ISSN 0012-2661, Differential Equations, 2009, Vol. 45, No. 8, pp. 1148–1167. © Pleiades Publishing, Ltd., 2009.
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PARTIAL DIFFERENTIAL EQUATIONS

Generalization of the Lions Theory for First-Order Evolution Differential Equations with Discontinuous Operators and with $\alpha \in [1/2, 1]$

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Received March 5, 2007

Abstract—We prove existence, uniqueness, and smoothness theorems for weak solutions of the problem

$$du(t)/dt + A(t)u(t) = f(t), \qquad t \in]0, T[; \qquad u(0) = u_0 \in H,$$

where, for almost all t, the linear unbounded operators $A(t)$ with domains $D(A(t))$ depending on t are closed and maximal accretive and have bounded inverses $A^{-1}(t)$ discontinuous with respect to t in the Hilbert space H. There exists an $\alpha \in [1/2, 1]$ such that the following is true in H for almost all t: the power $A^{\alpha}(t)$ is subordinate to the power $A^{*\alpha}(t)$ of the adjoint operators $A^*(t)$, the operators $A^{\alpha}(t)$ and $A^{*\alpha}(t)$ do not form an obtuse angle, and the domains $D(A^{*\alpha}(t))$ of the operators $A^{*\alpha}(t)$ are not increasing with respect to t.

This paper is the first to prove the well-posedness of the mixed problem for the multidimensional linearized Korteweg–de Vries equation smooth in time with boundary conditions piecewise constant in time.

DOI: 10.1134/S0012266109080072

In the present paper, we generalize the well-known Theorems 5.1 and 5.2 in [1, pp. 62–65] on the existence and uniqueness of solutions of the Cauchy problem for first-order operator-differential equations with variable domains of discontinuous positive self-adjoint operator coefficients with exponent $\alpha = 1/2$ to the case of discontinuous accretive operator coefficients with exponents $1/2 \leq \alpha \leq 1$. We prove a local smoothness increasing theorem for the weak solutions of this Cauchy problem.

1. STATEMENT OF THE CAUCHY PROBLEM

In a Hilbert space H with inner product $(\cdot\,,\cdot)$ and norm $|\cdot|$, we consider the operator-differential equation

$$\frac{du(t)}{dt} + A(t)\,u(t) = f(t), \qquad t \in]0, T[, \tag{1}$$

with the initial condition

$$u(0) = u_0, \tag{2}$$

where u and f are functions of the variable t with values in H and the $A(t)$ are linear unbounded operators in H that are defined for almost all $t \in]0, T[$ and have domains $D(A(t))$ depending on t. The operators $A(t)$ are subjected to the following conditions.

I. For almost all t, the closed operators $A(t)$ are maximal accretive in H, or, which is the same, the closed operators $-A(t)$ are maximal dissipative in H; i.e., for almost all t the operators $A(t)$ with dense domains $D(A(t))$ in H and their adjoint operators $A^*(t)$ in H with domains $D(A^*(t))$ depending on t satisfy the inequalities

$$\operatorname{Re}(A(t)u, u) \geq 0 \qquad \forall u \in D(A(t)), \tag{3}$$

$$\operatorname{Re}(A^*(t)v, v) \geq 0 \qquad \forall v \in D(A^*(t)). \tag{4}$$

For almost all t, the operators $A(t)$ have bounded inverses

$$A^{-1}(t) \in L_\infty(]0,T[,\mathfrak{L}(H)).$$

These inequalities are necessary and sufficient conditions for the maximal dissipativity (accretivity) of the closed operators $-A(t)$ and $-A^*(t)$ [respectively, $A(t)$ and $A^*(t)$] in H [2, p. 109]. By [3], the maximal accretive operators $A(t)$ and $A^*(t)$ have maximal accretive fractional powers $A^\beta(t)$ and $A^{*\beta}(t)$, $0<\beta<1$, with domains $D(A^\beta(t))$ and $D(A^{*\beta}(t))$, respectively.

II. There exists a number $\alpha \in [1/2,1]$ (the case in which $\alpha = 1/2$ and the $A(t)$ are self-adjoint operators was considered in [1]) such that the operators $A^\alpha(t)$ are subordinate to the operators $A^{*\alpha}(t)$ for almost all t; i.e., there exists a constant $c_1(\alpha)>0$ independent of w and t such that

$$|A^\alpha(t)w| \le c_1(\alpha)|A^{*\alpha}(t)w| \qquad \forall w \in D(A^{*\alpha}(t)) \tag{5}$$

for almost all t. The operators $A^\alpha(t)$ and $A^{*\alpha}(t)$ do not form an obtuse angle for almost all t; i.e., by Definition 1 below and condition (5),

$$\operatorname{Re}(A^\alpha(t)w, A^{*\alpha}(t)w) \ge 0 \qquad \forall w \in D(A^{*\alpha}(t)) \tag{6}$$

for almost all t.

Definition 1. We say that operators B_1 and B_2 with domains $D(B_1)$ and $D(B_2)$, respectively, in a Hilbert space H *do not form an obtuse angle* if

$$\operatorname{Re}(B_1 w, B_2 w) \ge 0 \qquad \forall w \in D(B_1)\cap D(B_2).$$

III. For almost all t, the operators $A^{2\alpha-1}(t)$ and $A^{*2\alpha-1}(t)$ are the restrictions to their domains $D(A^{2\alpha-1}(t))$ and $D(A^{*2\alpha-1}(t))$ of some linear operators $\tilde{A}^{2\alpha-1}(t)$ and $\tilde{A}^{*2\alpha-1}(t)$, $t\in[0,T]$, whose domains $D(\tilde{A}^{2\alpha-1})$ and $D(\tilde{A}^{*2\alpha-1})$, respectively, are independent of t and which have strong derivatives in $H^+_{1-\alpha,t}$ with respect to t in the strong sense [2, p. 218] for almost all t,

$$\begin{gathered} d\tilde{A}^{2\alpha-1}(t)/dt \in L_\infty(]0,T[,\mathfrak{L}(D(A^\alpha(t)),H^+_{1-\alpha,t})), \\ d\tilde{A}^{*2\alpha-1}(t)/dt \in L_\infty(]0,T[,\mathfrak{L}(D(A^{*\alpha}(t)),H^+_{1-\alpha,t})), \end{gathered}$$

such that

$$\begin{gathered} |(((d\tilde{A}^{2\alpha-1}(t)/dt) + (d\tilde{A}^{*2\alpha-1}(t)/dt))w,w)| \le c_2'(\alpha)((A^{2\alpha-1}(t)+A^{*2\alpha-1}(t))w,w) \\ \forall w \in D(A^{2\alpha-1}(t))\cap D(A^{*2\alpha-1}(t)), \end{gathered} \tag{7}$$

$$\begin{gathered} [(d\tilde{A}^{2\alpha-1}(t)/dt)u]_{1-\alpha,t} \le c_2(\alpha)|A^\alpha(t)u| \qquad \forall u \in D(A^\alpha(t)), \\ [(d\tilde{A}^{*2\alpha-1}(t)/dt)v]_{1-\alpha,t} \le c_2(\alpha)|A^{*\alpha}(t)v| \qquad \forall v \in D(A^{*\alpha}(t)) \end{gathered} \tag{8}$$

for almost all t, where the Hilbert spaces $H^+_{1-\alpha,t}$ are the domains $D(A^{1-\alpha}(t))$ of the powers $A^{1-\alpha}(t)$ equipped with the Hermitian norms $[u]_{1-\alpha,t} = |A^{1-\alpha}(t)u|$, equivalent to the corresponding graph norms, and the constants $c_2'(\alpha)\ge 0$ and $c_2(\alpha)\ge 0$ are independent of w, u, v, and t.

IV. There exists a constant $c^*>0$ such that $|A^{*\alpha}(t)v| \le c^*|A^{*\alpha}(s)v|$ for all $v\in D(A^{*\alpha}(s))$ and for almost all $s\in\,]t,T]$, $t\in[0,T]$. This means that the family of domains $D(A^{*\alpha}(t))$ is nonincreasing uniformly for almost all t; i.e.,

$$D(A^{*\alpha}(t_1)) \supset D(A^{*\alpha}(t_2)), \qquad 0<t_1<t_2<T, \tag{9}$$

uniformly for almost all t_1 and t_2. The operators $A^{1-\alpha}(t)$ and $A^{*\alpha}(t)$ for almost all t are obtained as the restriction to $D(A^{1-\alpha}(t))$ and $D(A^{*\alpha}(t))$, respectively, of some linear operators

$\tilde{A}^{1-\alpha}(t) \in \mathcal{B}([0,T], \mathfrak{L}(D(\tilde{A}^{1-\alpha}), H))$ and $\tilde{A}^{*\alpha}(t) \in \mathcal{B}([0,T], \mathfrak{L}(D(\tilde{A}^{*\alpha}), H))$ whose domains $D(\tilde{A}^{1-\alpha})$ and $D(\tilde{A}^{*\alpha})$ are independent of t and which have weak derivatives

$$d\tilde{A}^{1-\alpha}(t)/dt \in L_\infty(]0,T[, \mathfrak{L}(D(\tilde{A}^{1-\alpha}), H)), \qquad d\tilde{A}^{*\alpha}(t)/dt \in L_\infty(]0,T[, \mathfrak{L}(D(\tilde{A}^{*\alpha}), H))$$

with respect to t in the weak sense [4, p. 172 of the Russian translation] for almost all t in H such that

$$\begin{aligned} |(d\tilde{A}^{1-\alpha}(t)/dt)u| &\le c_3(\alpha)|A^{1-\alpha}(t)u| \qquad \forall u \in D(A^{1-\alpha}(t)), \\ |(d\tilde{A}^{*\alpha}(t)/dt)v| &\le c_4(\alpha)|A^{*\alpha}(t)v| \qquad \forall v \in D(A^{*\alpha}(t)), \end{aligned} \tag{10}$$

where the constants $c_3(\alpha) \ge 0$ and $c_4(\alpha) \ge 0$ are independent of u, v, and t. For almost all t, the domains $D(A^{*2}(t))$ of the squares $A^{*2}(t)$ of the operators $A^*(t)$ are dense in H, and the powers $A^{*2\alpha+1}(t)$, $A^{*2\alpha-3}(t)$, and $A^{*2}(t)$ are accretive on $D(A^{*2}(t))$ in H; i.e.,

$$\operatorname{Re}(A^{*2\alpha+1}(t)v, v) \ge 0, \qquad \operatorname{Re}(A^{*2\alpha-3}(t)v, v) \ge 0, \qquad \operatorname{Re}(A^{*2}(t)v, v) \ge 0 \tag{11}$$

for almost all t and for arbitrary $v \in D(A^{*2}(t))$. Here the Banach spaces $D(\tilde{A}^{1-\alpha})$ and $D(\tilde{A}^{*\alpha})$ are the same domains with the graph norms of the operators $\tilde{A}^{1-\alpha}(t_0)$ and $\tilde{A}^{*\alpha}(t_0)$, respectively, for some $t_0 \in [0,T]$, and $\mathcal{B}([0,T], E)$ is the set of all bounded functions of $t \in [0,T]$ ranging in a Banach space E with the norm $\|h(t)\|_{\mathcal{B}} = \sup_{t\in[0,T]} \|h(t)\|_E$.

In the present paper, we first prove the existence Theorem 1, the uniqueness Theorem 3, and Theorem 4 on the local smoothness of weak solutions (in the sense of Definition 2) of the Cauchy problem (1), (2). Then, by using them, in a similar way, we suggest to prove the existence Theorem 5 and the uniqueness Theorem 6 for weak solutions (in the sense of Definition 4) of the mixed problem (36)–(38) for the multidimensional linear KdV equation.

Remark 1. For positive self-adjoint boundedly invertible operators $A(t)$, conditions I and II are always satisfied for $c_1(\alpha) = 1$ and for any $\alpha \in [1/2, 1]$, condition III disappears for $\alpha = 1/2$, and in condition IV inequalities (11) hold for any $\alpha \in [1/2, 1]$, the embeddings (9) with $\alpha = 1/2$ coincide with the embeddings (5.8) in [1, p. 63], and inequalities (10) with $\alpha = 1/2$ degenerate into a single inequality that provides the validity of the estimate in [1, p. 64] for the derivative $a'(t; u, v)$ of the sesquilinear forms $a(t; u, v)$ with respect to t. In the case of inequalities (10), we have only the inessential difference that in [1] the constraints on the smoothness of the operators $A(t)$ with respect to t are expressed only by constraints on the t-smoothness of the forms $a(t; u, v)$ generating the operators $A(t)$. However, if $\alpha = 1/2$, then, instead of the subordination of the norm of values of the operators $A^{1/2}(t)$ on $D(A^{1/2}(t))$ in condition IV, the more restrictive condition of their equality for all $t \in [0,T]$ is imposed in [1]. Therefore, conditions I–IV indeed generalize the Lions assumptions for positive self-adjoint operators $A(t)$ and $\alpha = 1/2$ to the case of maximal accretive operators $A(t)$ and $\alpha \in [1/2, 1]$.

2. EXISTENCE THEOREM

First, we introduce the spaces and the notion of weak solutions of the original Cauchy problem. Suppose that the family of Hilbert spaces $H^+_{1-\alpha,t}$ in Section 1 is measurable and square integrable with respect to $t \in]0,T[$ [1, p. 62]. Then the space of weak solutions is the Hilbert space $\mathcal{H}^+_{1-\alpha} = L_2(]0,T[, H^+_{1-\alpha,t})$. Let the set of antidual Hilbert spaces $H^{*-}_{\alpha,t}$ of the Hilbert spaces $H^{*+}_{\alpha,t}$ obtained by equipping the domains $D(A^{*\alpha}(t))$ with the Hermitian norms $]v[_{\alpha,t} = |A^{*\alpha}(t)v|$ equivalent to the graph norms of the operators $A^{*\alpha}(t)$ be measurable and square integrable with respect to $t \in]0,T[$. For the space of right-hand sides of Eq. (1), we take the antidual Hilbert space $\mathcal{H}^{*-}_\alpha = L_2(]0,T[, H^{*-}_{\alpha,t})$ of the Hilbert space $\mathcal{H}^{*+}_\alpha = L_2(]0,T[, H^{*+}_{\alpha,t})$. Set $\mathcal{H} = L_2(]0,T[, H)$.

Definition 2. A function $u \in \mathcal{H}^+_{1-\alpha}$ is called a *weak solution* of the Cauchy problem (1), (2) for $f \in \mathcal{H}^{*-}_\alpha$ and $u_0 \in H$ if the integral identity

$$\int_0^T \left\{ (A^{1-\alpha}(t)u(t), A^{*\alpha}(t)\varphi(t)) - \left(u(t), \frac{d\varphi(t)}{dt} \right) \right\} dt = \int_0^T \langle f(t), \varphi(t) \rangle_{\alpha,t}\, dt + (u_0, \varphi(0))$$

holds for any function $\varphi \in \tilde{\Phi} \equiv \{\tilde{\varphi} \in \mathcal{H} : \ \tilde{\varphi}(t) \in D(A^{*\alpha}(t))$ for almost all t; the weak derivative $d\tilde{\varphi}/dt$ and $A^{*\alpha}(t)\tilde{\varphi}$ lie in $\mathcal{H}$; $\tilde{\varphi}(T) = 0\}$, where the $\langle \cdot\,, \cdot\rangle_{\alpha,t}$ are the sesquilinear forms specifying the antiduality of the spaces $H^{*+}_{\alpha,t}$ and $H^{*-}_{\alpha,t}$.

Let us now prove the existence theorem for weak solutions of the Cauchy problem (1), (2), whose special case, by virtue of Remark 1, is given by Theorem 5.1 in [1, p. 63].

Theorem 1. *If conditions* I–III *are valid, then for any* $f \in \mathcal{H}^{*-}_{\alpha}$ *and* $u_0 \in H$, *there exists a weak solution* $u \in \mathcal{H}^{+}_{1-\alpha}$ *of the Cauchy problem* (1), (2).

Proof. To prove the theorem, it suffices to use the projection Theorem 2 in [1, p. 37].

Theorem 2. *Let* F *be a Hilbert space with Hermitian norm* $\|\cdot\|_F$, *and let* Φ *be a pre-Hilbert space with Hermitian norm* $|||\cdot|||$ *continuously embedded in* F; *i.e., there exists a constant* $c_5 > 0$ *such that* $\|\varphi\|_F \le c_5|||\varphi|||$ *for all* $\varphi \in \Phi$. *Let* $E(w, \varphi)$ *be a given sesquilinear form on* $F \times \Phi$ *continuous with respect to* w *on* F *for each* $\varphi \in \Phi$, *and suppose that there exists a constant* $c_6 > 0$ *such that* $|E(\varphi, \varphi)| \ge c_6|||\varphi|||^2$ *for all* $\varphi \in \Phi$. *If the antilinear functional* $L(\varphi)$ *is continuous with respect to* φ *on* Φ, *then there exists an element* $w \in F$ *satisfying the equation* $E(w, \varphi) = L(\varphi)$ *for all* $\varphi \in \Phi$.

On the Hilbert space $F = \mathcal{H}^{+}_{1-\alpha}$ with Hermitian norm $\|w\|_F = \left(\int_0^T [w(t)]^2_{1-\alpha,t}\, dt\right)^{1/2}$ and on the pre-Hilbert space Φ formed by all functions $\varphi(t) \in \mathcal{H}$ such that $\varphi^*(t) = [\aleph_{2\alpha-1}(t) + I]^{-1}\varphi(t) \in \Phi^* \equiv \{\varphi^*(t) \in \mathcal{H} : \ \varphi^*(t) \in D(A^{*\alpha}(t))$ for almost all t; the weak derivative $d\varphi^*(t)/dt$ and $A^{*\alpha}(t)\varphi^*(t)$ belong to $\mathcal{H}$; $\varphi^*(T) = 0\}$ with the Hermitian norm $|||\varphi||| = \left(\int_0^T |A^{*\alpha}(t)\varphi^*(t)|^2\, dt + |\varphi^*(0)|^2\right)^{1/2}$, we choose the following sesquilinear form and antilinear functional:

$$E(w, \varphi) = \int_0^T e^{2ct}\left\{(A^{1-\alpha}(t)w(t), A^{*\alpha}(t)\varphi^*(t)) - \left(w(t), \frac{d\varphi^*(t)}{dt}\right)\right\} dt,$$

$$L(\varphi) = \int_0^T \langle f(t), \varphi^*(t)\rangle_{\alpha,t}\, dt + (u_0, \varphi^*(0)).$$

Here the self-adjoint operators

$$\aleph_{2\alpha-1}(t) = (A^{2\alpha-1}(t) + A^{*2\alpha-1}(t))/2$$

in H with domains $D(A^{2\alpha-1}(t)) \cap D(A^{*2\alpha-1}(t))$ are the so-called real parts of the corresponding sesquilinear forms [2, pp. 122–127; 3], and the operators $[\aleph_{2\alpha-1}(t) + I]^{-1}$ are the bounded inverses of $\aleph_{2\alpha-1}(t) + I$ in H.

By virtue of the estimates (5) and the existence of bounded inverses $A^{-1}(t) \in L_\infty(]0, T[, \mathfrak{L}(H))$ and hence of the powers $A^{-\beta}(t), A^{*-\beta}(t) \in L_\infty(]0, T[, \mathfrak{L}(H))$, $0 < \beta < 1$ [3], for almost all t, the continuous embedding of the spaces Φ in F with the constant

$$c_5 = (1/2)c_1(\alpha) + ((1/2) + \operatorname*{ess\,sup}_{0<t<T} \|A^{*1-2\alpha}(t)\|_{\mathfrak{L}(H)}) \tan(\pi(3 - 2\alpha)/4)$$

for $1/2 < \alpha \le 1$ follows from the inequalities

$$\begin{aligned} |A^{1-\alpha}(t)\varphi| &= |A^{1-\alpha}(t)[\aleph_{2\alpha-1}(t) + I]\varphi^*| \\ &\le (1/2)|[A^{2\alpha-1}(t) + A^{*2\alpha-1}(t)]\varphi^*| + |A^{1-\alpha}(t)\varphi^*| \\ &\le (1/2)|A^{\alpha}(t)\varphi^*| + (1/2)\tan(\pi(3 - 2\alpha)/4)|A^{*\alpha}(t)\varphi^*| \\ &\quad + \tan(\pi(3 - 2\alpha)/4)|A^{*1-\alpha}(t)\varphi^*| \\ &\le [(1/2)c_1(\alpha) + (1/2)\tan(\pi(3 - 2\alpha)/4) \\ &\quad + \operatorname*{ess\,sup}_{0<t<T} \|A^{*1-2\alpha}(t)\|_{\mathfrak{L}(H)} \tan(\pi(3 - 2\alpha)/4)]|A^{*\alpha}(t)\varphi^*|, \end{aligned} \tag{12}$$

since, by Theorem 1.1 in [3], the closed maximal accretive operators $A(t)$ satisfy the inequalities

$$|A^{1-\alpha}(t)w| \le \tan(\pi(3-2\alpha)/4)|A^{*1-\alpha}(t)w| \quad \forall w \in D(A^{*1-\alpha}(t)) \tag{13}$$

for $1-\alpha < 1/2$ and for almost all t. Obviously, in the case of positive self-adjoint operators $A(t)$ and $\alpha = 1/2$, the embedding of the spaces Φ in F takes place with the constant $c_5 = 2$.

The form $E(w,\varphi)$ is obviously continuous with respect to w on $F = \mathcal{H}^+_{1-\alpha}$ for each $\varphi \in \Phi$, since the operators $A^{\alpha-1}(t) \in L_\infty(]0,T[,\mathfrak{L}(H))$ are bounded and the functional $L(\varphi)$ is obviously continuous with respect to φ on Φ for any $f \in \mathcal{H}^{*-}_\alpha$ and $u_0 \in H$. Let us estimate the form

$$\operatorname{Re} E(\varphi,\varphi) = \operatorname{Re} \int_0^T e^{2ct} \left\{ (A^{1-\alpha}(t)\varphi(t), A^{*\alpha}(t)\varphi^*(t)) - \left(\varphi(t), \frac{d\varphi^*(t)}{dt}\right) \right\} dt \quad \forall \varphi \in \Phi$$

from below for $1/2 < \alpha \le 1$.

For the first inner product, we have

$$\begin{aligned}\operatorname{Re}(A^{1-\alpha}(t)\varphi(t), A^{*\alpha}(t)\varphi^*(t)) &= \operatorname{Re}(A^{1-\alpha}(t)[\aleph_{2\alpha-1}(t)+I]\varphi^*(t), A^{*\alpha}(t)\varphi^*(t)) \\ &= (1/2)\operatorname{Re}(A^{\alpha}(t)\varphi^*(t), A^{*\alpha}(t)\varphi^*(t)) \\ &\quad + (1/2)\operatorname{Re}(A^{1-\alpha}(t)A^{*2\alpha-1}(t)\varphi^*(t), A^{*1-\alpha}(t)A^{*2\alpha-1}(t)\varphi^*(t)) \\ &\quad + \operatorname{Re}(A^{1-\alpha}(t)\varphi^*(t), A^{*\alpha}(t)\varphi^*(t)).\end{aligned} \tag{14}$$

For the closed maximal accretive operators $A(t)$, the fractional powers $A^{1-\alpha}(t)$ and $A^{*1-\alpha}(t)$ with $1-\alpha < 1/2$ form an acute angle [3, Th. 1.1]; i.e.,

$$\operatorname{Re}(A^{1-\alpha}(t)w, A^{*1-\alpha}(t)w) \ge \cos\pi(1-\alpha)|A^{1-\alpha}(t)w|\,|A^{*1-\alpha}(t)w| \qquad \forall w \in D(A^{*1-\alpha}(t)) \tag{15}$$

for almost all t. By using inequalities (6), (15), inequalities (13) with $A^{*1-\alpha}(t)$ instead of $A^{1-\alpha}(t)$ and $A^{1-\alpha}(t)$ instead of $A^{*1-\alpha}(t)$ [3, Th. 1.1], and inequality (4) on the right-hand side in relation (14), for almost all t, we have the estimate

$$\operatorname{Re}(A^{1-\alpha}(t)\varphi(t), A^{*\alpha}(t)\varphi^*(t)) \ge (1/2)\cos\pi(1-\alpha)\operatorname{ctan}(\pi(3-2\alpha)/4)|A^{*\alpha}(t)\varphi^*(t)|^2. \tag{16}$$

In this case, before applying inequalities (4) to the last term in (14), we approximate the values of the power $A^{*\alpha}(t)$ by its values on elements in $D(A^*(t))$ [2, p. 140; 3, Lemma A3].

For the second inner product in the form $\operatorname{Re} E(\varphi,\varphi)$, we use the self-adjointness of the operators $\aleph_{2\alpha-1}(t)$ in H, perform integration by parts with respect to t, and obtain the relation

$$\begin{aligned}&-\operatorname{Re}\int_0^T e^{2ct}\left(\varphi(t), \frac{d\varphi^*(t)}{dt}\right) dt \\ &\qquad = \frac{1}{2}([\aleph_{2\alpha-1}(0)+I]\varphi^*(0),\varphi^*(0)) + c\int_0^T e^{2ct}([\aleph_{2\alpha-1}(t)+I]\varphi^*(t),\varphi^*(t))\,dt \\ &\qquad\quad + \frac{1}{4}\int_0^T e^{2ct}\left(\frac{d[\tilde{A}^{2\alpha-1}(t)+\tilde{A}^{*2\alpha-1}(t)]}{dt}\varphi^*(t),\varphi^*(t)\right) dt,\end{aligned}$$

whose right-hand side, by virtue of the accretivity of the operators $A^{2\alpha-1}(t)$ and $A^{*2\alpha-1}(t)$ in H and inequality (7), can be estimated from below by the expression

$$\frac{1}{2}|\varphi^*(0)|^2 + \left(c - \frac{c_2'(\alpha)}{2}\right)\int_0^T e^{2ct}([\aleph_{2\alpha-1}(t)+I]\varphi^*(t),\varphi^*(t))\,dt. \tag{17}$$

As a result, the estimates (16) and (17) with $c = c_2'(\alpha)/2$ lead to the inequality $\operatorname{Re} E(\varphi,\varphi) \geq c_6|||\varphi|||^2$ for all $\varphi \in \Phi$ with the constant $c_6 = (1/2)\cos\pi(1-\alpha)\operatorname{ctan}(\pi(3-2\alpha)/4)$ for $1/2 < \alpha \leq 1$. One can readily see that, in the case of positive self-adjoint operators $A(t)$ and $\alpha = 1/2$, this inequality holds with the constant $c_6 = 2$.

In view of Definition 2, to complete the proof of the fact that the Cauchy problem (1), (2) has a weak solution $u = \exp\{c_2'(\alpha)t\}w$, where $w \in \mathcal{H}_{1-\alpha}^{+}$ by Theorem 2, it remains to prove the embedding $\tilde{\Phi} \subset \Phi^*$, where $\tilde{\Phi}$ is the set occurring in Definition 2. In turn, to this end, it suffices to show that the function $[\aleph_{2\alpha-1}(t)+I]\tilde{\varphi}$ belongs to the space $\mathcal{H}$ for any function $\tilde{\varphi} \in \tilde{\Phi}$. This is indeed the case, since, by inequalities (5) and the boundedness of the inverses $A^{\alpha-1}(t), A^{*\alpha-1}(t) \in L_\infty(]0,T[,\mathfrak{L}(H))$, for $2\alpha-1 \leq \alpha$ and, for any function $\tilde{\varphi} \in D(A^{*\alpha}(t))$, we have the estimates

$$\begin{aligned}|A^{2\alpha-1}(t)\tilde{\varphi} + A^{*2\alpha-1}(t)\tilde{\varphi}| &\leq \|A^{\alpha-1}(t)\|_{\mathfrak{L}(H)}|A^{\alpha}(t)\tilde{\varphi}| + \|A^{*\alpha-1}(t)\|_{\mathfrak{L}(H)}|A^{*\alpha}(t)\tilde{\varphi}| \\ &\leq (c_1(\alpha)\operatorname*{ess\,sup}_{0<t<T}\|A^{\alpha-1}(t)\|_{\mathfrak{L}(H)} + \operatorname*{ess\,sup}_{0<t<T}\|A^{*\alpha-1}(t)\|_{\mathfrak{L}(H)})|A^{*\alpha}(t)\tilde{\varphi}|, \qquad 1/2 \leq \alpha \leq 1,\end{aligned}$$

for almost all t.

3. THE UNIQUENESS THEOREM

Let us prove the uniqueness theorem for weak solutions of this Cauchy problem, whose special case is given by Theorem 5.2 in [1, p. 64] by virtue of Remark 1.

Theorem 3. *If conditions* I–IV *are satisfied, then the weak solution* $u \in \mathcal{H}_{1-\alpha}^{+}$ *of the Cauchy problem* (1), (2) *is unique for any* $f \in \mathcal{H}_\alpha^{*-}$ *and* $u_0 \in H$.

Proof. First, let us show that, for almost all t, the domains $D(A^{*\alpha}(t))$ are closed subspaces of the Banach space $V_\alpha^* = D(\tilde{A}^{*\alpha})$ occurring in condition IV. If $v_n(t) \in D(A^{*\alpha}(t))$ for almost all t and $v_n(t) \to v_0(t)$ in V_α^* as $n \to \infty$, then, for almost all t, by virtue of the boundedness of the operators $\tilde{A}^{*\alpha}(t) \in \mathfrak{L}(V_\alpha^*, H)$, we find that $\tilde{A}^{*\alpha}(t)v_n(t) \to g_0(t)$ in H as $n \to \infty$. The operators $A^{*\alpha}(t)$ have bounded inverses $A^{*-\alpha}(t) \in L_\infty(]0,T[,\mathfrak{L}(H))$ for almost all t, since the maximal accretive operators $A(t)$ have bounded inverses $A^{-1}(t) \in L_\infty(]0,T[,\mathfrak{L}(H))$ [3] for almost all t. Therefore,

$$A^{*-\alpha}(t)g_0(t) = A^{*-\alpha}(t)\lim_{n\to\infty}\tilde{A}^{*\alpha}(t)v_n(t) = \lim_{n\to\infty} v_n(t) = v_0(t) \in D(A^{*\alpha}(t))$$

for almost all t.

Now let $\mathcal{H}_{1-\alpha}^{+} \ni u$ be a weak solution of the Cauchy problem (1), (2) for $f = 0$ and $u_0 = 0$. Then from Definition 2, we have the identity

$$\int_0^T \left\{(A^{1-\alpha}(t)u(t), A^{*\alpha}(t)\varphi(t)) - \left(u(t), \frac{d\varphi(t)}{dt}\right)\right\} dt = 0 \qquad \forall\varphi \in \tilde{\Phi}. \tag{18}$$

Here one can set

$$\varphi(t) = -\int_t^T e^{-2cs}[\aleph_{2\alpha-1}(s)+I]^{-1}u(s)\,ds, \tag{19}$$

$$\varphi(t) = -\int_t^T e^{-2cs}A^{*-\alpha}(s)\overline{A^{*\alpha-1}(s)[\aleph_{2\alpha-1}(s)+I]^{-1}A^{\alpha}(s)}A^{1-\alpha}(s)u(s)\,ds, \tag{20}$$

where the bounded operator $\overline{B}_1(s) = \overline{A^{*\alpha-1}(s)[\aleph_{2\alpha-1}(s)+I]^{-1}A^{\alpha}(s)}$ is the strong closure (with respect to continuity in H) of the product $B_1(s) = A^{*\alpha-1}(s)[\aleph_{2\alpha-1}(s)+I]^{-1}A^{\alpha}(s)$ for almost all s. In what follows, we derive the boundedness of the operators $\overline{B}_1(s)$ in H for almost all s from the boundedness of the operators $B_2(s) = A^{*\alpha}(s)[\aleph_{2\alpha-1}(s)+I]^{-1}A^{\alpha-1}(s)$ in H for almost all s.

Lemma 1. *The operators $A^{*\alpha}(s)[\aleph_{2\alpha-1}(s)+I]^{-1}A^{\alpha-1}(s)$ are bounded in H for almost all s.*

Proof. Let us show that the product

$$B_2(s): \; H \supset D(B_2(s)) \to R(B_2(s)) \subset H$$

of three operators $A^{\alpha-1}(s)$, $[\aleph_{2\alpha-1}(s)+I]^{-1}$, and $A^{*\alpha}(s)$ is continuous in H for almost all s and its domain is $D(B_2(s)) = H$.

First, for almost all s, the operators $B_2(s)$ are continuous in H by virtue of the inequality

$$|A^{*\alpha}(s)[\aleph_{2\alpha-1}(s)+I]^{-1}A^{\alpha-1}(s)g| \\ \le (2/[\cos\pi(1-\alpha)\,\mathrm{ctan}(\pi(3-2\alpha)/4)])|g| \quad \forall g \in D(B_2(s)), \tag{21}$$

which can readily be derived from the estimate (16).

Second, it follows from inequality (12) that the inverses $B_2^{-1}(s) = A^{1-\alpha}(s)[\aleph_{2\alpha-1}(s)+I]A^{*-\alpha}(s)$ of the operators $B_2(s)$ are bounded in H for almost all s; in particular, the domains of the operators $B_2^{-1}(s)$ are $R(B_2(s)) = H$.

Third, the domains $D(B_2(s))$ of the operators $B_2(s)$ are at least dense in H for almost all s. Suppose that there exists an element $v \in H$ such that $(u,v) = 0$ for all $u \in D(B_2(s))$ and for almost all s. Then, by virtue of the relation $R(B_2(s)) = H$, for almost all s here, one can set $u = A^{1-\alpha}(s)[\aleph_{2\alpha-1}(s)+I]A^{*-\alpha}(s)v$, take the real part, and obtain the relations

$$\mathrm{Re}\,(A^{1-\alpha}(s)[\aleph_{2\alpha-1}(s)+I]A^{*-\alpha}(s)v, v) \\ = \mathrm{Re}\,(A^{1-\alpha}(s)[\aleph_{2\alpha-1}(s)+I]w, A^{*\alpha}(s)w) = 0, \qquad w = A^{*-\alpha}(s)v,$$

which, together with the estimate (16), imply that $w = 0$ for almost all s and hence $v = 0$.

Since the operators $B_2(s)$, being the inverses of the bounded operators $B_2^{-1}(s)$, are bounded for almost all s, it follows from (21) that $D(B_2(s)) = H$ for almost all s. The proof of Lemma 1 is complete.

The adjoints of bounded operators are always bounded. By the lemma in [5, p. 201], for almost all s, the adjoint $B_2^*(s)$ of the operator $B_2(s)$ is equal to the strong closure in H of the product of the adjoints of the unbounded operator $A^{*\alpha}(s)$ and the bounded operator $[\aleph_{2\alpha-1}(s)+I]^{-1}A^{\alpha-1}(s)$; i.e.,

$$(A^{*\alpha}(s)[\aleph_{2\alpha-1}(s)+I]^{-1}A^{\alpha-1}(s))^* \\ = \overline{([\aleph_{2\alpha-1}(s)+I]^{-1}A^{\alpha-1}(s))^*A^{\alpha}(s)} = \overline{A^{*\alpha-1}(s)[\aleph_{2\alpha-1}(s)+I]^{-1}A^{\alpha}(s)} \tag{22}$$

for almost all s, which implies that the operators $\overline{B}_1(s)$ are bounded in H for almost all s.

The functions φ of the form (19) and (20) belong to the set $\tilde{\Phi}$. Their strong and hence weak derivatives with respect to t in H belong to $\mathcal{H}$, and $\varphi(T) = 0$ since, by virtue of relations (21) and (22), the integrals (19) and (20) are convergent in H for all $t \in [0,T]$. The function φ of the form (20) belongs to the set $D(A^{*\alpha}(t))$ for almost all t, and $A^{*\alpha}(t)\varphi \in \mathcal{H}$, since the integral (20) is convergent in the space $D(A^{*\alpha}(t))$ with the norm $]\cdot[_{\alpha,t}$ for almost all t. The convergence of the integral follows from the fact that the domains $D(A^{*\alpha}(t))$ are nonincreasing with respect to t in view of the embeddings (9), the closedness of the operators $A^{*\alpha}(t)$ and the above-proved closedness of the sets $D(A^{*\alpha}(t))$ for almost all t, and the boundedness of the operators $A^{*\alpha}(t)A^{*-\alpha}(s) \in L_\infty(]t,T[,\mathfrak{L}(H))$ in H uniformly for almost all $s \ge t$ for almost all $t \in [0,T]$ in view of the inequalities written out before the embeddings (9) in condition IV and relations (21) and (22). Moreover, $\varphi(t) \in D(A^{*\alpha}(\tau))$ for almost all $\tau \in [0,t]$, but $\varphi(t)$ does not belong to $D(A^{*\alpha}(\tau))$ for all $\tau \in]t,T[$ in the case of the strict embeddings (9). For the same reason, the same is true for the function φ of the form (19) since the operator in the integrand can be represented in the form $[\aleph_{2\alpha-1}(s)+I]^{-1} = A^{*-\alpha}(s)A^{*\alpha}(s)[\aleph_{2\alpha-1}(s)+I]^{-1}A^{\alpha-1}(s)A^{1-\alpha}(s)$, where the operators $B_2(s) = A^{*\alpha}(s)[\aleph_{2\alpha-1}(s)+I]^{-1}A^{\alpha-1}(s)$ are bounded in H uniformly with respect to almost all $s \in]0,T[$ by Lemma 1.

In identity (18), we successively set the function φ equal to the expressions (19) and (20); i.e., we have first $u = \exp\{2ct\}[\aleph_{2\alpha-1}(t) + I](d\varphi(t)/dt)$, $\varphi(T) = 0$ and then

$$u = \exp\{2ct\}A^{\alpha-1}(t)(\overline{A^{*\alpha-1}(t)[\aleph_{2\alpha-1}(t) + I]^{-1}A^{\alpha}(t)}\,)^{-1}A^{*\alpha}(t)(d\varphi(t)/dt), \qquad \varphi(T) = 0.$$

We add the results of substitutions, take the real part, and obtain the relation

$$\begin{aligned}\operatorname{Re}\int\limits_0^T e^{2ct}\Bigg\{ &-\left(A^{1-\alpha}(t)[\aleph_{2\alpha-1}(t) + I]\frac{d\varphi(t)}{dt}, A^{*\alpha}(t)\varphi(t)\right)\\ &-\left(\overline{A^{-\alpha}(t)[\aleph_{2\alpha-1}(t) + I]A^{*1-\alpha}(t)}A^{*\alpha}(t)\frac{d\varphi(t)}{dt}, A^{*\alpha}(t)\varphi(t)\right)\\ &+\left([\aleph_{2\alpha-1}(t) + I]\frac{d\varphi(t)}{dt}, \frac{d\varphi(t)}{dt}\right)\\ &+\left(A^{\alpha-1}(t)\overline{A^{-\alpha}(t)[\aleph_{2\alpha-1}(t) + I]A^{*1-\alpha}(t)}A^{*\alpha}(t)\frac{d\varphi(t)}{dt}, \frac{d\varphi(t)}{dt}\right)\Bigg\}\,dt, = 0 \end{aligned} \tag{23}$$

since, by Lemma 2 (see Theorem 3 in [6]), the inverse of the closure is equal to the closure of the inverse operator,

$$\begin{aligned}(\,\overline{A^{*\alpha-1}(t)[\aleph_{2\alpha-1}(t) + I]^{-1}A^{\alpha}(t)}\,)^{-1} &= \overline{(A^{*\alpha-1}(t)[\aleph_{2\alpha-1}(t) + I]^{-1}A^{\alpha}(t))^{-1}}\\ &= \overline{A^{-\alpha}(t)[\aleph_{2\alpha-1}(t) + I]A^{*1-\alpha}(t)},\end{aligned} \tag{24}$$

where the bar over an operator stands for the strong closure by continuity in H.

Lemma 2. *Let X and Z be Banach spaces, let $S:\ X \supset D(S) \to Z$ be a linear operator, and let $S^{-1}:\ Z \supset R(S) \to X$ be the inverse operator. If the operators S and S^{-1} admit closures $\overline{S}$ and $\overline{S^{-1}}$, respectively, then there exists an inverse operator $\overline{S}^{-1}$ of the closure $\overline{S}$, and $\overline{S}^{-1} = \overline{S^{-1}}$.*

The use of Lemma 2 for $X = Z = H$ in relations (24) is justified, since the closability of the operator $S = A^{*\alpha-1}(t)[\aleph_{2\alpha-1}(t) + I]^{-1}A^{\alpha}(t)$ in H for almost all t follows from relations (22), and the closability of the operator $S^{-1} = A^{-\alpha}(t)[\aleph_{2\alpha-1}(t) + I]A^{*1-\alpha}(t)$ in H is a consequence of the boundedness of the adjoint operator $(S^{-1})^* = A^{1-\alpha}(t)[\aleph_{2\alpha-1}(t) + I]A^{*-\alpha}(t)$ in H by the theorem in [7, pp. 198–199 of the Russian translation]. This operator $(S^{-1})^*$ is computed below in formula (25).

For the second inner product in relation (23), we pass to the adjoint operators in H first with the use of Theorem 3 in [4, p. 271 of the Russian translation] for the product of the unbounded operator $[\aleph_{2\alpha-1}(t)+I]A^{*1-\alpha}(t)$ by the bounded operator $A^{-\alpha}(t)$ and then with the use of Lemma 3 in [8, p. 43, Th. 11.4] for the product of two unbounded operators $S = A^{*1-\alpha}(t)$ and $P = \aleph_{2\alpha-1}(t)+I$ and obtain

$$\begin{aligned}&\left(\overline{A^{-\alpha}(t)[\aleph_{2\alpha-1}(t) + I]A^{*1-\alpha}(t)}A^{*\alpha}(t)\frac{d\varphi(t)}{dt}, A^{*\alpha}(t)\varphi(t)\right)\\ &\qquad= \left(A^{*\alpha}(t)\frac{d\varphi(t)}{dt}, ([\aleph_{2\alpha-1}(t) + I]A^{*1-\alpha}(t))^*\varphi(t)\right)\\ &\qquad= \left(A^{*\alpha}(t)\frac{d\varphi(t)}{dt}, A^{1-\alpha}(t)[\aleph_{2\alpha-1}(t) + I]\varphi(t)\right)\end{aligned} \tag{25}$$

for almost all t, since for almost all t, the function $\varphi(t)$ belongs to the set $D(A^{*\alpha}(t))$, the operator

$$A^{1-\alpha}(t)A^{2\alpha-1}(t) = A^{\alpha}(t)$$

is subordinate to the operator $A^{*\alpha}(t)$ by inequality (5), and

$$|A^{1-\alpha}(t)A^{*2\alpha-1}(t)\varphi| \le \tan(\pi(3-2\alpha)/4)|A^{*\alpha}(t)\varphi|$$

by inequality (13); i.e., the function $\varphi(t)$ belongs to the domain $D(A^{1-\alpha}(t)[\aleph_{2\alpha-1}(t)+I])$ of the product $A^{1-\alpha}(t)[\aleph_{2\alpha-1}(t)+I]$.

Lemma 3. *Let* X, Y, *and* Z *be Banach spaces, and let*

$$S: \ X \supset D(S) \to Y, \qquad P: \ Y \supset D(P) \to Z$$

be linear closed operators with dense domains $D(S)$ *and* $D(P)$ *in* X *and* Y, *respectively. If* S *is a* d*-normal operator* [*its range* $R(S)$ *is closed in* Y, *and its deficiency is finite*, $\dim \operatorname{Coker} S < +\infty$], *then the adjoint operator*

$$(P\cdot S)^*: \ Z' \supset D((P\cdot S)^*) \to X'$$

of their product $(P\cdot S): \ X \supset D(P\cdot S) \to Z$ *exists and is equal to* $(P\cdot S)^* = S^*\cdot P^*$, *where* S^* *and* P^* *are the adjoints of* S *and* P, *respectively.*

The use of Lemma 3 for $X = Y = Z = H$ in relations (25) is justified, since the closedness of the range $R(S)$ in $Y = H$ of the closed operator $S = A^{*1-\alpha}(t)$ in H for almost all t is provided by the boundedness of its inverse operator $S^{-1} = A^{*\alpha-1}(t)$ in H, and the finite dimension of the deficiency of the operator $S = A^{*1-\alpha}(t)$ is provided by the relations $R(S)^\perp = N(S^*) = N(A^{1-\alpha}(t)) = \{0\}$ [8, p. 14].

For the last inner product in relation (23), in a similar way, we obtain

$$\begin{aligned}\left(A^{\alpha-1}(t)\,\overline{A^{-\alpha}(t)[\aleph_{2\alpha-1}(t)+I]A^{*1-\alpha}(t)}A^{*\alpha}(t)\frac{d\varphi(t)}{dt}, \frac{d\varphi(t)}{dt}\right)&\\ = \left(A^{*\alpha}(t)\frac{d\varphi(t)}{dt}, A^{1-\alpha}(t)[\aleph_{2\alpha-1}(t)+I]A^{*-1}(t)\frac{d\varphi(t)}{dt}\right)&\end{aligned} \tag{26}$$

for almost all t.

By taking into account the Hermitian property of the sum of the first two inner products in relation (23) and by using relations (25) for the single integration by parts with respect to t in both inner products, we conclude that the real part of the integral of the sum of the first two inner products in relation (23) is equal to the expression

$$\begin{aligned}&\operatorname{Re}(A^{1-\alpha}(0)[\aleph_{2\alpha-1}(0)+I]\varphi(0), A^{*\alpha}(0)\varphi(0))\\ &\quad + 2c\operatorname{Re}\int_0^T e^{2ct}(A^{1-\alpha}(t)[\aleph_{2\alpha-1}(t)+I]\varphi(t), A^{*\alpha}(t)\varphi(t))\,dt\\ &\quad + \operatorname{Re}\int_0^T e^{2ct}\left(\frac{d\tilde{A}^{1-\alpha}(t)}{dt}[\aleph_{2\alpha-1}(t)+I]\varphi(t), A^{*\alpha}(t)\varphi(t)\right)dt\\ &\quad + \frac{1}{2}\operatorname{Re}\int_0^T e^{2ct}\left(A^{1-\alpha}(t)\left[\frac{d\tilde{A}^{2\alpha-1}(t)}{dt} + \frac{\tilde{A}^{*2\alpha-1}(t)}{dt}\right]\varphi(t), A^{*\alpha}(t)\varphi(t)\right)dt\\ &\quad + \operatorname{Re}\int_0^T e^{2ct}\left(A^{1-\alpha}(t)[\aleph_{2\alpha-1}(t)+I]\varphi(t), \frac{d\tilde{A}^{*\alpha}(t)}{dt}\varphi(t)\right)dt.\end{aligned} \tag{27}$$

By relations (14) and inequalities (16), (10), (8), (5), and (12), this expression for $1/2 < \alpha \le 1$ can be estimated from below by the quantity

$$
\begin{aligned}
&c\cos\pi(1-\alpha)\,\mathrm{ctan}\left(\frac{\pi}{4}(3-2\alpha)\right)\int\limits_0^T e^{2ct}|A^{*\alpha}(t)\varphi(t)|^2\,dt \\
&\quad -\frac{1}{2}(c_2(\alpha)+c_3(\alpha)+c_4(\alpha))\Big[c_1(\alpha)+(1+2\operatorname{ess}\sup_{0<t<T}\|A^{*1-2\alpha}(t)\|_{\mathfrak{L}(H)}) \\
&\quad \times\tan\left(\frac{\pi}{4}(3-2\alpha)\right)\Big]\int\limits_0^T e^{2ct}|A^{*\alpha}(t)\varphi(t)|^2\,dt. \qquad (28)
\end{aligned}
$$

Obviously, in the case of positive self-adjoint operators $A(t)$ and $\alpha = 1/2$, the expression (27) can be estimated from below by the quantity

$$
4c\int\limits_0^T e^{2ct}|A^{1/2}(t)\varphi(t)|^2\,dt - 2\left(c_3\left(\frac{1}{2}\right)+c_4\left(\frac{1}{2}\right)\right)\int\limits_0^T e^{2ct}|A^{1/2}(t)\varphi(t)|^2\,dt. \qquad (29)
$$

The third inner product in relation (23) is nonnegative for almost all t by virtue of inequalities (3) and (4), which imply that the fractional powers $A^{2\alpha-1}(t)$ and $A^{*2\alpha-1}(t)$ are accretive in H for almost all t. Let us show that the real part of the last inner product in relation (23) is also nonnegative for almost all t. Since the values of the power $A^{*\alpha}(t)$, whose domain, by construction, contains the function $d\varphi/dt$, can be approximated by its values on elements of the domains $D(A^{*2}(t))$, it follows from (26) that the above-mentioned real part is equal to the sum

$$
\begin{aligned}
&\frac{1}{2}\operatorname{Re}\left(A^{*\alpha}(t)\frac{d\varphi(t)}{dt}, A^{\alpha}(t)A^{*-1}(t)\frac{d\varphi(t)}{dt}\right) \\
&\qquad +\frac{1}{2}\operatorname{Re}\left(A^{*\alpha}(t)\frac{d\varphi(t)}{dt}, A^{1-\alpha}(t)A^{*2\alpha-2}(t)\frac{d\varphi(t)}{dt}\right) \\
&\qquad +\operatorname{Re}\left(A^{*\alpha}(t)\frac{d\varphi(t)}{dt}, A^{1-\alpha}(t)A^{*-1}(t)\frac{d\varphi(t)}{dt}\right) \\
&\quad =\frac{1}{2}\operatorname{Re}\left(A^{*2\alpha+1}(t)A^{*-1}(t)\frac{d\varphi(t)}{dt}, A^{*-1}(t)\frac{d\varphi(t)}{dt}\right) \\
&\qquad +\frac{1}{2}\operatorname{Re}\left(A^{*2\alpha-3}(t)A^{*}(t)\frac{d\varphi(t)}{dt}, A^{*}(t)\frac{d\varphi(t)}{dt}\right) \\
&\qquad +\operatorname{Re}\left(A^{*2}(t)A^{*-1}(t)\frac{d\varphi(t)}{dt}, A^{*-1}(t)\frac{d\varphi(t)}{dt}\right) \qquad (30)
\end{aligned}
$$

for almost all t, each of whose terms is nonnegative for almost all t by inequalities (11). Obviously, in the case of positive self-adjoint operators $A(t)$ and $\alpha = 1/2$, the last two inner products in relation (23) are nonnegative for almost all t.

Therefore, we can claim that, as a result of integration by parts on the left-hand side in relation (23), we have an expression that, by (27) and (30), is estimated from below by the quantity (28) for $1/2 < \alpha \le 1$ and by the quantity (29) in the case of positive self-adjoint operators $A(t)$ and $\alpha = 1/2$. In turn, by estimating these last quantities from below for

$$
c > \frac{(c_2(\alpha)+c_3(\alpha)+c_4(\alpha))[c_1(\alpha)+(1+2\operatorname{ess}\sup_{0<t<T}\|A^{*1-2\alpha}(t)\|_{\mathfrak{L}(H)})\tan(\pi(3-2\alpha)/4)]}{2\cos\pi(1-\alpha)\,\mathrm{ctan}(\pi(3-2\alpha)/4)},
$$

we pass from relation (23) to the inequality $c_7(\alpha)\int_0^T e^{2ct}|A^{*\alpha}(t)\varphi(t)|^2\,dt \le 0$, $c_7(\alpha) > 0$, for $1/2 < \alpha \le 1$ and the inequality $c_8(\alpha)\int_0^T e^{2ct}|A^{1/2}(t)\varphi(t)|^2\,dt \le 0$, $c_8(\alpha) > 0$, for $c > (c_3(1/2)+c_4(1/2))/2$

in the case of positive self-adjoint operators $A(t)$ and $\alpha = 1/2$. The last two inequalities imply that $\varphi = 0$ in $\mathcal{H}$ for all $1/2 \le \alpha \le 1$ and hence $u = 0$ in $\mathcal{H}^+_{1-\alpha}$. The proof of Theorem 3 is complete.

Remark 2. Under the assumptions of Theorem 3, all weak solutions $u \in \mathcal{H}^+_{1-\alpha}$ of the Cauchy problem (1), (2) satisfy the a priori estimate (see [1, p. 38, Remark 1.2])

$$\int_0^T [u(t)]^2_{1-\alpha,t}\,dt \le c_9^2(\alpha)\left(\int_0^T]f(t)[^2_{\alpha,-t}\,dt + |u_0|^2\right), \qquad c_9(\alpha) = \frac{c_5(\alpha)}{c_6(\alpha)},$$

where the $]\cdot[_{\alpha,-t}$ are norms of the antidual Hilbert spaces $H^{*-}_{\alpha,t}$,

$$c_9(\alpha) = \frac{c_1(\alpha) + (1 + 2\operatorname{ess\,sup}_{0<t<T} \|A^{*1-2\alpha}(t)\|_{\mathfrak{L}(H)})\tan(\pi(3-2\alpha)/4)}{\cos\pi(1-\alpha)\operatorname{ctan}(\pi(3-2\alpha)/4)}$$

for $1/2 < \alpha \le 1$, and $c_9(\alpha) = 1$ in the case of positive self-adjoint operators $A(t)$ and $\alpha = 1/2$.

4. SMOOTHNESS THEOREM

On any interval $J =]a,b[\subset]0,T[$, we analyze the local smoothness of weak solutions of the Cauchy problem (1), (2) in the Hilbert scale of spaces $\mathcal{H}^s(J,H) = L_2(J, W^s(t))$, where the Hilbert spaces $W^s(t)$ are the domains $D(A^s(t))$ of the power $A^s(t)$ of the operators $A(t)$ equipped with the Hermitian norms $|v|_{s,t} = |A^s(t)v|$ equivalent to the graph norms of the operators $A^s(t)$, $s \ge 0$. We have the following smoothness increasing theorem for the weak solutions of the original Cauchy problem in the time-dependent Hilbert scale $\{W^s(t)\}$ induced by the accretive operators $A(t)$ with t-dependent domains $D(A(t))$, $t \in J$.

Theorem 4. *Let conditions* I–IV *be satisfied. If, for some $q \ge 0$, the bounded inverse operators $A^{-q}(t)$ are strongly continuous with respect to $t \in J$ in H and, for almost all $t \in J$, have a bounded weak derivative $dA^{-q}(t)/dt \in L_\infty(J, \mathfrak{L}(H))$ in H such that*

$$|([\aleph_{2\alpha-1}(t) + I]\varphi, (dA^{*-q}(t)/dt)A^{*q}(t)\varphi)| \le c_{10}([\aleph_{2\alpha-1}(t) + I]\varphi,\varphi)^{1/2}|A^{*\alpha}(t)\varphi| \tag{31}$$

for all $\varphi \in D(A^{*\alpha_0}(t))$, where $dA^{*-q}(t)/dt$ is the weak derivative of the operators $A^{*-q}(t)$ in H, $\alpha_0 = \max\{\alpha, q\}$, and $c_{10} \ge 0$ is a constant independent of φ and t, then, for any right-hand side $f \in \mathcal{H}^q(J,H)$ of the equation and any initial data $u_a \in D(A^q(a))$, the Cauchy problem (1), (2) *has a weak solution $u \in \mathcal{H}^+_{1-\alpha}$ satisfying the inclusion $u \in \mathcal{H}^{1-\alpha+q}(J,H)$, and its generalized derivative satisfies the inclusion*

$$du/dt \in \mathcal{H}^{-\alpha+q}(J,H) \tag{32}$$

for $q \ge \alpha$.

Proof. One can readily see that if the operators $A^{-q}(t)$ in H are strongly continuous with respect to $t \in J$ and, for almost all $t \in J$, have a bounded weak derivative $dA^{-q}(t)/dt$, then their adjoint operators $A^{*-q}(t)$ in H are weakly continuous with respect to $t \in J$ and have a bounded weak derivative $dA^{*-q}(t)/dt = (dA^{-q}(t)/dt)^*$, where $(dA^{-q}(t)/dt)^*$ is the adjoint operator of the bounded operator $dA^{-q}(t)/dt$.

On the interval J, we introduce the auxiliary Cauchy problem [9]

$$\frac{dw(t)}{dt} + A(t)w(t) + A^q(t)\,\frac{dA^{-q}(t)}{dt}\,w(t) = \tilde{f}(t), \qquad t \in J, \tag{33}$$

$$w(a) = \tilde{u}_a, \tag{34}$$

and consider its weak solutions instead of strong solutions in [9].

Definition 3. A function $w \in \mathcal{H}^{1-\alpha}(J,H)$ is called a *weak solution* of the Cauchy problem (33), (34) for $\tilde f \in \mathcal{H}^{*\alpha-}(J,H) = L_2(J, H^{*-}_{\alpha,t})$ and $\tilde u_a \in H$ if the integral identity

$$\int\limits_a^b \left\{ (A^{1-\alpha}(t)w(t), A^{*\alpha}(t)\varphi(t)) - \left(w(t), \frac{d\varphi(t)}{dt} - \frac{dA^{*-q}(t)}{dt} A^{*q}(t)\varphi(t) \right) \right\} dt$$

$$= \int\limits_a^b \langle \tilde f(t), \varphi(t)\rangle_{\alpha,t}\, dt + (\tilde u_a, \varphi(a))$$

holds for all functions $\varphi \in \tilde\Phi_{\alpha_0} \equiv \{\tilde\varphi \in L_2(J,H):\ \tilde\varphi(t) \in D(A^{*\alpha_0}(t))$ for almost all $t \in J$; the weak derivative $d\tilde\varphi/dt$ and $A^{*\alpha_0}(t)\tilde\varphi$ belong to $L_2(J,H)$; $\tilde\varphi(b) = 0\}$.

First, just as in the proof of Theorem 1, we use Theorem 2 to show that, for any $\tilde f \in \mathcal{H}^{*\alpha-}(J,H)$ and $\tilde u_a \in H$, there exists a weak solution $w \in \mathcal{H}^{1-\alpha}(J,H)$ of the Cauchy problem (33), (34). The only difference is that, unlike the proof of Theorem 1, the form $E(w,\varphi)$ with an integral over J instead of $]0,T[$ contains the additional term

$$\int\limits_a^b e^{2ct} \left(w(t), \frac{dA^{*-q}(t)}{dt} A^{*q}(t)\varphi^*(t) \right) dt;$$

in the form $\operatorname{Re} E(\varphi,\varphi)$, by virtue of inequality (31) and the δ-inequality $-2ab \ge -\delta^{-1}a^2 - \delta b^2$ for sufficiently small $\delta > 0$, this term, together with the remaining terms in the form $\operatorname{Re} E(\varphi,\varphi)$, can be estimated from below by $c_{11}|||\varphi|||^2$, $c_{11} > 0$, for sufficiently large $c > 0$ and for all $\varphi \in \tilde\Phi_{\alpha_0}$. Therefore, there exist weak solutions $w \in \mathcal{H}^{1-\alpha}(J,H)$ of the Cauchy problem (33), (34).

If we set $\tilde f = A^q(t)f$, $\tilde u_a = A^q(a)u_a$, and $\varphi = A^{*-q}(t)\psi$ in the integral identity in Definition 3 for all ψ in the set $\tilde\Phi(]a,b[)$ obtained from the set $\tilde\Phi$ in Definition 2 by the replacement of the interval $]0,T[$ with the interval $]a,b[$, then we obtain the identity

$$\int\limits_a^b \left\{ (A^{1-\alpha-q}(t)w(t), A^{*\alpha}(t)\psi(t)) - \left(A^{-q}(t)w(t), \frac{d\psi(t)}{dt} \right) \right\} dt = \int\limits_a^b \langle f(t), \psi(t)\rangle_{\alpha,t}\, dt + (u_a, \psi(a)),$$

since $\langle f,\psi\rangle_{\alpha,t} = (f,\psi)$ for $f \in H$ and $\psi \in D(A^{*\alpha}(t))$. This identity, together with Theorem 3 on the uniqueness of weak solutions of the Cauchy problem (1), (2) and the identity in Definition 2, implies that

$$u(t) = A^{-q}(t)w(t), \qquad t \in J. \tag{35}$$

Relation (35) means that $u(t) \in D(A^{1-\alpha+q}(t))$ for almost all $t \in J$ and $u \in \mathcal{H}^{1-\alpha+q}(J,H)$, since $w \in \mathcal{H}^{1-\alpha}(J,H)$. But if $q \ge \alpha$, then, to prove the inclusion (32), we transpose the operators $A^{*\alpha}(t)$ in the integral identity in Definition 2 from the left-hand side to the right-hand side in the first inner product, use elementary estimates, and obtain the inequality

$$\left| \int\limits_a^b \left(u(t), \frac{d\varphi(t)}{dt} \right) dt \right| \le c_{12} \left(\int\limits_a^b |\varphi(t)|^2\, dt \right)^{1/2}, \qquad c_{12} > 0, \qquad \forall\, \varphi(t) \in C_0^\infty(J,H),$$

where $C_0^\infty(J,H)$ is the set of all infinitely differentiable functions of the variable $t \in J$ that take values in H and are compactly supported in J. This inequality implies that a weak solution u has a regular generalized derivative $du/dt \in L_2(J,H)$ [10, p. 19 of the Russian translation]; consequently, u satisfies Eq. (1) for almost all $t \in J$. Then, by using Eq. (1), we obtain $A^{-\alpha+q}(t)(du/dt) = A^{-\alpha+q}(t)f - A^{1-\alpha+q}(t)u \in L_2(J,H)$, since $f \in \mathcal{H}^q(J,H)$. The proof of Theorem 4 is complete.

Remark 3. If the operators $A(t)$ are positive self-adjoint operators, $\alpha = q = 1/2$, and $dA^{-1/2}(t)/dt \in L_\infty(J, \mathfrak{L}(H))$ under the assumptions of Theorem 4, then inequality (31) holds, and the weak solution $u \in \mathcal{H}^+_{1-\alpha}$ of the Cauchy problem (1), (2) has the local smoothness $d^k u/dt^k \in \mathcal{H}^{1-k}(J, H)$, $k = 0, 1$. The smoothness increasing theorem for weak solutions of the Cauchy problem (1), (2) is lacking in [1].

5. APPLICATION

As an example of a boundary value problem whose well-posedness can be proved on the basis of the abstract results obtained above, we consider the following mixed problem for the multidimensional linearized KdV equation.

Statement of the Mixed Problem

In the domain $G =]0, T[\times \Omega$, where $\Omega = \prod_{k=1}^n]\beta_k, \gamma_k[$ is a bounded parallelepiped in $\mathbb{R}^n$, $n \geq 2$, of the variables $x = \{x_1, \ldots, x_n\} \in \Omega$ with boundary $S = \partial\Omega$, consider the differential equation

$$\frac{\partial u(t,x)}{\partial t} - \sum_{k=1}^{n} a_k(t) \frac{\partial^3 u(t,x)}{\partial x_k^3} = f(t,x), \qquad (t,x) \in G, \tag{36}$$

where the coefficients satisfy the conditions $a_k(t) \in C[0,T]$ and $a_k(t) \geq a_0 > 0$ and have bounded first derivative $\partial a_k(t)/\partial t$, $k = 1, \ldots, n$, for almost all t, with the boundary conditions

$$u|_{S_t} = 0; \qquad \left.\frac{\partial u}{\partial x_k}\right|_{S_k^-} = 0, \qquad k = 1, \ldots, n; \qquad \sum_{k=1}^{n} a_k(t) \frac{\partial^2 u}{\partial x_k^2} \cos(\vec{\nu}(x'), \vec{e}_k)|_{S \setminus S_t} = 0, \tag{37}$$

where $\{S_t\}$ is a set of parts of the boundary S with positive surface measure and S_k^- is the set of all points $x' \in S$ with negative cosines of the angles between the unit outward normal vectors $\vec{\nu}\,(x')$ to S at the point $x' \in S$ and the unit vector $\vec{e}_k$ of the axis Ox_k, and with the initial condition

$$u(0,x) = u_0(x), \qquad x \in \Omega. \tag{38}$$

A straightforward application of the abstract Theorems 1 and 3 to this mixed problem is rather difficult owing to the involved nature of the fractional powers and their variable domains, especially for nonself-adjoint differential operators. Therefore, we suggest to derive the well-posed solvability of some mixed problems [for example, (36)–(38)] by replacing the abstract integro-differential operators specifying the fractional powers (for example, with exponent $\alpha = 2/3$) of the original differential operators $A(t)$ by some close dominating differential operators and by reproving the existence and uniqueness theorems on the basis of the above-given proofs of the corresponding abstract theorems (for example, Theorems 1 and 3). This technique permits one to avoid the rather difficult assumptions in conditions III and IV that some fractional powers of the corresponding linear unbounded operators with t-dependent domains are obtained by the restriction of some fractional-degree unbounded operators with t-independent domains. Thus, we illustrate the suggested approach by the above-posed mixed problem.

Existence of Solutions

To eliminate fractional powers of operators, we first need to modify Definition 2 of weak solutions for $\alpha = 2/3$. Let the Hilbert spaces $\tilde{H}^+_{1/3,t}(\Omega)$ (for almost all t) be the closure of the set of all functions in the Sobolev space $W_2^3(\Omega)$ satisfying the boundary conditions (37) in the Hermitian norms

$$[|u|]_{1/3,t} = \left(\int_\Omega \sum_{k=1}^{n} a_k(t) |\partial u(t,x)/\partial x_k|^2 \, dx \right)^{1/2},$$

equivalent to the norm $\|\cdot\|_1$ of the Sobolev space $W_2^1(\Omega)$ (see an estimate with a constant c_{14} below). We denote the norms of the Sobolev spaces $W_2^p(\Omega)$ by $\|\cdot\|_p$, $p \in \mathbb{Z}$. Note that the functions

$u \in \tilde{H}^{+}_{1/3,t}(\Omega)$ satisfy the boundary condition $u|_{S_t} = 0$ in the classical sense for almost all t. Let the Hilbert spaces $\tilde{H}^{*-}_{1/3,t}(\Omega)$ with the norms

$$]|v|[_{1/3,-t} = \left(\int\limits_{\Omega}]v[_{1/3,-t}\, dx \right)^{1/2}$$

(for almost all t) be the antidual spaces of the Hilbert spaces $\tilde{H}^{*+}_{1/3,t}(\Omega)$ obtained by the closure of the set of all functions $v \in W^3_2(\Omega)$ satisfying the adjoint boundary conditions

$$v|_{S_t} = 0; \qquad \left.\frac{\partial v}{\partial x_k}\right|_{S_k^+} = 0, \quad k = 1, \dots, n; \qquad \sum_{k=1}^{n} a_k(t) \frac{\partial^2 v}{\partial x_k^2} \cos(\vec{\nu}(x'), \vec{e}_k)|_{S \setminus S_t} = 0 \tag{39}$$

in the norms $[|\cdot|]_{1/3,t}$, where S_k^+ is the set of all points $x' \in S$ with $\cos(\vec{\nu}(x'), \vec{e}_k) > 0$.

We assume that the families of Hilbert spaces $\tilde{H}^{+}_{1/3,t}(\Omega)$ and $\tilde{H}^{*-}_{1/3,t}(\Omega)$ are measurable and square integrable with respect to $t \in]0, T[$. We denote the spaces by the symbols

$$\tilde{\mathcal{H}}^{+}_{1/3}(G) = L_2(]0, T[, \tilde{H}^{+}_{1/3,t}(\Omega)), \qquad \tilde{\mathcal{H}}^{*-}_{1/3}(G) = L_2(]0, T[, \tilde{H}^{*-}_{1/3,t}(\Omega)).$$

Definition 4. A function $u \in \tilde{\mathcal{H}}^{+}_{1/3}(G)$ is called a *weak solution* of the mixed problem (36)–(38) for $f \in \tilde{\mathcal{H}}^{*-}_{1/3}(G)$ and $u_0 \in L_2(\Omega)$ if

$$-\int\limits_0^T \int\limits_{\Omega} \left\{ \sum_{k=1}^{n} a_k(t) \frac{\partial u}{\partial x_k} \frac{\partial^2 \overline{\varphi}}{\partial x_k^2} + u \frac{\partial \overline{\varphi}}{\partial t} \right\} dx\, dt = \int\limits_0^T \langle f, \overline{\varphi} \rangle_{1/3,t}\, dt + \int\limits_{\Omega} u_0(x) \overline{\varphi}\,(0, x)\, dx \tag{40}$$

for any function $\varphi \in \Phi(G) \equiv \{\varphi \in W^{0,3}_2(G) : \ \varphi(t) \in (39)$ for almost all $t \in]0, T[$; $\partial\varphi/\partial t \in L_2(G)$, $\varphi(T, x) = 0$, $x \in \Omega\}$, where $\langle \cdot\,, \cdot \rangle_{1/3,t}$ are the sesquilinear forms of antiduality of the Hilbert spaces $\tilde{H}^{*+}_{1/3,t}(\Omega)$ and $\tilde{H}^{*-}_{1/3,t}(\Omega)$ and the bar stands for complex conjugation.

One can readily see that Definition 3 of weak solutions is an extension of the notion of classical solutions of problem (36)–(38).

Lemma 4. *If the embedding*

$$S_t \supset \bigcup_{k=1}^{n} S_k^- \tag{41}$$

holds in the parallelepiped Ω for almost all t, then, for each function $v \in W^{1,2}_2(G)$, the condition $(\partial v/\partial x_k)|_{S_k^+} = 0$, $k = 1, \dots, n$, provides the validity of the condition

$$\sum_{k=1}^{n} a_k(t)(\partial v/\partial x_k) \cos(\vec{\nu}\,(x'), \vec{e}_k)|_{S \setminus S_t} = 0$$

for almost all t.

Proof. The boundary of any parallelepiped satisfies $S = (\bigcup_{k=1}^{n} S_k^-) \cup (\bigcup_{k=1}^{n} S_k^+)$, and hence $(S \setminus \bigcup_{k=1}^{n} S_k^-) \subset \bigcup_{k=1}^{n} S_k^+$. This inclusion, together with (41), implies that

$$(S \setminus S_t) \subset \bigcup_{k=1}^{n} S_k^+ \tag{42}$$

for almost all t. For each $k = 1, \ldots, n$, we have

$$a_k(t)(\partial v(t,x')/\partial x_k)\cos(\vec{\nu}\,(x'), \vec{e}_k) = 0$$

for any $x' \in S_m^+$ and $m = 1, \ldots, n$, $m \neq k$, since $\cos(\vec{\nu}\,(x'), \vec{e}_k) = 0$ for all $x' \in S_m^+$ and $m = 1, \ldots, n$, $m \neq k$. Therefore,

$$\sum_{k=1}^{n} a_k(t)\frac{\partial v(t,x')}{\partial x_k}\cos(\vec{\nu}\,(x'), \vec{e}_k) = a_m(t)\frac{\partial v(t,x')}{\partial x_m}\cos(\vec{\nu}\,(x'), \vec{e}_m)$$
$$= a_m(t)\frac{\partial v(t,x')}{\partial x_m} = 0, \qquad m = 1, \ldots, n,$$

for all $x' \in S_m^+$, since $(\partial v/\partial x_m)|_{S_m^+} = 0$, $m = 1, \ldots, n$. In particular, this, together with the inclusion (42), implies that

$$\sum_{k=1}^{n} a_k(t)(\partial v(t,x')/\partial x_k)\cos(\vec{\nu}\,(x'), \vec{e}_k)|_{S\setminus S_t} = 0$$

for almost all t and for any $x' \in S\setminus S_t$. The proof of Lemma 4 is complete.

Theorem 5. *If $a_k(t) \in C[0,T]$, $a_k(t) \geq a_0 > 0$, $\partial a_k(t)/\partial t \in L_\infty(0,T)$, $k = 1, \ldots, n$, and the set $\{S_t\}$ satisfies the inclusion* (41) *for almost all $t \in]0,T[$, then for any $f \in \tilde{\mathcal{H}}^{*-}_{1/3}(G)$ and $u_0 \in L_2(\Omega)$, there exists a weak solution $u \in \tilde{\mathcal{H}}^{+}_{1/3}(G)$ of the mixed problem* (36)–(38).

Proof. In the projection Theorem 2, we use the Hilbert space $F = \tilde{\mathcal{H}}^{+}_{1/3}(G)$ with the Hermitian norm

$$\|w\|_F = \left(\int_0^T\!\!\int_\Omega \sum_{k=1}^{n} a_k(t)|\partial w/\partial x_k|^2\,dx\,dt\right)^{1/2}$$

and the pre-Hilbert space Φ that is the set $\Phi(\Omega)$ in Definition 3 with the Hermitian norm

$$|||\varphi||| = \left(\int_0^T\!\!\int_\Omega \sum_{k=1}^{n} a_k(t)|\partial \varphi/\partial x_k|^2\,dx\,dt + \int_\Omega |\varphi(0,x)|^2\,dx\right)^{1/2}.$$

On these spaces, we consider the sesquilinear form and the antilinear functional

$$E(w,\varphi) = -\int_0^T\!\!\int_\Omega e^{c\langle x\rangle}\left\{\sum_{k=1}^{n} a_k(t)\frac{\partial w}{\partial x_k}\,\frac{\partial^2\overline{\varphi}}{\partial x_k^2} + c\sum_{k=1}^{n} a_k(t)\,w\,\frac{\partial^2\overline{\varphi}}{\partial x_k^2} + w\,\frac{\partial\overline{\varphi}}{\partial t}\right\}dx\,dt, \qquad c > 0,$$

$$L(\varphi) = \int_0^T \langle f(t,x), \overline{\varphi}\,(t,x)\rangle_{1/3,t}\,dt + \int_\Omega u_0(x)\overline{\varphi}\,(0,x)\,dx, \qquad \langle x\rangle = x_1 + \cdots + x_n.$$

The continuous embedding of the spaces Φ in F with constant $c_5 = 1$ is obvious. Obviously, the form $E(w,\varphi)$ is continuous with respect to w on F for any function $\varphi \in \Phi$. For arbitrary $\varphi \in \Phi(G)$, the form $\operatorname{Re} E(\varphi,\varphi)$ with sufficiently small $c = c_{13} \in]0,1]$ can be estimated below by $(c_{13}/2)\exp\{c_{13}\min_{x\in\overline{\Omega}}\langle x\rangle\}|||\varphi|||^2$; i.e., the inequality $|E(\varphi,\varphi)| \geq c_6|||\varphi|||^2$, $\varphi \in \Phi(G)$, holds for $c_6 = (c_{13}/2)\exp\{c_{13}\min_{x\in\overline{\Omega}}\langle x\rangle\}$. This fact can readily be proved by single integration by parts with respect to x_k and t in the three double inequalities

$$-\operatorname{Re}\int_0^T\!\!\int_\Omega e^{c\langle x\rangle}\sum_{k=1}^n a_k(t)\frac{\partial\varphi}{\partial x_k}\frac{\partial^2\overline{\varphi}}{\partial x_k^2}\,dx\,dt \ge \frac{c}{2}\int_0^T\!\!\int_\Omega e^{c\langle x\rangle}\sum_{k=1}^n a_k(t)\left|\frac{\partial\varphi}{\partial x_k}\right|^2 dx\,dt$$
$$\ge c_6\int_0^T\!\!\int_\Omega \sum_{k=1}^n a_k(t)\left|\frac{\partial\varphi}{\partial x_k}\right|^2 dx\,dt,$$

$$-\operatorname{Re}\int_0^T\!\!\int_\Omega e^{c\langle x\rangle}\sum_{k=1}^n a_k(t)\varphi\frac{\partial^2\overline{\varphi}}{\partial x_k^2}\,dx\,dt$$
$$=\int_0^T\!\!\int_\Omega e^{c\langle x\rangle}\sum_{k=1}^n a_k(t)\left|\frac{\partial\varphi}{\partial x_k}\right|^2 dx\,dt + c\operatorname{Re}\int_0^T\!\!\int_\Omega e^{c\langle x\rangle}\sum_{k=1}^n a_k(t)\varphi\frac{\partial\overline{\varphi}}{\partial x_k}\,dx\,dt$$
$$\ge\left(\int_0^T\!\!\int_\Omega e^{c\langle x\rangle}\sum_{k=1}^n a_k(t)\left|\frac{\partial\varphi}{\partial x_k}\right|^2 dx\,dt\right)^{1/2}$$
$$\times\left[\left(\int_0^T\!\!\int_\Omega e^{c\langle x\rangle}\sum_{k=1}^n a_k(t)\left|\frac{\partial\varphi}{\partial x_k}\right|^2 dx\,dt\right)^{1/2} - c\left(\int_0^T\!\!\int_\Omega e^{c\langle x\rangle}\sum_{k=1}^n a_k(t)|\varphi|^2\,dx\,dt\right)^{1/2}\right],$$

$$-\operatorname{Re}\int_0^T\!\!\int_\Omega e^{c\langle x\rangle}\varphi\,\frac{\partial\overline{\varphi}}{\partial t}\,dx\,dt = \frac{1}{2}\int_\Omega e^{c\langle x\rangle}|\varphi(0,x)|^2\,dx \ge c_6\int_\Omega |\varphi(0,x)|^2\,dx \qquad \forall c\le 1$$

with the use of the second boundary condition in (39) in the first inequality, Lemma 4 and the Cauchy–Schwarz inequality in the second one, and the condition $\varphi(T,x)=0$, $x\in\Omega$, in the third inequality. Here, by virtue of the well-known estimate

$$\int_\Omega |\varphi|^2\,dx \le c_{14}\int_\Omega\sum_{k=1}^n |\partial\varphi/\partial x_k|^2\,dx, \qquad c_{14}>0,$$

which is valid for almost all t in view of the first boundary condition in (39) and the assumption that $\mu(S_t)\ne 0$ for almost all t, the right-hand side of the second inequality can be estimated from below by the quantity

$$\left(\int_0^T\!\!\int_\Omega e^{c\langle x\rangle}\sum_{k=1}^n a_k(t)\left|\frac{\partial\varphi}{\partial x_k}\right|^2 dx\,dt\right)^{1/2}$$
$$\times\left[\sqrt{a_0}\exp\left\{\frac{c}{2}\min_{x\in\overline{\Omega}}\langle x\rangle\right\} - c\left(\max_{t\in[0,T]}\sum_{k=1}^n a_k(t)\right)^{1/2}\exp\left\{\frac{c}{2}\max_{x\in\overline{\Omega}}\langle x\rangle\right\}\sqrt{c_{11}}\right]$$
$$\times\left(\int_0^T\!\!\int_\Omega\sum_{k=1}^n\left|\frac{\partial\varphi}{\partial x_k}\right|^2 dx\,dt\right)^{1/2},$$

which is nonnegative for sufficiently small $c=c_{13}>0$ and hence can be omitted in the estimate from below. Obviously, the functional $L(\varphi)$ is continuous with respect to φ on $\Phi(G)$.

Therefore, by Theorem 2, there exists a solution $w\in\tilde{\mathcal{H}}^+_{1/3}(G)$ of the equation

$$E(w,\varphi)=L(\varphi) \qquad \forall\varphi\in\Phi(G),$$

and consequently, there exists a weak solution $u = \exp\{c_{13}\langle x\rangle\}w \in \tilde{\mathcal{H}}^+_{1/3}(G)$ of the mixed problem (36)–(38), since the left-hand side of identity (40) can be reduced by the change of variables $u = \exp\{c_{13}\langle x\rangle\}w$ to the form $E(w,\varphi)$ with $c = c_{13}$ used in the proof. The proof of Theorem 5 is complete.

Uniqueness of Solutions

In addition, we require that the family $\{S_t\}$ be nondecreasing almost everywhere with respect to t; i.e., the inclusion $S_{t_1} \subset S_{t_2}$, $0 < t_1 < t_2 < T$, be valid for almost all t_1 and t_2. Then the family of Hilbert spaces $\{\tilde{H}^+_{1/3,t}(\Omega)\}$ is nonincreasing almost everywhere with respect to t; i.e., the inclusion $\tilde{H}^+_{1/3,t_1}(\Omega) \supset \tilde{H}^+_{1/3,t_2}(\Omega)$, $0 < t_1 < t_2 < T$, holds for almost all t_1 and t_2.

Theorem 6. *Let the assumptions of Theorem* 5 *hold, and let the family* $\{S_t\}$ *be nondecreasing and piecewise constant almost everywhere with respect to* $t \in\,]0,T[$. *Then, for any* $f \in \tilde{\mathcal{H}}^{*-}_{1/3}(G)$ *and* $u_0 \in L_2(\Omega)$, *the weak solution* $u \in \tilde{\mathcal{H}}^+_{1/3}(G)$ *of the mixed problem* (36)–(38) *is unique.*

Proof. Let $\tilde{\mathcal{H}}^+_{1/3}(G) \ni u$ be a weak solution of the mixed problem (36)–(38) for $f = 0$ and $u_0 = 0$; i.e., from identity (40), we have

$$\int_0^T\!\!\int_\Omega \left\{\sum_{k=1}^n a_k(t)\frac{\partial u}{\partial x_k}\frac{\partial^2\overline{\varphi}}{\partial x_k^2} + u\frac{\partial\overline{\varphi}}{\partial t}\right\}dx\,dt = 0 \qquad \forall\varphi \in \Phi(G). \tag{43}$$

If $\{S_t\} = S_0$ for almost all $t \in [0,t_1[$, $0 < t_1 < T$, then one can set

$$\varphi(t,x) = -\int_t^{t_1} e^{-2cs}A^{*-1}(s)u(s,x)\,ds, \qquad 0 \le t < t_1, \qquad x \in \Omega,$$

and $\varphi(t,x) = 0$, $t_1 \le t \le T$, $x \in \Omega$, or, which is the same, $u = \exp\{2ct\}A^*(t)(\partial\varphi/\partial t)$, $t \in [0,t_1[$, and $\varphi(t,x) = 0$, $t \in [t_1,T]$, $x \in \Omega$, where $A^{*-1}(t)$ are the bounded inverses in $L_2(\Omega)$ of the operators $A^*(t)$ generated in $L_2(\Omega)$ by the differential expressions $\tilde{A}^*(t)v = \sum_{k=1}^n a_k(t)(\partial^3 v/\partial x_k^3)$ on all functions of the set $D(A^*(t)) = \{v \in L_2(\Omega) : v \in (39)$ for almost all $t \in\,]0,T[,\ \tilde{A}^*(t)v \in L_2(\Omega)\}$. Since $\{S_t\}$ is a constant set for almost all $t \in [0,t_1[$, it follows that this function φ satisfies the boundary condition (39) for almost all $t \in [0,t_1[$ by virtue of the relation

$$A^*(t)\int_t^{t_1} e^{-2cs}A^{*-1}(s)u(s,x)\,ds = \int_t^{t_1} e^{-2cs}A^*(t)A^{*-1}(s)u(s,x)\,ds$$

valid for almost all $t < t_1$, the closedness of the operators $A^*(t)$, and the boundedness of the operators

$$A^*(t)A^{*-1}(s) \in L_\infty(\,]0,t_1[\,\times\,]0,t_1[,\mathfrak{L}(L_2(\Omega)))$$

[2, p. 176, Lemma 7.1]. For this function φ, from identity (43), we have

$$\int_0^{t_1}\!\!\int_\Omega \left\{\sum_{k=1}^n a_k(t)\frac{\partial u}{\partial x_k}\frac{\partial^2\overline{\varphi}}{\partial x_k^2} + u\frac{\partial\overline{\varphi}}{\partial t}\right\}dx\,dt = 0,$$

which, after one integration by parts with respect to x_k, acquires the form

$$\int_0^{t_1}\!\!\int_\Omega e^{2ct}\left\{-A^*(t)\frac{\partial\varphi}{\partial t}A^*(t)\overline{\varphi} + A^*(t)\frac{\partial\varphi}{\partial t}\frac{\partial\overline{\varphi}}{\partial t}\right\}dx\,dt = 0. \tag{44}$$

Since the operators $A^*(t)$ are the restrictions of the operators $\tilde{A}^*(t)$ to $D(A^*(t))$, we have

$$
\begin{aligned}
&- \operatorname{Re} \int_0^{t_1}\int_\Omega e^{2ct} A^*(t) \frac{\partial \varphi}{\partial t} A^*(t)\overline{\varphi}\, dx\, dt \\
&\quad = \frac{1}{2}\int_\Omega |A^*(0)\varphi(0,x)|^2\, dx + c\int_0^{t_1}\int_\Omega e^{2ct}|A^*(t)\varphi|^2\, dx\, dt + \operatorname{Re}\int_0^{t_1}\int_\Omega e^{2ct}\,\frac{\partial \tilde{A}^*(t)}{\partial t}\,\varphi A^*(t)\overline{\varphi}\, dx\, dt \\
&\quad \ge (c - c_{15})\int_0^{t_1}\int_\Omega e^{2ct}|A^*(t)\varphi|^2\, dx\, dt,
\end{aligned}
$$

because there exists a constant $c_{15} > 0$ such that $\int_\Omega |(\partial \tilde{A}^*(t)/\partial t) A^{*-1}(t) g|^2\, dx \le c_{15}^2 \int_\Omega |g|^2\, dx$ for all $g \in L_2(\Omega)$ and for almost all $t \in\,]0, t_1[$ by [2, p. 176, Lemma 7.1]. Therefore, by estimating the real part of the left-hand side of relation (44) from below, we obtain the inequality

$$
(c - c_{15})\int_0^{t_1}\int_\Omega e^{2ct}|A^*(t)\varphi|^2\, dx\, dt \le 0,
$$

which, for $c > c_{15}$, implies that $\varphi = 0$ and hence $u = 0$ for almost all $t \in\,]0, t_1[$.

Then, from identity (43), we obtain the identity

$$
\int_{t_1}^{T}\int_\Omega \left\{ \sum_{k=1}^n a_k(t) \frac{\partial u}{\partial x_k}\, \frac{\partial^2 \overline{\varphi}}{\partial x_k^2} + u\, \frac{\partial \overline{\varphi}}{\partial t} \right\} dx\, dt = 0 \qquad \forall \varphi \in \Phi(G). \tag{45}
$$

If $\{S_t\} = S_{t_1}$ for almost all $t \in [t_1, t_2[$, $t_1 < t_2 < T$, then one can set

$$
\varphi(t,x) = \begin{cases} 0 & \text{for} \quad t \in [t_2, T], \ \ x \in \Omega \\ -\int_t^{t_2} e^{-2cs} A^{*-1}(s) u(s,x)\, ds & \text{for} \quad t \in [t_1, t_2[, \ \ x \in \Omega \\ \psi(t,x) & \text{for} \quad t \in [0, t_1[, \ \ x \in \Omega, \end{cases} \tag{46}
$$

where $\psi(t,x)$ is a weak solution of the mixed problem in reverse time,

$$
\begin{aligned}
&\frac{\partial \psi(t,x)}{\partial t} - \sum_{k=1}^n a_k(0) \frac{\partial^3 \psi(t,x)}{\partial x_k^3} = 0, \qquad (t,x) \in\,]0, t_1[\, \times\, \Omega, \\
&\varphi \in (39) \quad \text{for almost all} \quad t \in\,]0, t_1[; \\
&\psi(t_1, x) = -\int_{t_1}^{t_2} e^{-2cs} A^{*-1}(s) u(s,x)\, ds, \qquad x \in \Omega,
\end{aligned}
$$

which can be reduced by the change of variables $t' = t_1 - t$ to the corresponding mixed problem with direct time. By using Theorem 4 on the smoothness in the Hilbert spaces $H = L_2(\Omega)$ and $\mathcal{H}^s(J, H) = L_2(]0, t_1[, W^s(\Omega))$, where $W^s(\Omega)$ is the domain $D(A^{*s}(0))$ of the operator $A^{*s}(0)$ with the norm $|v|_s = \|A^{*s}(0)v\|_0$, for $J = \,]0, t_1[$ and $q = \alpha = 2/3$, we find that its weak solution ψ has the smoothness $\psi \in W_2^{0,3}(]0, t_1[\, \times \Omega)$, ψ satisfies the boundary conditions (39) for almost all $t \in\,]0, t_1[$, and $\partial\psi/\partial t \in L_2(]0, t_1[\, \times \Omega)$, since $dA^{-2/3}(t)/dt = 0$ on the interval $]0, t_1[$, the initial data satisfies the inclusion $\psi(t_1, x) \in W^{2/3}(\Omega)$, and $H^+_{1/3,0}(\Omega) \subset W^{1/3}(\Omega)$.

Let us show that the function $\psi(t_1, x)$ indeed belongs to the set $D(A^{*2/3}(0))$. The obvious inequalities $\|A^*(0)v\|_0 \le c_{16}\|v\|_3$, $c_{16} > 0$, and $\|A^{*-1}(0)A^*(0)v\|_0 = \|v\|_0$ for arbitrary $v \in D(A^*(0))$ permit one to claim that the linear continuous operator $\pi = A^*(0)$ satisfies the relation

$$\pi = A^*(0) \in \mathfrak{L}(W_2^3(\Omega), L_2(\Omega)) \cap \mathfrak{L}(L_2(\Omega), W^{-1}(\Omega)),$$

and therefore, by Theorem 5.1 on interpolation, the operator π belongs to $\mathfrak{L}([W_2^3(\Omega), L_2(\Omega)]_\theta, [L_2(\Omega), W^{-1}(\Omega)]_\theta)$, $0 < \theta < 1$ [10, p. 41 of the Russian translation]. If $\theta = 1/3$, then for the intermediate spaces, we have $[W_2^3(\Omega), L_2(\Omega)]_{1/3} = W_2^2(\Omega)$ by Theorem 9.1 in [10, p. 56 of the Russian translation] and

$$[L_2(\Omega), W^{-1}(\Omega)]_{1/3} = [W^0(\Omega), W^{-1}(\Omega)]_{1/3} = W^{-1/3}(\Omega)$$

by Definition 2.1 in [10, p. 23 of the Russian translation]. Therefore, the linear continuous operator π belongs to $\mathfrak{L}(W_2^2(\Omega), W^{-1/3}(\Omega))$; i.e.,

$$\|A^{*2/3}(0)v\|_0 = \|A^{*-1/3}(0)A^*(0)v\|_0 \le c_{17}\|v\|_2, \qquad c_{17} > 0, \qquad \forall v \in D(A^{*2/3}(0)). \tag{47}$$

In addition, first, by the definition of powers of operators, for any $v \in D(A^{*2/3}(0))$, there exists a sequence $v_n \in D(A^*(0))$ such that $v_n \to v$ and $A^{*2/3}(0)v_n \to A^{*2/3}(0)v$ in $L_2(\Omega)$ as $n \to \infty$ [2, p. 140]. Second, it is known that the closure of the domain $D(A^*(0))$ in the norm $\|\cdot\|_2$ is equal to the set

$$\mathcal{D}(t) = \left\{v \in W_2^2(\Omega) : \ v|_{S_t} = 0; \quad (\partial v/\partial x_k)|_{S_k^+} = 0, \quad k = 1, \dots, n\right\}$$

for $t = 0$. Since $\psi(t_1, x) \in \mathcal{D}(t_1)$ by construction and $\mathcal{D}(0) \supset \mathcal{D}(t_1)$ because the set $\{S_t\}$ is nondecreasing with respect to t, it follows from (47) that $\psi(t_1, x) \in D(A^{*2/3}(0))$.

If $1 - \alpha = 1/3$, then, in a similar way, one can prove the embedding $H^+_{1/3,0}(\Omega) \subset D(A^{*1/3}(0))$ for almost all $t \in]0, t_1[$ and for all functions in the space $L_2(]0, t_1[, H^+_{1/3,0}(\Omega))$ of weak solutions.

Therefore, the function φ of the form (46) indeed belongs to the set $\Phi(G)$, and for it identity (45) becomes the relation

$$\int_{t_1}^{t_2}\int_\Omega e^{2ct}\left\{-A^*(t)\frac{\partial\varphi}{\partial t}\,A^*(t)\overline{\varphi} + A^*(t)\frac{\partial\varphi}{\partial t}\,\frac{\partial\overline{\varphi}}{\partial t}\right\}dx\,dt = 0,$$

which, just as in the case of relation (44), implies that $u = 0$ for almost all $t \in]t_1, t_2[$, and so on. As a result, in finitely many steps, we find that $u = 0$ for almost all $t \in]0, T[$. The case of countably many intervals of constant value of the family $\{S_t\}$ with respect to t can be reduced to their finite number; it is only necessary to additionally use the definition of a set of zero measure in $[0, T]$. The proof of Theorem 6 is complete.

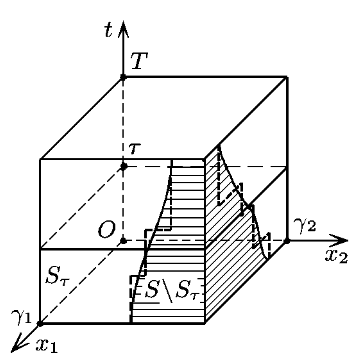

Figure.

Remark 4. In the rectangle $]0, \gamma_1[\, \times\,]0, \gamma_2[\, \subset \mathbb{R}^2$, the set S_τ nondecreasing with respect to τ and satisfying the inclusion (41) (respectively, the assumptions of Theorem 6) for each $\tau \in [0, T]$ (respectively, for almost all $\tau \in [0, T]$) is bounded in the figure by the solid (respectively, dashed) line. The sets $\{S_\tau\}$ and $\{S \backslash S_\tau\}$ are the projections onto the plane $t = 0$ of the intersections of the nonshaded and shaded parts of the lateral area of the direct prism and the secant plane $t = \tau$, $\tau \in [0, T]$, respectively. By analogy with Remark 2, one can claim that, under the assumptions of Theorem 6, all weak solutions $u \in \tilde{\mathcal{H}}^+_{1/3}(G)$ of the mixed problem (36)–(38) satisfy the estimate

$$\int_0^T \int_\Omega \sum_{k=1}^n a_k(t) \left| \frac{\partial u(t,x)}{\partial x_k} \right|^2 dx\, dt$$

$$\leq \frac{4c_{13}^{-2}}{\exp\{2c_{13} \min_{x \in \overline{\Omega}} \langle x \rangle\}} \left(\int_0^T \int_\Omega]|f(t,x)|[^2_{1/3,-t}\, dx\, dt + \int_\Omega |u_0(x)|^2\, dx \right).$$

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