ON THE NORMAL STRUCTURE OF ISOTROPIC REDUCTIVE GROUPS OVER RINGS

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Let K be a domain and let G be a reductive group scheme over K. We study the normal structure of the group of points $G(R)$ of G over a commutative ring R. For $G = GL_n$, it was described by J. Wilson ($n \geq 4$) and I. Golubchik ($n \geq 3$). For a Chevalley groups the standard normal structure was established by L. Vaserstein with invertible structure constants and by E. Abe in the general case.

Theorem (L. Vaserstein). *Let G be a Chevalley–Demazure group scheme with a reduced irreducible root system $\Phi \neq A_1$ and let R be a commutative ring. Given a normal subgroup $H \triangleleft G(R)$ there exists an ideal $\mathfrak{a}$ of R such that*

$$E(R, \mathfrak{a}) \leq H \leq C(R, \mathfrak{a}),$$

where $C(R, \mathfrak{a})$ is the preimage of the center under the reduction homomorphism $G(R) \to G(R/\mathfrak{a})$.

The ideal $\mathfrak{a}$ is called the level of H. By standard arguments using standard commutator formulas and elementary computations to describe the level, a proof reduces to extraction of unipotents. This is a difficult part of Vaserstein's and Abe's proofs.

A substantial progeress in understanding the structure theory of isotropic but nonsplit reductive algebraic groups was made by V. Petrov and A. Stavrova. In particular, they proved that the elementary subgroup $E_P(R)$ does not depend of the choice of a parabolic subgroup P of G. Recently A. Stavrova computed the level of a normal subgroup H of $G(R)$ provided that the structure constants are invertible. It turns out that the level is defined by an ideal $\mathfrak{a}$ of R, similarly to the case of Chevalley groups.

Another ingredient is a new proof of the Vaserstein theorem obtained by A. Stepanov. The main idea of this proof consists of 2 parts. First, we reformulate a way of extracting transvections in $\mathrm{GL}_n(R)$ in terms of parabolic subgroups and extract a unipotent element from the generic element of G. Second, we show that if this unipotent vanishes for all elements of H, then H lies in a subscheme, which is proper over any K-algebra. After that by standard arguments one proves that such H is central. In this way we obtain the following result.

Theorem. *Let G be a simple algebraic group scheme of constant type Φ defined over a connected commutative ring K such that the structure constants of Φ are invertible in K. Assume moreover that G contains at least two distinct parabolic subgroups $P_1 < P_2 < G$ over K.*

Let R be a K-algebra. Given a subgroup $H \leq G(R)$, normalized by the elementary subgroup $E(R)$, there exists an ideal $\mathfrak{a}$ of R such that

$$E(R, \mathfrak{a}) \leq H \leq C(R, \mathfrak{a}),$$

where $C(R, \mathfrak{a})$ is the preimage of the center under the reduction homomorphism $G(R) \to G(R/\mathfrak{a})$.