A RECURRENCE FORMULA FOR JACK CONNECTION COEFFICIENTS

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This report is devoted to Jack connection coefficients, a generalization of the connection coefficients of the classical subalgebras of the group algebra of the symmetric group closely related to the theory of Jack symmetric functions. First introduced by Goulden and Jackson (1996) these numbers indexed by three partitions of a given integer n and the Jack parameter α are defined as the coefficients in the power sum expansion of some Cauchy sum for Jack symmetric functions. Goulden and Jackson [1] conjectured that they are polynomials in $\beta = \alpha - 1$ with non negative integer coefficients of combinatorial significance, the *Matchings-Jack conjecture.*

We show that Jack connection coefficients satisfy a recurrence formula when two of the partitions are equal to the single part (n) and prove the Matchings-Jack conjecture in this case.

The ring of symmetric functions Λ has the following bases indexed by partitions λ of integer numbers: monomial basis m_λ, power sums p_λ, Schur polynomials s_λ, zonal polynomials Z_λ (see [2]). Let $\langle\cdot\,,\cdot\rangle$ be the scalar product on Λ such that $\langle p_\lambda, p_\mu\rangle = z_\lambda \delta_{\lambda\mu}$ where $z_\lambda = \prod_i i^{m_i(\lambda)} m_i(\lambda)!$ ($m_i(\lambda)$ is a number of i-parts of λ). The Schur polynomials are characterized by the fact that they form an orthonormal basis of Λ for $\langle\cdot\,,\cdot\rangle$ and the transition matrix between Schur and monomial symmetric functions is upper triangular. The zonal polynomials directly linked with the theory of the zonal spherical functions and verify the same properties as the s_λ (except the unit length property) if the scalar product is replaced by $\langle\cdot,\cdot\rangle_2$ with $\langle p_\lambda, p_\mu\rangle_2 = 2^{\ell(\lambda)} z_\lambda \delta_{\lambda\mu}$ where $\ell(\lambda)$ is the number of parts of λ. These functions are linked with classical subalgebras of the group algebra:

I. Class algebra, i. e. the center of $\mathbb{C}S_n$ with the basis $(C_\lambda)_{\lambda\vdash n}$ where C_λ is a sum of permutations of cycle type λ.

II. Double coset algebra, i. e. **Hecke algebra of the Gelfand pair** (S_{2n}, B_n), where B_n is a centralizer of $f_\star = (12)(34)\dots(2n-1, 2n)$, with basis $(K_\lambda)_{\lambda\vdash n}$ where K_λ is a sum of all permutations $\omega \in S_{2n}$ such that $f_\star \omega f_\star \omega^{-1}$ has cycle type $\lambda\lambda$ (see [2]).

Let $c^\lambda_{\mu\nu}$ and $b^\lambda_{\mu\nu}$ be the connection coefficients of these algebras:

$$C_\mu C_\nu = \sum\nolimits_{\lambda\vdash n} c^\lambda_{\mu\nu} C_\lambda, \qquad K_\mu K_\nu = \sum\nolimits_{\lambda\vdash n} b^\lambda_{\mu\nu} K_\lambda.$$

Using an additional parameter α, Henry Jack [3] introduced the bases of **Jack symmetric functions** J^α_λ which are characterized by two properties: (1) $\langle J^\alpha_\lambda, J^\alpha_\mu\rangle_\alpha = j_\lambda(\alpha)\delta_{\lambda\mu}$ where the scalar product $\langle\cdot\,,\cdot\rangle_\alpha$ is defined by $\langle p_\lambda, p_\mu\rangle_\alpha = \alpha^{\ell(\lambda)} z_\lambda \delta_{\lambda\mu}$ and $j_\lambda(\alpha)$ is some normalizing factor; (2) the transition matrix between J^α_λ and m_λ is upper triangular. So $J^1_\lambda = \sqrt{j_\lambda(1)} s_\lambda$ and $J^2_\lambda = Z_\lambda$.

Goulden and Jackson [1] showed that

$$\sum_{\lambda,\mu,\nu\vdash n} z_\lambda^{-1} c^\lambda_{\mu\nu} p_\lambda(x) p_\lambda(y) p_\lambda(z) = \sum_{\gamma\vdash n} h_\gamma(1) s_\gamma(x) s_\gamma(y) s_\gamma(z),$$

$$\sum_{\lambda,\mu,\nu\vdash n} 2^{-\ell(\lambda)} z_\lambda^{-1} \frac{b^\lambda_{\mu\nu}}{|B_n|} p_\lambda(x) p_\lambda(y) p_\lambda(z) = \sum_{\gamma\vdash n} \frac{Z_\gamma(x) Z_\gamma(y) Z_\gamma(z)}{\langle Z_\gamma, Z_\gamma\rangle_2}$$

and introduced the coefficients $a^\lambda_{\mu\nu}(\alpha)$ by the equality

$$\sum_{\lambda,\mu,\nu\vdash n} \alpha^{-\ell(\lambda)} z_\lambda^{-1} a^\lambda_{\mu\nu}(\alpha) p_\lambda(x) p_\mu(y) p_\nu(z) = \sum_{\gamma\vdash n} \frac{J^\alpha_\gamma(x) J^\alpha_\gamma(y) J^\alpha_\gamma(z)}{\langle J^\alpha_\gamma, J^\alpha_\gamma\rangle_\alpha}.$$

Computations of $a^{\lambda}_{\mu\nu}(\alpha)$ for all $\lambda, \mu, \nu \vdash n \leq 8$ showed that the $a^{\lambda}_{\mu\nu}(\alpha)$ are polynomials in $\beta = \alpha - 1$ with non negative integer coefficients and of degree at most $n - \min\{\ell(\mu), \ell(\nu)\}$. Goulden and Jackson conjectured this property for arbitrary λ, μ, ν. Moreover, following the combinatorial interpretation in Proposition 1, they also suggest the stronger **Matchings-Jack conjecture.** Dołęga and Féray [4] proved that the $a^{\lambda}_{\mu\nu}(\alpha)$ are polynomials in α with rational coefficients.

Graph interpretation. For a partition $\lambda = (\lambda_1, \ldots, \lambda_p) \vdash n$, consider the graph G on $2n$ vertices consisting of p cycles of lengths $2\lambda_1, \ldots, 2\lambda_p$. A matching in G is a set of edges without common vertices that contains all the vertices of G. Coloring successively the edges of the cycles of G in gray and black colors, we get two matchings: $\mathbf{g}$ (gray edges) and $\mathbf{b}$ (black edges). We also colour successively vertices of G in two colours and call a matching of G by *bipartite* if the ends of each its edge have different colors. We call such graph induced by λ **a λ-graph**.

Proposition 1 [1]. *The quantity $b^{\lambda}_{\mu\nu}/|B_n|$ (resp. $c^{\lambda}_{\mu\nu}$) is the number of matchings (resp. bipartite matchings) δ such that $\mathbf{b} \cup \delta$ is a μ-graph and $\mathbf{g} \cup \delta$ is a ν-graph.*

Matchings-Jack Conjecture [1]. *There exists a function* wt *on matchings such that*

$$a^{\lambda}_{\mu\nu}(\beta + 1) = \sum\nolimits_{\delta} \beta^{\mathrm{wt}_{\lambda}(\delta)}$$

for all $\lambda, \mu, \nu \vdash n$ where the summation is over all matchings as in Proposition 1, and $\mathrm{wt}_{\lambda}(\delta) \in \{0, 1, \ldots, n - \min\{\ell(\mu), \ell(\nu)\}\}$, $\mathrm{wt}_{\lambda}(\delta) = 0 \iff \delta$ *is bipartite.*

We prove Matchings-Jack conjecture in the case $\mu = \nu = (n)$ (Theorem 2) using the recurrence formula in Theorem 1. Define the following operations on matchings:

$$\lambda_{\downarrow(k)} = \lambda \setminus k \cup (k-1), \quad \lambda_{\downarrow(k,l)} = \lambda \setminus (k,l) \cup (k+l-1), \quad \lambda^{\uparrow(k,l)} = \lambda \setminus (k+l+1) \cup (k,l).$$

Theorem 1. *For integer n and partition $\lambda \vdash n+1$, the Jack connection coefficients a^{λ}_{nn} verify the following recurrence formula for any $i \in \{1, \ldots, \ell(\lambda)\}$:*

$$a^{\lambda}_{n+1,n+1}(\alpha) = (\alpha - 1)(\lambda_i - 1) a^{\lambda_{\downarrow(\lambda_i)}}_{nn}(\alpha) + \sum_{d=1}^{\lambda_i - 2} a^{\lambda^{\uparrow(\lambda_i - 1 - d, d)}}_{nn}(\alpha) + \alpha \sum_{j \neq i} \lambda_j a^{\lambda_{\downarrow(\lambda_i, \lambda_j)}}_{nn}(\alpha).$$

This formula admits a nice combinatorial interpretation in terms of graphs in the special cases $\alpha = 1$, 2 that provided us with the intuition for the general case and that we used in the proof of Theorem 2. We call a matching δ such that both $\mathbf{b} \cup \delta$ and $\mathbf{g} \cup \delta$ are $2n$-cycles ((n)-graphs) *a good matching.* Denote by $\mathcal{G}(\lambda)$ the set of all good matchings of the λ-graph G.

Theorem 2. *Let λ be a partition of n and G a λ-graph. Then there exists a function* $\mathrm{wt}_{\lambda} \colon \mathcal{G}(\lambda) \to \{0, 1, \ldots, n-1\}$ *such that*

$$a^{\lambda}_{nn}(\beta + 1) = \sum_{\delta \in \mathcal{G}(\lambda)} \beta^{\mathrm{wt}_{\lambda}(\delta)}, \quad \mathrm{wt}_{\lambda}(\delta) = 0 \iff \delta \text{ is bipartite.}$$

As a consequence, the quantity $a^{\lambda}_{nn}(\beta+1)$ is a nonnegative integer polynomial in β with constant term $c^{\lambda}_{nn} = a^{\lambda}_{nn}(1)$ and sum of coefficients equal to $b^{\lambda}_{nn}/|B_n| = a^{\lambda}_{nn}(2)$.

References

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