{\partial^2 A^{-1}(t)}{\partial t_2\partial t_1}v - \sum_{i=1}^{2}A_i(t)\frac{\partial A^{-1}(t)}{\partial t_i}v - v, \delta J\left(\frac{\partial v}{\partial t_1}+\frac{\partial v}{\partial t_2}\right)\right).$$

Since the operators $\partial^2 A^{-1}(t)/\partial t_2 \partial t_1$ and $A_i(t)\partial A^{-1}(t)/\partial t_i$, $i = 1, 2$, are bounded in $L_\infty(\mathcal{T}, H)$, and, by virtue of the estimate (13), the functions $\gamma^{1/2}(t)v_\delta(t)$, $i = 1, 2$, are bounded uniformly with respect to δ in $\mathcal{H}$, it follows from Corollary 4 and Eq. (10) that, by passing in inequality (35) to the limit as $\delta \to 0$, we obtain the inequality $(1/4)\int_{\mathcal{T}} e^{-\theta(t)}\gamma(t)|A^{-1/2}(t)w|^2\,dt \le 0$. Hence $w = 0$. The proof of Lemma 2 is complete.

By using Lemma 2 in identity (22), we conclude that $v = 0$ and $(l_1 u, \psi_1)_1 + (l_2 u, \psi_2)_2 = 0$ for all $u \in D(L)$. By setting $u = A^{-1}(t)(\psi_1(t_1) + \psi_2(t_2) - \psi_1(0))$ in the last relation, we obtain

$$\sum_{i=1}^{2} \int_0^{T_i} \left[\left(\frac{\partial A^{-1}(t)}{\partial t_i}\,\psi_i(t_i), \frac{\partial \psi_i(t_i)}{\partial t_i} \right) + \left| A^{-1/2}(t)\frac{\partial \psi_i(t_i)}{\partial t_i} \right|^2 + |\psi_i(t_i)|^2 \right] \Bigg|_{\substack{t_j=0\\ j\ne i}} dt_i = 0.$$

It follows from the estimate (5) and the condition $\psi_1(0) = \psi_2(0)$ that

$$(1 - c_6^2/4)\sum_{i=1}^{2}\int_0^{T_i} |\psi_i|^2\,dt_i \le 0,$$

which, for $c_6 < 2$, implies that $\psi_1 = 0$ and $\psi_2 = 0$. The proof of Theorem 2 is complete.

Remark 3. If $B_i(t)$, $B(t)A^{-1/2}(t) \in L_\infty(\mathcal{T}, \mathfrak{L}(H))$, $i = 1, 2$, then the assertions of Theorems 1 and 2 remain valid (possibly, with a larger constant c_{10}) for the complete operator-differential equation

$$\mathcal{L}(t)u(t) + B_1(t)\frac{\partial u(t)}{\partial t_1} + B_2(t)\frac{\partial u(t)}{\partial t_2} + B(t)u(t) = f(t)$$

with lower-order terms, because the lower-order terms are bounded operators from E to $\mathcal{H}_\gamma$.

6. PARTIALLY CHARACTERISTIC BOUNDARY VALUE PROBLEM

Consider the problem of finding solutions of the complete hyperbolic equation

$$\begin{aligned}\mathcal{L}(t)u \equiv \frac{\partial^2 u(t,x)}{\partial t_2 \partial t_1} + a_1(t,x)\frac{\partial^2 u(t,x)}{\partial x\,\partial t_1} + a_1(t,x)\frac{\partial^2 u(t,x)}{\partial x\,\partial t_2} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u(t,x)}{\partial x}\right)\\ + b_1(t,x)\frac{\partial u(t,x)}{\partial t_1} + b_2(t,x)\frac{\partial u(t,x)}{\partial t_2} + b_0(t,x)\frac{\partial u(t,x)}{\partial x} + d_0(t,x)u(t,x) = f(t,x) \qquad (36)\end{aligned}$$

in the domain $G =]0, T_1[\times]0, T_2[\times]0, l[$ of the variables $t = \{t_1, t_2\}$ and x with the boundary conditions

$$u(t,0) = 0, \qquad \frac{\partial u(t,l)}{\partial x} + \beta(t)u(t,l) = 0, \qquad t \in \overline{\mathcal{T}} = [0,T_1]\times[0,T_2], \qquad (37)$$

depending on the two-dimensional time t and the coordinated Picard conditions

$$u(t,x)|_{t_2=0} = \varphi_1(t_1), \quad t_1 \in [0,T_1], \quad u(t,x)|_{t_1=0} = \varphi_2(t_2), \quad t_2 \in [0,T_2], \quad \varphi_1(0) = \varphi_2(0). \qquad (38)$$

One can show that the assumptions of Theorems 1 and 2 and Remark 3 hold for the boundary value problem (36)–(38) provided that the coefficients satisfy the conditions $a_1(t,x) \in C^{(1)}(\overline{G})$, $a_1(t,0) = 0$, $a_1(t,l) \ge 0$, $t \in \mathcal{T}$, $b_i(t,x)$, $d_0(t,x) \in C(\overline{G})$, $i = 0, 1, 2$, $\beta(t) \in C^{(2)}(\overline{\mathcal{T}})$, $\beta(t) \ge 0$, $t \in \mathcal{T}$, and

$$\left|\frac{\partial \beta}{\partial t_i}\right| \le 2l^{-1/2}c_{12}^{-1/2}\left(1 + a(l)\beta(t)\int_0^l \frac{ds}{a(s)}\right)^2 \left(a(l)\int_0^l \frac{ds}{a(s)}\right)^{-1}, \qquad t \in \mathcal{T}, \qquad i = 1, 2,$$

where $c_{12} = 2\max\Big\{a(l)\left(\int_0^l a(s)^{-1}\,ds\right)^2 \sup_{t\in\mathcal{T}}(1+\beta(t)),\ l a_0^{-1}\Big\}$.

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