ON AUTOMORPHISMS OF THE ENDOMORPHISM SEMIGROUP OF A FREE ABELIAN DIMONOID

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An algebra $(D, \dashv, \vdash)$ with two binary associative operations $\dashv$ and $\vdash$ is called a *dimonoid* [1] if for all $x, y, z \in D$ the following conditions hold:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \;\; (x \vdash y) \dashv z = x \vdash (y \dashv z), \;\; (x \dashv y) \vdash z = x \vdash (y \vdash z).$$

A dimonoid $(D, \dashv, \vdash)$ will be called *abelian* (in the same as a digroup in [2]) if $x \dashv y = y \vdash x$ for all $x, y \in D$. For example, any left zero and right zero dimonoid is abelian. More general information on dimonoids and examples of different dimonoids can be found, e.g., in [1–3].

Let X be a nonempty set and $FCm(X)$ be the free commutative monoid on X with the unity ε. We put $FAd(X) = X \times FCm(X)$ and define two binary operations $\dashv$ and $\vdash$ on $FAd(X)$ by

$$(x, u) \dashv (y, v) = (x, uyv), \;\; (x, u) \vdash (y, v) = (y, xuv).$$

Theorem 1. *The algebra* $\mathfrak{F}_X = (FAd(X), \dashv, \vdash)$ *is the free abelian dimonoid of rank* $|X|$.

We denote by $\mathfrak{F}_n$ the free abelian dimonoid $\mathfrak{F}_X$ on an n-element set X.

Let $(S, \circ)$ be an arbitrary semigroup and $a \in S$. Define on S a new binary operation $\circ_a$ by $x \circ_a y = x \circ a \circ y$ for all $x, y \in S$. Clearly, $(S, \circ_a)$ is a semigroup, it is called a *variant* of $(S, \circ)$.

Corollary 1. *The free abelian dimonoid* $\mathfrak{F}_1$ *is isomorphic to the variant* $(N^0, +_1)$ *of the additive semigroup of all nonnegative integers.*

Proposition 1. *The endomorphism monoid* $End(\mathfrak{F}_1)$ *of the free abelian dimonoid* $\mathfrak{F}_1$ *is isomorphic to the semigroup* $(N^0, \star)$, *where* $x \star y = x + y + x \cdot y$ *for all* $x, y \in N^0$.

Denote by $\mathbb{P}$ the set of all prime numbers.

Proposition 2. *Let* X *be a singleton set,* Y *be an arbitrary set and* $End(\mathfrak{F}_X) \cong End(\mathfrak{F}_Y)$. *Then* $|Y| = 1$ *and the isomorphisms of* $End(\mathfrak{F}_X)$ *onto* $End(\mathfrak{F}_Y)$ *are in a natural one-to-one correspondence with permutations of* $\mathbb{P}$.

From here it follows that the automorphism group $Aut(End(\mathfrak{F}_1))$ of the endomorphism monoid $End(\mathfrak{F}_1)$ is isomorphic to the symmetric group $S(\mathbb{P})$ on a countably infinite set $\mathbb{P}$.

Theorem 2. *Let* $(x, \varepsilon), (y, \omega) \in FAd(X)$, *where* $w = w_1^{\alpha_1} w_2^{\alpha_2} ... w_n^{\alpha_n} \neq \varepsilon$. *Every isomorphism* $End(\mathfrak{F}_X) \to End(\mathfrak{F}_Y)$ *is induced by the isomorphism* $\pi_\varphi : \mathfrak{F}_X \to \mathfrak{F}_Y$ *such that*

$$(x, \varepsilon)\pi_\varphi = (x\varphi, \varepsilon), \;\; (y, \omega)\pi_\varphi = (y\varphi, (w_1\varphi)^{\alpha_1}(w_2\varphi)^{\alpha_2} \ldots (w_n\varphi)^{\alpha_n}),$$

where $\varphi : X \to Y$ *is a uniquely determined bijection.*

Corollary 2. *For an arbitrary set* X *with* $|X| \geq 2$, *the automorphism group* $Aut(End(\mathfrak{F}_X))$ *is isomorphic to the symmetric group* $S(X)$.

We observe that the automorphism group of the endomorphism monoid of a free semigroup or a free monoid was described by Mashevitsky G. and Schein B.M. [4].

References

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