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RESEARCH ARTICLE

SEMIGROUPS WITH $\wedge$-SEMIDISTRIBUTIVE SUBSEMIGROUP LATTICES

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I. PRELIMINARIES

A lattice $(L, \wedge, \vee)$ is called $\wedge$-semidistributive (lower semidistributive) [9] iff it satisfies the following Jonsson's condition [8]:

$$(SD_\wedge) \quad a \wedge b = a \wedge c \implies a \wedge (b \vee c) = a \wedge b .$$

$\vee$-semidistributive lattices are defined dually. The condition $(SD_\wedge)$ is a natural generalization of distributivity and it was investigated by many authors. Some natural classes of lattices consist of $\wedge$-semidistributive ones, in particular, sublattices of free lattices [8], congruence lattices of semilattices [I3], [2I], et al. Another term ($\cap$-quasidistributive lattices) was used by M. Curzio [4] and F. Napolitani [I2].

In this paper, semigroups with $\wedge$-semidistriutive subsemigroup lattices are considered. Theorem 2.4 provides a description of such semigroups akin to the well-known one of semigroups with distributive or modular subsemigroup lattices obtained by L. N. Shevrin and M. Ego [5], [6], [I7], [I8], [I9] (See also the survey [20]).

Besides, each $\wedge$-semidistributive subgroup lattice of a group is proved to be distributive (Theorem 2.2).

2. NOTATION , MAIN THEOREMS

The terminology and notation will be that of semigroup theory [9], [I4], lattice theory [7], [8], and group theory [IO]. Some of notions and notations useful in this paper are furnished below.

Given a lattice L and an element $a \in L$, the principal filter of L generated by a is denoted by $\uparrow a$. The subsemigroup lattice [20] of a semigroup S is denoted by $\mathrm{Sub}\, S$ (with $\cap$ and $\vee$ the operations of this lattice). If S is a group, then $\mathrm{Subgr}\, S$ is the lattice of its subgroups. Throughout the paper, except Section 3, the subsemigroup of a semigroup S generated by a set $X \subset S$ is denoted by $\langle X \rangle$. In Section 3 , $\langle X \rangle$ is the subgroup of a group S generated by a set $X \subset S$. E_S is the set of all idempotents of a semigroup S. Given $e \in E_S$, G_e is the largest subgroup of a semigroup S containing e, and K_e denotes the torsion class in a periodic semigroup S corresponding to an idempotent e, i.e. the set of elements of S with some power equal to e. $\mathcal{K}_S$ denotes the equivalence relation on a periodic semigroup S corresponding to the torsion class partition of S, i.e.

$$\mathcal{K}_S = \bigcup_{e \in E_S} K_e \times K_e .$$

We denote by $|a|$ the index of an element a in a periodic semigroup S. By 2^S we denote the lattice of all subsets of a semigroup S (the Boolean of subsets of S). Let S be a periodic semigroup. For $X \in 2^S$ the set

$$\bigcup_{e \in X \cap E_S} G_e$$

is denoted by $Gr_S X$. A subsemigroup A of S is called a G-subsemigroup iff $Gr_S A \subset A$. The set of all non-empty G-subsemigroups of S will be denoted by $Sub_G S$. If the lattice Sub S is a sublat-lattice of the Boolean 2^S, then S is called an U-semigroup [I4]. In this vein, if $Sub_G S$ is a sublattice of 2^S, then S is called an UG-semigroup.

For the further considerations, we need the following

LEMMA 2.I. *Let* $(L, \wedge, \vee)$ *be an algebraic* $\wedge$*-semidistributive lattice. then for any* $a, b \in L$ *a set*

$$\{ c \in L \mid a \wedge b = a \wedge c \}$$

has the largest element.

In other words, an algebraic $\wedge$-semidistributive lattice is a semi-Browerian lattice in the sense of J. Schmidt [I6]. The assertion of this lemma follows easily from $(SD_\wedge)$ and meet-continuity [7] of any algebraic lattice (see also Theorem II.I.4 of [3]).

Now we formulate the main results.

THEOREM 2.2. *Given a group* S , *the lattice* Subgr S *is* $\wedge$*-semidistributive if and only if it is distributive and hence, iff* S *is a locally cyclic group.*

REMARK 2.3. The assertion of Theorem 2.2 was proved for finite groups by M. Curzio [4]. Distributivity of a $\vee$-semidistributive subgroup lattice of a group was shown by F. Napolitani [II]. He also explored some lattice properties of finite groups related to the condition $(SD_\wedge)$ [I2].

THEOREM 2.4. *The following conditions are necessary and sufficient for a semigroup* S *to have* $\wedge$*-semidistributive subsemigroup lattice:*

1) S *is a periodic semigroup and the index of each its element is less than* 6;

2) E_S *is a subsemigroup of* S *and it is an ordinal sum of singular bands*;

3) *the equivalence* $\mathcal{H}_S$ *is a congruence on* S;

4) *given* $e \in E_S$, *the torsion class* K_e *is an ideal extension of a periodic locally cyclic group by a* n-*nilpotent* U-*semigroup where* $n \leq 5$.

5) S *is an* UG-*semigroup*.

In Section 3 Theorem 2.2 is proved. A proof of Theorem 2.4 is given in Sections 4 and 5. In Section 6 the difference between classes of semigroups with $\wedge$-semidistributive subsemigroup lattices and with distributive ones is discussed.

3. PROOF OF THEOREM 2.2.

Let S be a group with $\wedge$-semidistributive subgroup lattice L = Subgr S. We are going to prove that L is distributive. Modularity of L is sufficient because in this case $\wedge$-semidistributivity implies distributivity of L [4, Proposition 2.6]. That will be done providing S is shown to be an abelian group.

Let $a, b \in S$. We want to show that

$$ab = ba . \qquad (1)$$

A lattice Subgr $\langle a,b \rangle$ is $\wedge$-semidistributive as a sublattice of the $\wedge$-semidistributive lattice Subgr S. So we may assume that $S = \langle a,b \rangle$, L = Subgr $\langle a,b \rangle$. Consider the subgroup

$$C = \langle a \rangle \cap \langle b \rangle . \qquad (2)$$

Obviously, C is contained in the center of S, hence C is a normal subgroup of S. A homomorphism

$$\zeta = \text{nat } C : S \to S/C$$

induces an isomorphism (which we denote by η) of the principal filter $\uparrow C$ of the lattice L onto the lattice Subgr (S/C). Hence the latter is $\wedge$-semidistributive. Now our aim is to show that S/C is a direct sum of groups $\langle \zeta(a) \rangle$ and $\langle \zeta(b) \rangle$. Define A as the largest subgroup of S among the subgroups A' satisfying the condition:

$$\langle a \rangle \cap A' = C .$$

Such subgroup A exists according to Lemma 2.I , because L is an algebraic lattice. Let us show that A is a normal subgroup of S.

In fact, in view of (2), $\langle b \rangle \subset A$, whence

$$bAb^{-I} \subset A . \tag{3}$$

For $d \in A$, we show that

$$\langle a \rangle \cap \langle a^{-I} d a \rangle C = C . \tag{4}$$

This implies, by the definition of A, that

$$a^{-I} d a \in \langle a^{-I} d a \rangle C \subset A .$$

Indeed,

$$C \subset \langle a \rangle \cap \langle a^{-I} d a \rangle C .$$

Conversely, let $x \in \langle a^{-I} d a \rangle C \cap \langle a \rangle$. Since $C \subset \langle a \rangle$, we have

$$x = (a^{-I} d a)^m a^k = a^l$$

for some $m,k,l \in Z$ (Z is the set of integers) with $a^k \in C$, but then $a^{-I} d^m a a^k = a^l$, whence $d^m = a^{l-k}$ and therefore

$$d^m \in \langle d \rangle \cap \langle a \rangle \subset A \cap \langle a \rangle = C .$$

Consequently,

$$x = (a^{-I} d a)^m a^k = a^{-I} d^m a a^k = d^m a^k \in CC \subset C .$$

Thus (4) is valid which yields together with (3) that A is normal in S.

Again, define B as the largest subgroup among the subgroups B' of S satisfying the condition:

$$B' \cap A = C.$$

We next have to show that B is normal in S. In fact, $\langle a \rangle \cap A = C$ implies $\langle a \rangle \subset B$ and

$$a^{-1} B a \subset B . \qquad (5)$$

For arbitrary $d \in B$, we state that

$$A \cap \langle b^{-1} d b \rangle C = C . \qquad (6)$$

This will imply that $b^{-1} d b \in B$ and so $b^{-1} B b \subset B$. In view of (5) and the equality $S = \langle a,b \rangle$ it will follow that B is normal in S.

Obviously, $C \subset \langle b^{-1} d b \rangle C \cap A$. To prove the converse, assume that $x \in \langle b^{-1} db \rangle C \cap A$. Since $C \subset \langle b \rangle$, there exist $m, k \in Z$ such that

$$x = (b^{-1} d b)^m b^k \in A ,$$

where $b^k \in C$. Since $x, b \in A$ we obtain

$$b^{-1} d^m b = (b^{-1} d b)^m \in A .$$

Since A is normal, we have

$$d^m \in b A b^{-1} \subset A .$$

Hence, $d^m \in B \cap A = C$ and then

$$x = b^{-1} d^m b b^k \in (b^{-1} C b) C \subset C .$$

Thus (6) holds, whence B is a normal subgroup of S.

Since the isomorphism η preserves normality of subgroups, the subgroups $\eta(A)$ and $\eta(B)$ are normal in S/C. At the same time

$$\eta(A) \cap \eta(B) = \eta(C), \quad \eta(A) \vee \eta(B) = S/C .$$

It follows that S/C is a direct sum of subgroups $\eta(A)$ and $\eta(B)$. Since $\zeta(b) \in \eta(A)$, $\zeta(a) \in \eta(B)$, this implies that the elements $\zeta(a)$ and $\zeta(b)$ commute, and taking into account the equality

$$S/C = \langle \zeta(a), \zeta(b) \rangle$$

we conclude that S/C is an abelian group. This property together with $\wedge$-semidistributivity of the lattice Subgr (S/C) yields the latter to be distributive and hence by Ore's theorem [IO] the group S/C is locally cyclic. Since it is 2-generated, we deduce that S/C is cyclic. Therefore, S is a central extension by a cyclic group, whence S is an abelian group. This completes the proof of Theorem 2.2.

4. PROOF OF THEOREM 2.4. NECESSITY.

Let S be a semigroup with $\wedge$-semidistributive subsemigroup lattice. Under this assumption we shall prove a number of lemmas.

LEMMA 4.I. *The semigroup S is periodic with the index of each element less than 6.*

PROOF. Let $a \in S$ and suppose, ex adverso, that none of the elements a, a^2, a^3, a^4, a^5 belong to any subgroup of $\langle a \rangle$. Define

$$A = \{a^4, a^5, a^6, a^7, \dots\},$$

$$B = \{a^2, a^4, a^6, a^7, \dots\},$$

$$C = \{a^3, a^4, a^6, a^7, \dots\}.$$

It is easy to see that

$$A \cap B = \{a^4, a^6, a^7, \dots\} = A \cap C .$$

and $\wedge$-semidistributivity of Sub S implies that

$$A \cap (B \vee C) = A \cap B = \{a^4, a^6, a^7, \dots\} .$$

However,

$$B \vee C = \{a^2, a^3, a^4, a^5, a^6, \ldots\} \supset A ,$$

whereupon

$$A \cap (B \vee C) = A \neq A \cap B .$$

A contradiction shows that index of a is less than 6. The assertion of the lemma follows.

LEMMA 4.2. *The set E_S is a subsemigroup of S and it is an ordinal sum of singular bands.*

PROOF. Let e, f E_S, $e \neq f$. Let us show that $ef \in \{e,f\}$. In fact, if $ef \notin \{e,f\}$, then either ef is an idempotent of a semigroup $\langle e,f \rangle$, or due to Proposition I of [I] there exists an idempotent

$$g \in \langle e,f \rangle \setminus \{e,f\} .$$

In the former case, set $g = ef$. But then

$$\langle g \rangle \cap \langle e \rangle = \emptyset = \langle g \rangle \cap \langle f \rangle$$

and hence $\wedge$-semidistributivity of Sub S leads to a contradiction:

$$\langle g \rangle = \langle g \rangle \cap \langle e,f \rangle =$$

$$= \langle g \rangle \cap (\langle e \rangle \vee \langle f \rangle) = \langle g \rangle \cap \langle e \rangle = \emptyset .$$

Consequently, $\langle e,f \rangle = \{e,f\}$ and $2^S = \text{Sub } E_S$. Thus Sub E_S is distributive and therefore E_S is an ordinal sum of singular bands according to the description of semigroups with distributive subsemigroup lattices (see the corresponding references in Section I).

COROLLARY 4.3. *Let S is a band. The lattice Sub S is $\wedge$-semidistributive iff S is an ordinal sum of singular bands.*

LEMMA 4.4. *Given $e \in E_S$, the torsion class K_e is a subsemigroup of S.*

PROOF. Let $e, f \in E_S$, $a, b \in K_e$, $ab \in K_f$. If $e \neq f$, then

$$\langle a\rangle \cap \langle f\rangle = \emptyset = \langle b\rangle \cap \langle f\rangle,$$

therefore, applying $(SD_\wedge)$ we obtain

$$\langle a,b\rangle \cap \langle f\rangle = (\langle a\rangle \vee \langle b\rangle) \cap \langle f\rangle = \langle a\rangle \cap \langle f\rangle = \emptyset$$

which contradicts to the fact that $f \in \langle a,b\rangle$. Thus $e = f$ and $ab \in K_e$.

LEMMA 4.5. The equivalence $\mathcal{K}_S$ is a congruence.

PROOF. Let $e,f \in E_S$, $e \neq f$. It is to be proved that

$$K_e \vee K_f = K_e \cup K_f .$$

If the equality is not valid, then for some $a \in K_e$, $b \in K_f$ we have $\{ab, ba\} \not\subset K_e \cup K_f$. For the sake of definiteness, suppose that $ab \notin K_e \cup K_f$. Then for some $g \in E_S$, we have $ab \in K_e$ whence

$$\langle a\rangle \cap \langle g\rangle = \emptyset = \langle b\rangle \cap \langle g\rangle .$$

Using $(SD_\wedge)$, we obtain $\langle a,b\rangle \cap \langle g\rangle = \emptyset$ which leads to a contradiction. Consequently,

$$K_e \vee K_f = K_e \cup K_f$$

is a semigroup with two idempotents precisely, but then due to Proposition 6 of [I5] $K_e \cup K_f$ is a band of semigroups K_e and K_f. By Lemma 4.2 we obtain $K_e K_f \subset K_f$ or $K_e K_f \subset K_e$. This proves the assertion of the lemma.

LEMMA 4.6. Given $e \in E_S$, the group G_e is locally cyclic.

PROOF. It follows immediately from Theorem 2.2 and Lemma 4.I.

LEMMA 4.7. Given $e \in E_S$, the lattice $\mathrm{Sub}\,(K_e/G_e)$ is $\wedge$-semidistributive.

PROOF. For $e \in E_S$, by Lemma 4.4, K_e is a subsemigroup with the ideal G_e. Define

$$F_e = K_e/G_e$$

(Rees quotient semigroup).

Let $\varphi : K_e \to F_e$ be a natural homomorphism. It is easy to see that the set $Sub_G K_e$ is a principal filter $\uparrow G_e$ of the lattice $Sub\, K_e$, hence $Sub_G K_e$ is a $\wedge$-semidistributive lattice. Furthermore, since F_e is a nilsemigroup with the zero G_e, the lattice $Sub_G F_e$ consists of all non-empty subsemigroups of F_e and it is a principal filter of $Sub\, F_e$. Since the homomorphism φ induces an isomorphism of the lattice $Sub_G K_e$ onto $Sub_G F_e$, the latter is $\wedge$-semidistributive. Because $Sub\, F_e$ is obtained from $Sub_G F_e$ by adjoining the zero, the former is $\wedge$-semidistributive. Lemma is proved.

LEMMA 4.8. Let S be a nilsemigroup. The lattice Sub S is $\wedge$-semidistributive iff S is an U-semigroup. In this case, S is n-nilpotent where $n \leq 5$.

PROOF. Sufficience is obvious. Let us prove necessity. Assume that Sub S is $\wedge$-semidistributive. By Lemma 4.I, the indexes of all elements are bounded by 5. Assume that S is not an U-semigroup. Let us pick two elements $a, b \in S$ such that $ab \notin \langle a \rangle \cup \langle b \rangle$ with the corresponding value of a function

$$(x,y) \mapsto |x| + |y|$$

being the least for pairs $(x,y) \in S \times S$ such that $xy \notin \langle x \rangle \cup \langle y \rangle$. Clearly, $a \neq 0$, $b \neq 0$. Define

$$C = \langle a^2, a^3 \rangle \cup \langle b^2, b^3 \rangle,$$

$$A = \{a\} \cup C,$$

$$B = \{b\} \cup C.$$

We show that $A \in Sub\, S$. Since for any $l \geqslant I$, $q > I$

we have

$$|b^q| < |b|, \quad aa^1 = a^1a \in C,$$

by the minimality of $|a| + |b|$, for every $x \in C$ we infer that $ax \in \langle a \rangle \cup \langle x \rangle$. Since S is a nilsemigroup, we have $ax \neq a$ and therefore

$$ax \in \langle a^2 \rangle \cup \langle a^3 \rangle \cup \langle x \rangle \subset C .$$

Similarly, $xa \in C$. Thus $A \in \mathrm{Sub}\, S$. By the same manner we obtain $B \in \mathrm{Sub}\, S$. Since $a \notin \langle ab \rangle$, $b \notin \langle ab \rangle$, we deduce

$$\langle ab \rangle \cap A = \langle ab \rangle \cap (\{a\} \cup C) =$$
$$= \langle ab \rangle \cap C = \langle ab \rangle \cap (\{b\} \cup C) = \langle ab \rangle \cap B .$$

Using $(SD_\wedge)$ for $\mathrm{Sub}\, S$, we have

$$ab \in \langle ab \rangle \cap (A \vee B) = \langle ab \rangle \cap A = \langle ab \rangle \cap C \subset C \subset \langle a \rangle \cup \langle b \rangle,$$

violating the hypothesis. Consequently, S is an U-semigroup. By Lemma 3.5 of [18], S is n-nilpotent where $n \leq 5$. This completes the proof of the lemma.

COROLLARY 4.9. *Let* $\mathrm{Sub}\, S$ *be a* $\wedge$-*semidistributive lattice. Then for every* $e \in E_S$, *the semigroup* K_e *is an ideal extension of a periodic locally cyclic group* G_e *by an n-nilpotent U-semigroup where* $n \leq 5$.

LEMMA 4.10. *Let* S *be a semigroup satisfying the conditions* 1) – 4) *of Theorem* 2.4. *Assume that* $e, f \in E_S$. *If* $\{e,f\}$ *is a singular band, then* $G_e \cup G_f$ *is an ideal of a semigroup* $K_e \cup K_f$; *if* $\{e,f\}$ *is a chain, say,* $f \leq e$, *then* G_f *is an ideal of a semigroup* $K_e \cup K_f$.

PROOF. First, assume that $\{e,f\}$ is a singular band, say, $ef = f$, $fe = e$. Let $x \in K_e$, $a \in G_f$. Then

$$xa = xaf, \quad ax = fax = efax = eax,$$

hence $xa \in G_f$, $ax \in G_e$. Therefore,

$$K_eG_f \cup G_fK_e \subset G_e \cup G_f \ .$$

By analogy,

$$K_fG_e \cup G_eK_f \subset G_e \cup G_f \ .$$

and this is the first part of the assertion.

To prove the second one, assume that $ef = fe = f$, $x \in K_e \cup K_f$, $a \in G_f$. Then $ax, xa \in K_f$ and

$$xa = xaf \ , \ ax = fax \ .$$

Consequently, G_f is an ideal of $K_e \cup K_f$.

LEMMA 4.11. *If S is a semigroup with a $\wedge$-semidistributive subsemigroup lattice, then S is an UG-semigroup.*

PROOF. Given $e,f \in E_S$, $a \in K_e$, $b \in K_f$, we prove that

$$ab \in \langle a \rangle \cup \langle b \rangle \cup G_e \cup G_f \ . \qquad (7)$$

If $e = f$, it follows from Corollary 4.9. Now assume that $e \neq f$. First, suppose that $\{e,f\}$ is a singular band. Then $G_e \cup G_f$ is an ideal of $K_e \cup K_f$ by Lemma 4.10. Denote the Rees quotient semigroup $(K_e \cup K_f)/(G_e \cup G_f)$ by P with

$$\varphi : K_e \cup K_f \to P$$

being the natural homomorphism. Evidently, P is a nil-semigroup. The homomorphism φ induces an isomorphism of the principal filter $\uparrow(G_e \cup G_f)$ of the lattice $\mathrm{Sub}\,(K_e \cup K_f)$ onto the filter $\mathrm{Sub}_G P$ of the lattice $\mathrm{Sub}\,P$. As in the proof of Lemma 4.7, one can deduce that $\mathrm{Sub}\,P$ is $\wedge$-semidistributive. By Lemma 4.8 we obtain

$$\varphi(ab) = \varphi(a)\varphi(b) \in \langle \varphi(a) \rangle \cup \langle \varphi(b) \rangle \ .$$

This implies that (7) holds if $\{e,f\}$ is a singular band.

Now assume that $ef = fe = f$. By induction on $|b|$ we show that

$$ab, ba \in \langle b \rangle \cup G_f \ .$$

If $|b| = I$, then it stems from Lemma 4.IO. Let $b \notin G_f$. Then for $k > I$, $|b^k| < |b|$ and, by induction,

$$ab^k,\ b^k a \in \langle b^k \rangle \cup G_f \ .$$

We claim that $ab \in \langle b \rangle \cup G_f$. In fact, define

$$C = (\langle b \rangle \cup G_f) \cap (\langle ab \rangle \cup G_f) \ .$$

At first suppose that $C = G_f$. Set

$$A = \langle a \rangle \cup G_f \ .$$

By Lemma 4.IO, $A \in \mathrm{Sub}\ S$. Clearly,

$$(\langle ab \rangle \cup G_f) \cap A = G_f \ .$$

In view of $\wedge$-semidistributivity of Sub S we infer that

$$ab \in (\langle ab \rangle \cup G_f) \cap (A \vee (\langle b \rangle \cup G_f)) = G_f \ .$$

Assume for the remainder that $C \neq G_f$. If $ab \in C$ then $ab \in \langle b \rangle \cup G_f$ because of the inclusion

$$C \subset \langle b \rangle \cup G_f \ .$$

Now we consider the case $ab \notin C$. If $b \in C$ then $b \in C \setminus G_f \subset \langle ab \rangle$ and $b = (ab)^k$ for some $k \geqslant I$. The case $k > I$ is impossible, otherwise

$$b = (ab)^k = (ab)^{k-I} ab = ((ab)^{k-I} a) b \in G_f \ ,$$

violating the hypothesis. Hence $b = ab$ and

$$ab \in \langle b \rangle \cup G_f \ .$$

The possibility $b \notin C$ remains to be considered. Thus we have the following situation:

$$C = (\langle b \rangle \cup G_f) \cap (\langle ab \rangle \cup G_f) \neq G_f \ ,$$

$b \notin C$, $ab \notin C$. Let us show that it leads to a contradiction. Set

$$A = \langle a \rangle \cup C .$$

We have to prove that

$$A \cap (\langle ab \rangle \cup G_f) = C . \qquad (8)$$

Indeed, from the definition of C we deduce that

$$C \subset A \cap (\langle ab \rangle \cup G_f).$$

For the converse, let

$$x \in A \cap (\langle a \rangle \cup G_f).$$

Assume that $x \notin G_f$. Then

$$x \in (\langle a \rangle \cup C) \setminus G_f .$$

Since $C \subset \langle b \rangle \cup G_f$ and G_f is an ideal of $K_e \cup K_f$, the element x may be represented as

$$x = w(a, b^{k_I}, b^{k_2}, \ldots, b^{k_m})$$

where $k_i > I$ ($I \leqslant i \leqslant m$) and $w(x_I, x_2, \ldots, x_{m+I})$ is a semigroup word of $m+I$ variables $x_I, x_2, \ldots, x_{m+I}$.

Recalling the inductive hypothesis we deduce that $x \in \langle b \rangle \cup G_f$. Consequently,

$$x \in (\langle b \rangle \cup G_f) \cap (\langle ab \rangle \cup G_f) = C .$$

Thus, (8) is valid. Applying $(SD_\wedge)$ for Sub S we have

$$ab \in (\langle ab \rangle \cup G_f) \cap (A \vee (\langle b \rangle \cup G_f)) = C .$$

This contradicts the assumption $ab \notin C$. Therefore, in all cases $ab \in \langle b \rangle \cup G_f$. Lemma is proved.

The necessity of Theorem 2.4 now follows from the aforesaid lemmas.

5. PROOF OF THEOREM 2.4. SUFFICIENCY

In this section, S will be a semigroup satisfying the conditions I) - 5) of Theorem 2.4. For short, the set of all idempotents of S will be denoted by E.

LEMMA 5.I. *Let* $x \in S$, $e \in E$ *and* $x \in G_e$. *If*

$$x = x_I x_2 x_3 \ldots x_n$$

for some $x_I, x_2, x_3, \ldots, x_n \in S$, *then*

$$x = (ex_I e)(ex_2 e)(ex_3 e)\ldots(ex_n e).$$

PROOF. By induction on n. If $n = II$, the assertion is clear. Let $n > I$. Set

$$y = x_2 x_3 \ldots x_n .$$

Then $x = x_I y = (ex_I)(ye)$. By means of the conditions 2) and 3), $ex_I \in K_e$ or $ye \in K_e$.

Let $ex_I \in K_e$. Then $ex_I \in G_e$ and $ex_I = ex_I e$, whence

$$x = ex_I eye = (ex_I e)(eye). \qquad (9)$$

Let us show that $eye \in G_e$. Indeed, if $y \in K_f$, $e \neq f$ and $\{e,f\}$ is a singular band or $ef = fe = e$, then it is obvious. Now assume that $ef = fe = f$. Then $eye \in K_f$ which contradicts the fact that $ex_I e \in K_e$ in view of (9). Thus $eye \in G_e$ and the inductive assumption yields

$$eye = ex_2 x_3 \ldots x_n e = (ex_2 e)(ex_3 e) \ldots (ex_n e).$$

Now using (9), we obtain the needed decomposition for x.

The case $ye \in K_e$ is treated similarly.

LEMMA 5.2. *Let* $A \in \mathrm{Sub}\, S$, $A \neq \emptyset$. *Define* A' *as follows:*

$$A' = A \cup Gr_S A . \qquad (IO)$$

Then $A' \in \mathrm{Sub}_G S$ *and*

$$A \setminus Gr_A A = A' \setminus Gr_{A'} A'.$$

PROOF. Let us show that A' is a subsemigroup of S. Let $e,f \in E$, $a \in K_e \cap A'$, $b \in K_f \cap A'$. According to (I0), we have $e,f \in E_A$. Owing to the condition 4),

$$\langle a \rangle \cup G_e , \langle b \rangle \cup G_f \in \mathrm{Sub}_G S .$$

Applying the condition 5) and (I0), we obtain

$$ab \in (\langle a \rangle \cup G_e) \vee (\langle b \rangle \cup G_f) =$$

$$= (\langle a \rangle \cup G_e) \cup (\langle b \rangle \cup G_f) \subset A' .$$

Thus, $A' \in \mathrm{Sub}\, S$ and A' contains A. It is easy to see that these subsemigroups have the same non-group elements. Evidently, $A' \in \mathrm{Sub}_G S$. Lemma is proved.

Now we have sufficient auxillary means for proving $\wedge$-semidistributivity of $\mathrm{Sub}\, S$. Let $A,B,C \in \mathrm{Sub}\, S$ and

$$A \cap B = A \cap C . \tag{II}$$

We have to show that

$$A \cap (B \vee C) \subset A \cap B . \tag{I2}$$

Assume that $x \in A \cap (B \vee C)$, $x \in K_e$, $e \in E$. Using condition 5) and Lemma 5.2 one can obtain:

$$x \in A \cap (B \vee C) \subset A \cap (B' \vee C') =$$

$$= A \cap (B' \cup C') = (A \cap B') \cup (A \cap C').$$

For definiteness, suppose that $x \in A \cap C'$. If $x \notin G_e$, then by (I0) and (II) we have

$$x \in A \cap C = A \cap B .$$

Now let $x \in G_e$. Since G_e is a periodic group according to condition I), we have $e \in A \cap C'$ in view of $x \in A \cap C'$. Taking into account (I0) and (II) we then obtain $e \in A \cap C = A \cap B$, whence

$$e \in A \cap B \cap C . \tag{I3}$$

Since $x \in B \vee C$, for some $x_I, x_2, \ldots, x_n \in B \cup C$, we have by Lemma 5.I:

$$x = x_1 x_2 \ldots x_n = (ex_1e)(ex_2e)\ldots(ex_ne). \qquad (14)$$

When proving Lemma 5.1 it was established that all elements ex_ie ($1 \leqslant i \leqslant n$) in the decomposition (14) belong to G_e , where from using (13) we must have

$$ex_ie \in G_e \cap (B \cup C) = (G_e \cap B) \cup (G_e \cap C). \qquad (15)$$

Subsemigroups $A \cap G_e$, $B \cap G_e$, $C \cap G_e$ are subgrouops of G_e because the latter is a periodic group. Since Subgr G_e is distributive by the condition 4), we obtain applying (14),(15) and (11):

$$x \in (A \cap G_e) \cap ((B \cap G_e) \vee (C \cap G_e)) =$$
$$= (A \cap B \cap G_e) \vee (A \cap C \cap G_e) = A \cap B \cap G_e \subset A \cap B ,$$

as is required. Thus the inclusion (12) holds. This completes the proof of Theorem 2.4.

6. CRITERION OF DISTRIBUTIVITY OF $\wedge$-SEMIDISTRIBUTIVE SUBSEMIGROUP LATTICE.

It is natural to reveal a difference between the class of semigroups described by Theorem 2.4 and that of semigroups with distributive subsemigroup lattices. The main result of this section (Theorem 6.2) due to S.I.Kacman is published here under his kind permission.

We begin by proving the before-mentioned classes do not coincide. Let $\mathfrak{M}$ be the class of all semigroups isomorphic to direct products of the form $A \times J$ with A a nontrivial finite cyclic group and J a two-elemented singular band. Now let φ be a nontrivial homomorphism of a nontrivial finite cyclic group A onto a finite cyclic group B. This homomorphism defines [2] an ideal extension S of the semigroup B by the semigroup A^0 (A^0 is the group A with the zero adjoined and φ may be considered as a partial homomorphism of $A^0 \setminus \{0\}$ into B). Let $\mathfrak{L}$ be the class of all semigroups isomorphic to such semigroups S. The union of classes $\mathfrak{N}$ and $\mathfrak{B}$ will be denoted by $\mathcal{L}$.

LEMMA 6.I. *Let* S *be a semigroup of the class* $\mathcal{L}$. *Then the lattice* Sub S *is* $\wedge$-*semidistributive but non-distributive.*

PROOF. $\wedge$-semidistributivity of Sub S follows immediately from Theorem 2.4. We show that Sub S fails to be distributive. First, let $S = A \times J$ with $A = \langle a \rangle$ being a finite nontrivial cyclic group generated by a and $J = \{e,f\}$ being a singular band, say, right zero semigroup. Then, denoting by i the identity of the group A, we get

$$(a,e)\,(i,f) = (ai,ef) = (a,f) \in \langle (i,f) \rangle .$$

Hence the semigroup S is not a strong band (in the sence of L. N. Shevrin [I8]) of its torsion classes $A \times \{e\}$ and $A \times \{f\}$. Consequently, Sub S is not distributive. The case when S lies in the class $\mathfrak{B}$, is treated similarly.

THEOREM 6.2 (S. I. Kacman). *Let* S *be a semigroup with* $\wedge$-*semidistributive subsemigroup lattice. Then* Sub S *is non-distributive iff* S *contains a subsemigroup which belongs to* $\mathcal{L}$.

PROOF. Sufficiency follows from Lemma 6.I. We shall prove necessity. Assume that Sub S is $\wedge$-semidistributive and any subsemigroup of S does not belong to the class $\mathcal{L}$. To prove distributivity of Sub S, it is sufficient to show that S is a strong band of its torsion classes.

Let e,f E_S, $e \neq f$, $x \in K_e$, $y \in K_f$ be given. If $\{e,f\}$ is a singular band, then the groups G_e and G_f are trivial. In fact, suppose that $G_f \neq \{f\}$, $ef = f$, $fe = e$. Pick in G_f an element a with order $n > I$. (It exists owing to Theorem 2.4). Then

$ae \in G_e$ and for every $k = 1,2,\ldots$

$$(ae)^k f = (ae)\ldots(ae)f =$$

$$= afefafef\ldots fef = afaf\ldots af = a^k.$$

Also

$$a^k e = aa\ldots ae = afaf\ldots fae =$$

$$= aefaef\ldots ae = aeae\ldots ae = (ae)^k.$$

Thereof we deduce that functions $(ae)^k \mapsto (ae)^k f$ and $a^k \mapsto a^k e$ are mutually inverse isomorphisms of cyclic groups $\langle a \rangle \subset G_f$ and $\langle ae \rangle \subset G_e$. But then the semigroup $\langle a,e \rangle$ is isomorphic to $\langle a \rangle \times \{e,f\}$, i.e. belongs to the class $\mathfrak{M}$ which contradicts to the hypothesis. Thus, the groups G_e and G_f are trivial. Since S is an UG-semigroup, we infer that

$$xy \in \langle x \rangle \cup \langle y \rangle \cup G_e \cup G_f = \langle x \rangle \cup \langle y \rangle .$$

This is what had to be shown.

Finally, let $\{e,f\}$ be a chain, say, $ef = fe = f$. Suppose that $xy \notin \langle x \rangle \cup \langle y \rangle$. Since S is an UG-semigroup, it follows that

$$xy \in G_f \setminus \langle y \rangle . \qquad (16)$$

Then

$$xy = xyf = xfyf = xefyf = (xe)(yf). \qquad (17)$$

Put $G = \langle xe \rangle$. If $xe = e$, then

$$xy = xeyf = eyf = efyf = fyf = yf \in \langle y \rangle ,$$

which contradicts (16). Therefore, G is non-trivial group. Define a map $\varphi : G \to G_f$ as follows:

$$\varphi(a) = af \quad \text{for any} \quad a \in G .$$

Note that $\varphi(xe) \neq f$, otherwise from (17) we get

$$xy = (xe)(yf) = xefyf = \varphi(xe)yf = fyf \in \langle y \rangle ,$$

which leads to a contradiction.

It is easy to check that φ is a homomorphism of

G into G_f . Set

$$H = \langle yf, \ (xe) \rangle .$$

Since by Theorem 2.4 G_f is a locally cyclic group, H is a finite cyclic group and $\varphi : G \to H$ is a nontrivial homomorphism. Further, from the equalities

$$(xe)(yf) = xefyf = \varphi(xe)yf ,$$
$$(yf)(xe) = yfxef = yf\varphi(xe) ,$$
$$xe\varphi(xe) = xexef = xefxef =$$
$$= \varphi(xe)\varphi(xe) = xefxe = \varphi(xe)xe$$

it follows that φ is a nontrivial partial homomorphism of $G^o \setminus \{0\}$ into the semigroup H which defines semigroup $G \cup H$ as an ideal extension of H by G^o . Thus the semigroup $G \cup H$ belongs to $\mathfrak{B}$ which contradicts the hypotheses. Consequently, $xy \in \langle x \rangle \cup \langle y \rangle$ what completes the proof of the theorem.

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REFERENCES

I. Benzaken, C. and H.C. Mayr, Notion de demi-bande. Demi-bandes de type deux, Semigroup Forum, IO (I975), II5-I28.

2. Clifford, A.H. and G.B. Preston, The Algebraic Theory of Semigroups, Amer. Math. Soc., Providence, R. I., I (I96I).

3. Cohn, P., Universal Algebra, Harper & Row, New York, Evanston and London (I965).

4. Curzio, M., *Alcune osservazione sul reticolo dei sottogruppi d'un gruppo finito*, Ricerche mat. 6 (I957), 96-II0.

5. Ego, M., *Structure des demi-groupes dont le treillis des sous-demi-groupes est distributif*, C. r. Acad. Sci. 252 (I96I), 2490-2492.

6. Ego, M., *Structure des demi-groupes dont le treillis des sous-demi-groupes est modulaire ou semi-modulaire*, C. r. Acad. Sci. 254 (I962), I723-I725.

7. Gierz, G., K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin, Heidelberg, New-York (I980).

8. Gratzer, G., *General Lattice Theory*, Akademie-Verlag, Berlin (I978).

9. Igoshin, V.I., *On lattices of algebraic systems quasivarieties*, Ordered sets and lattices, 5 Saratov (I978), 44-55 (in Russian).

I0. Kurosh, A.G., *Group Theory*, Nauka, Moscow (I967) (in Russian).

II. Napolitani, F., *Elementi $\cup$-quasidistributivi nel reticolo dei sottogruppi di un gruppo*, Ricerche mat. I4 (I965), 93-I0I.

I2. Napolitani, F., *Elementi $\cap$-quasidistributivi ed u.c.r. elementi del reticolo dei sottogruppi di un gruppo finito*, Ricerche mat. I7 (I968), 95-I08.

I3. Papert, D., *Congruences in semilattices*, J. of The London Math. Soc. 39 (I964), 723-729.

I4. Petrich, M., *Lectures in Semigroups*, Akademie-Verlag, Berlin (I977).

15. Prosvirov, A.S., On periodic semigroups, Mat. Zap. Ural. Univ., Sverdlovsk, 8 (1971), 77-94.

16. Schmidt, J., Binomial pairs, semi-Browerian and Browerian semilattices, Notre Dame J. of Formal Logic, 19 (1978), 421-434.

17. Shevrin, L.N., Semigroups with certain classes of subsemigroup lattices, Dokl. Akad. Nauk SSSR, 138 (1961), 796-798 (in Russian).

18. Shevrin, L.N., On lattice properties of semigroups, Sib. Mat. J., 3 (1962), 446-470 (in Russian).

19. Shevrin, L.N., Semigroups with Dedekind subsemigroup lattices, Dokl. Akad. Nauk SSSR, 148 (1963), 292-295 (in Russian).

20. Shevrin, L.N. and A.J. Ovsyannikov, Semigroups and their subsemigroup lattices, Semigroup Forum, 27 (1983), 1-154.

21. Varlet, J., Congruences dans les demi-lattis, Bull. Soc. roy. sci. Liege 34 (1965), 231-240.

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