ON GROUPS OF PERIOD 12

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We consider groups of period 12. In particular, we give a criterion for such groups to be locally finite.

It is well known that groups of period 4 and groups of period 6 are locally finite [1–4]. Local finiteness of groups of period 12 with some additional conditions has been proved in [1, 5–7].

We reduce a question of local finiteness of groups of period 12 to a question of finiteness of their subgroups generated by three elements of order 3. Our main result is as follows.

Theorem. *A group of period* 12 *is locally finite iff every subgroup H of G is finite, given that one of the following conditions is true.*

(1) *H is generated by and element a of order* 3 *and elements b and c of order* 2*, such that* $(ab)^3 = (bc)^3 = 1$.

(2) *H is generated by elements a and b of order* 3 *and an element c of order* 2*, such that* $(ac)^2 = 1$.

For the proof of the theorem we first prove that the following results are true.

Lemma 1. *If G is a finite group of period* 12 *and $p \in \{2, 3\}$, then the p-length of G is at most two, and that bound is exact. If besides the* 2*-length of the group G equals* 2 *and the* 2*-length of every proper subgroup of G is less than two, then G is isomorphic to either S_4, or the semidirect product of a non-cyclic group of order* 4 *by the group $B = \langle a, x \mid a^3 = x^4 = 1, a^x = a^{-1}\rangle$. In particular, G contains a subgroup isomorphic to A_4.*

Lemma 2. *If G is a localy finite group of period* 12*, then*

$$G = O_{2,3,2,3,2}(G) = O_{3,2,3,2,3}(G).$$

The proof of the theorem also uses computations with the help of GAP [8]. A good example is given by

Lemma 3. *Suppose that G is a group of period* 12 *generated by an element a of order* 3 *and involutions b, c, such that $(ab)^3 = (bc)^3 = 1$. Then G is a semidirect product of the subgroup $H = \langle (bc)^G\rangle$ coinciding with its derived subgroup, and a group $A = \langle a, b\rangle$ isomorphic to A_4. The subgroup H is generated by elements $x_1 = bc$, $x_2 = x_1^a$, $x_3 = x_2^a$, $x_4 = x_2^b$, $x_5 = x_3^b$, $x_6 = x_5^a$, and the action of A on H is defined by the following equalities:*

$$x_1^a = x_2, x_2^a = x_3, x_3^a = x_1, x_4^a = x_5^{-1}, x_5^a = x_6, x_6^a = x_4; \quad (1)$$

$$x_1^b = x_1^{-1}, x_2^b = x_4, x_3^b = x_5, x_4^b = x_2, x_5^b = x_3, x_6^b = x_6^{-1}. \quad (2)$$

Proof of Lemma 3. Computations in GAP [8] show that in the group

$$K = \langle a, b, c \mid 1 = a^3 = b^2 = c^2 = (ab)^3 = (abc)^3 = (ac)^{12} = (abc)^{12}\rangle$$

the subgroup $H = \langle (bc)^K\rangle$ coincides with its derived subgroup, and $K/H \simeq A_4$. It is clear that G is a homomorphic image of K, and the kernel of the corresponding homomorphism lies in H. The equality $x_1^b = x_1^{-1}$ follows from the fact that b and c are involutions and $x_1 = bc$. Other equalities from (1) and (2) follow from the definitions of elements x_i, $i = 1, \ldots, 6$, and defining relations of the group A.

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