CRITICAL HEREDITARY CLASSES FOR ALGORITHMIC GRAPH PROBLEMS

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A *hereditary graph class* is a set of simple graphs closed under isomorphism and deletion of vertices. It is well-known that each hereditary class $\mathcal{X}$ can be defined by a set of forbidden induced subgraphs $\mathcal{Y}$, written $\mathcal{X} = Free(\mathcal{Y})$. If a hereditary class can be defined by a finite set of forbidden induced fragments, then it is called *finitely defined*. For example, the sets of line graphs and bounded degree graphs are finitely defined, but the sets of planar graphs and bipartite graphs do not.

Given an algorithmic graph problem, the problem of its complexity classification in the family of hereditary classes is of our attention. How to classify hereditary graph classes with respect to the complexity of a given problem? The first natural idea coming to mind is to consider phase transition between easy and hard hereditary classes under some definitions of easiness and hardness. It is like determining critical temperatures, when ice melts to water or water turns to steam. We know that the answer is zero and one hundred degrees Celsius, respectively. The phase-transition approach seams to be unsuccessful. For a given NP-complete graph problem Π, natural definitions of easy and hard instances are the following. A hereditary class is Π-*easy* if Π can be solved for its graphs in polynomial time. If Π is NP-complete for a hereditary class, then it is said to be Π-*hard*. In fact, these notions were introduced in the paper of V.E. Alekseev [1], where the absence of maximal easy classes was also proved for any NP-complete graph problem. Minimal hard classes may exist or may not exist. It is easy to prove that the set of all complete graphs is a minimal hard case for the travelling salesman problem. The following result was proved in [2].

Theorem 1. *For each k, there are no minimal hard classes for the vertex and edge k-colorability problems.*

So, minimal hard classes could be called critical, as they play a specific role in the analysis of the computational complexity. But, they may be absent at all. In other words, both sets of easy and hard classes can be open with respect to the inclusion relation. That is why we consider the phase-transition approach to be useless.

To solve the major problem, we have to take into account the fact of the existence of infinite monotonically decreasing chains of hard classes. Intuitively, the limits of such kind sequences have a special role in the analysis of computational complexity. It is really true. This leads to the notion of a boundary class. A class $\mathcal{X}$ is Π-*limit* if there is an infinite monotonically decreasing chain $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \ldots$ of Π-hard classes such that $\mathcal{X} = \bigcap\limits_{i=1}^{\infty} \mathcal{X}_i$. A minimal Π-limit class is said to be Π-*boundary*. This notion was introduced by V.E. Alekseev [1], who also proved the following theorem certifying its significance.

Theorem 2. *A finitely defined class is Π-hard if and only if it contains a Π-boundary class.*

By the theorem, if the set of all Π-boundary classes, called the Π-*boundary system*, is known, then, for the problem Π, we have a complete complexity dichotomy in the family of finitely defined classes. Moreover, any NP-complete graph problem has boundary classes. One more corollary is the fact that there are no finitely defined classes with an intermediate complexity, i.e. different from polynomial-time solvability and NP-completeness.

Assuming $P \neq NP$, one boundary class is known for the independent set problem [1], four boundary classes are known for the dominating set problem [3,4], two boundary classes are known for the Hamiltonian cycle problem [5]. Unfortunately, for all of the mentioned problems, there is no a complete description of a boundary system. Only one result of this type exists [6]. An idea arises that obtaining a complete description of the boundary system of a given graph problem may be a problem impossible to solve, because the structure of the answer is too complex. This is certified by the following result [5,7].

Theorem 3. *For each $k > 2$, the boundary systems for the vertex and edge k-colorability problems have the continuum cardinality.*

Sometimes, a known subset of a boundary system is enough to obtain a complexity dichotomy for some simple subfamily of the hereditary graph classes family. This idea really works sometimes. There is a dichotomy for the independent set problem within the family of classes defined by induced subgraphs with at most five vertices. Similar results exist for one forbidden induced structure for the dominating set problem [8] and the coloring problem [9]. There are some dichotomies for the vertex 3- and 4-colorability problems and one small forbidden induced structure [10,11]. There exist dichotomies for the vertex and edge 3-colorability problems and several small forbidden induced fragments [12,13].

The talk will be devoted to the results above and some other results in the theory of critical hereditary graph classes.

The article was prepared within the framework of the Academic Fund Program at the National Research University Higher School of Economics (HSE) in 2015–2016 (grant 15-01-0010) and supported within the framework of a subsidy granted to the HSE by the Government of the Russian Federation for the implementation of the Global Competitiveness Program.

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