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Some Extremal Properties of the Solutions of Ordinary Differential Equations Systems

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Based on variables doubling procedure the extremals flow immersion into the trajectories bunch has been considered. The conditions for existence of extremals have been obtained. The Lyapunov function for a doubled linearized system has been constructed.

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1. Introduction

Let $f = (f_1, \ldots, f_m) \subset U(x_0)\colon \mathbb{E}^m \to \mathbb{E}^m$ be a C^1-vector field, i.e. $f \in C^1(U(x_0))$ and moreover $f \in \mathrm{Lip}^{loc}(U(x_0)$ is a function being an endomorphism of the neighbourhood $U(x_0)$ on $\mathbb{E}^m$.

A system of differential equations $\dot{x} = f(x)$ is canonic iff $\operatorname{div} f(x) = 0$ (Liouville's condition) [1]. It is evident that in general this condition is not fulfilled. In what follows we will consider a system with additive perturbation y of the vector field $f(x)$ of the form $\dot{x} = f(x+y)$ where $x + y \in U(x_0) \subset \mathbb{E}^m$.

2. General consideration

The *doubled system* for the system $\dot{x} = f(x)$, $x, y \in \mathbb{E}^m$ has been introduced in [1] as

$$\dot{x} = f(x+y), \quad \dot{y} = f(x) - f(x+y). \qquad (1)$$

It is clear that there exists a map $\Psi\colon U(x_0) \to \mathbb{R}$, such that $f(x) = \nabla\Psi$. Then the system (1) is a canonic one, and the function

$$E(x, y) = \Psi(x+y) - \Psi(x), \quad E(x, 0) = 0$$

plays the part of the Hamiltonian.

As far as it is known, for the first time the doubling variables procedure as a Hamiltonian system constructing method has been proposed by Yu. G. Pavlenko [1] exactly in terms of perturbation. This technique was successfully applied to solve problems of the variation calculus, for inversion of series, maps and others (see also [2]).

For system (1) it is clear that:

(i) if $x(0) - x_0 = y(0) - y_0$ is an initial conditions for (1) and $y_0 = 0$ then $y = 0\ \forall\, t \in T \subset \mathbb{R}$;

(ii) if $f \in \mathrm{Lip}^{loc}(U(x_0))$ and one is monotone non-decreasing function then $\forall\, y_0 \in U(x_0)$, such that $y(t, y_0) \xrightarrow[t \in T]{} 0$;

(iii) if $||y|| \ll ||x||$ then second equation of the system (1) coincides with the equation in variations.

System (1) in the last case becomes

$$\dot{x}_i = f_i(x), \quad \dot{y}_i = -(y, \frac{\partial f(x)}{\partial x_i}), \quad i = \overline{1, m} \qquad (2)$$

and one has a non-degenerated Hamiltonian

$$E(x, y) = (y, f(x)).$$

The function being dual to $E(x, y)$ (in terms of Jung-Legendre) is the distribution density of the field on T

$$\Lambda(x, \dot{x}) = \sup_y((y, \dot{x}) - E(x, y)).$$

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Then, in real motion

$$\delta S(x) \stackrel{def}{=} \int_T \delta\Lambda dt = 0$$

is faithful, where δ is isochronic variation and $S(x)$ is the action (the field function limited value).

The value $\delta K \stackrel{def}{=} (y, \delta x) - E\delta t$ along the extremals is invariant in respect to the action (Cartan invariant) and $\delta K = 0$ on the ends of the extremals (transversability condition). Let $y \to 0$. Then the equality

$$\left(\frac{\partial E}{\partial y}\right)_{y=0} = \left(\frac{dx}{dt}\right)_{y=0}$$

is faithful along the flow of the system $\dot{x} = f(x)$.

Thus, it is proved the following

Proposition 1 *If the Hamiltonian $E(x, y)$ is non-degenerate then the system* (1) *delivers the necessary conditions of the extremum for $S(x)$.*

Ad hoc it is clear that

$$\Lambda(x, \dot{x}) = (\dot{x}, \varphi(\dot{x}) - x) - \Psi(\varphi(\dot{x})) + \Psi(x).$$

It is also clear that $\varphi = f^{-1}\colon \mathbb{E}^m \to U(x_0)$ and let us define a map $\Phi\colon \mathbb{E}^m \to \mathbb{R}$, such that $\varphi = \nabla\,\Phi$. The Lagrange equation being necessary extremum condition has the form

$$\ddot{x}\frac{\partial\varphi(\dot{x})}{\partial\dot{x}} = f(x)$$

or

$$\dot{x} = u, \quad \dot{u} = f(x)\left(\frac{\partial\varphi(u)}{\partial u}\right)^{-1}$$

with the initial conditions

$$x(0) - x_0 = \dot{x}(0) - u_0, \quad u_0 \stackrel{def}{=} f(x_0 + y_0).$$

The Cauchy problem first integral reads

$$(u, \varphi(u)) - (u_0, \varphi(u_0)) =$$
$$\Psi(x) - \Psi(x_0) + \Phi(u) - \Phi(u_0).$$

Let $y_0 = 0$ and $y(t, y_0) = 0$. It is evident that $\Lambda = 0$ in this case. Then $u_0 = f(x_0)$, $\varphi(\dot{x}) = x$ and the first integral is a partial integration identity

$$(x, f(x)) - (x_0, f(x_0)) =$$
$$\Psi(x) - \Psi(x_0) - \Phi(f(x)) + \Phi(f(x_0)).$$

Theorem 1 *There exists the extremals field for non-degenerate positive functional such that one contains the original equation characteristics.*

Proof The theorem statement is a corollary of the proposition 1. Moreover monotonic nondecreasing of $f(x)$ ensures convergence $y(t,\, y_0)$ to zero on T and extremal concerns the characteristics at the point $(x,\, 0)$ of the phase space $(x,\, y)$. ■

Otherwise characteristics is "envelope curve" of the extremals field.

3. Example: linear system [3, 4]

Let us consider the linear system with a constant coefficient matrix

$$\dot{x} = a \cdot x.$$

It is evident that the system can be transformed to the Jordan form by a non-singular transformation

$$\dot{\xi} = j \cdot \xi$$

where j-matrix consists from Jordan cell. Let the number of Segre matrix a is equal to $(m, m, 1)$ for brevity. This means a coincidence full and algebraic multiplicities

$$\dim\ker(a - \lambda E)^m = m.$$

Otherwise the matrix j is reduced to one maximum Jordan cell and the matrix a characteristic polynomial coincides with its minimal polynomial.

Procedure of additive doubling allows two options. In the first variant the partial (diagonalizable) doubling is executed by the scheme

$$\dot{\xi}_i = \lambda(\xi_i + \eta_i) + \xi_{i+1}, \quad i = \overline{1, m-1},$$
$$\dot{\xi}_m = \lambda(\xi_m + \eta_m)$$

where $\eta = (\eta_1, \ldots, \eta_m)$ is a perturbations vector.

Hamiltonian is defined by the formula

$$E(\xi, \eta) = \lambda(\xi, \eta) + \frac{\lambda||\eta||^2}{2} + \sum_{1 \leqslant i \leqslant m-1} \xi_{i+1}\eta_i$$

for the doubling system

$$\text{colon}(\dot{\xi}, \dot{\eta}) = \alpha \cdot \text{colon}(\xi, \eta)$$

with the matrix α given as

$$\begin{pmatrix} \lambda & 1 & 0 & \ldots & \lambda & \ldots & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -\lambda & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \ldots & -1 & -\lambda & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \ldots & \ldots & -1 & -\lambda \end{pmatrix}.$$

It is clear that

$$\alpha = \begin{pmatrix} j & \text{diag}\lambda \\ 0 & -j^T \end{pmatrix}.$$

The momentum components on the coefficient matrix main diagonal and on the top border of the diagonal with full doubling are added, i.e.

$$\dot{\xi}_i = \lambda(\xi_i + \eta_i) + \xi_{i+1} + \eta_{i+1}, \quad i = \overline{1, m-1},$$
$$\dot{\xi}_m = \lambda(\xi_m + \eta_m)$$

in which connection

$$\tilde{E}(\xi, \eta) = E(\xi, \eta) + \sum_{1 \leqslant i \leqslant m-1} \eta_i \eta_{i-1}$$

where

$$E(\xi, \eta) = \lambda(\xi, \eta) + \frac{\lambda||\eta||^2}{2} + \sum_{1 \leqslant i \leqslant m-1} \xi_{i+1}\eta_i$$

Otherwise, the full and partial doubling Hamiltonians differ by the allocated sums amount containing only gyroscopic terms. It is clear that one does not affect the recording of the canonical equations for the components of the perturbation vector η, such that only the top-right block matrix α first coefficients of the double system is deformed. In this case the system coefficients matrix reads

$$\tilde{\alpha} = \begin{pmatrix} \lambda & 1 & \ldots & \lambda & 1 & \ldots & \ldots & 0 \\ 0 & \lambda & \ldots & 0 & \lambda & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \ldots & -\lambda & 0 & \ldots \\ 0 & 0 & \ldots & \ldots & \ldots & -1 & -\lambda & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & -1 & -\lambda \end{pmatrix}$$

Therefore the following assertion is faithful

Theorem 2 *If the matrix trace is zero equation of system $\dot{x} = ax$ then quadrics*

$$L(x) = \left| ||x||_a^2 \right| \stackrel{def}{=} |(ax, x)|$$

is the system Lyapunov function in the point $(0, 0)$.

To the Hamiltonian cases belong also the systems with the matrix a for incomplete and full doubling. The role of the Lyapunov function plays the absolute value of square α-norm of the vector x

$$L(x) \stackrel{def}{=} |(\alpha x, x)|.$$

4. Conclusion

1° The dynamic system trajectories space is weakly dense immersed in the bundle of extremals. Based on variables doubling procedure the extremals flow immersion into the trajectories bunch has been considered.

2° If a perturbation vector is small then the double system coincides with the equation in variations and the same as the original system with a vanishing variations.

3° It turns out that every ordinary differential equation (system of equations) is generated by some condition of an extremal problem.

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