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PARTIAL DIFFERENTIAL EQUATIONS

The Cauchy Problem for Complete Second-Order Hyperbolic Differential Equations with Variable Domains of Operator Coefficients

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Complete second-order hyperbolic differential equations with constant domains of operator coefficients were investigated in [1, 2]. In the case of variable domains of operator coefficients, second-order hyperbolic differential equations with a two-term leading part were analyzed in [3–5]. In the present paper, we investigate second-order hyperbolic differential equations with a three-term leading part in the case of variable domains of operator coefficients.

1. STATEMENT OF THE PROBLEM

Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. On a bounded interval $]0,T[$ of the real axis, we consider the Cauchy problem for the differential equation

$$Lu \equiv d^2u/dt^2 + B(t)du/dt + A(t)u = f, \qquad t \in]0,T[, \tag{1}$$

with the homogeneous initial conditions

$$u|_{t=0} = 0, \qquad du/dt|_{t=0} = 0, \tag{2}$$

where u and f are functions of the variable t ranging in H, $A(t)$ and $B(t)$, $t \in \Theta$, are linear unbounded operators in H with domains $D(A(t))$ and $D(B(t))$, respectively, depending on t, and Θ is some set of full measure on $[0,T]$.

We assume that the operator $A(t)$, $t \in \Theta$, satisfy the following conditions.

Condition A1. The operators $A(t)$, $t \in \Theta$, are self-adjoint in H and satisfy the inequalities

$$(A(t)u,u) \geq c_1(t)|u|^2 \qquad \forall u \in D(A(t)), \tag{3}$$

where $c_1(t) > 0$ are constants independent of u.

Condition A2. The inverse operators $A^{-1}(t)$, $t \in \Theta$, have the strong regular derivative $dA^{-1}(t)/dt \in L_\infty(]0,T[,\mathscr{L}(H))$ on $]0,T[$, which satisfies the inequalities

$$-\left(\frac{dA^{-1}(t)}{dt}g,\ g\right) \leq c_2\left(A^{-1}(t)g,g\right) \qquad \forall g \in H, \tag{4}$$

where $c_2 \geq 0$ is a constant independent of g and t.

The notion of strong regular derivative was introduced in [6] for operators defined almost everywhere.

We assume that the operators $B(t)$, $t \in \Theta$, satisfy the following conditions.

Condition B1. $D(A(t)) \subset D(B(t))$, $t \in \Theta$; $B(t)A^{-1/2}(t) \in L_\infty(]0,T[,\mathscr{L}(H))$, where $A^{-1/2}(t)$ are the inverses of the square roots $A^{1/2}(t)$ of $A(t)$, $t \in \Theta$, and

$$-\operatorname{Re}(B(t)u,u) \leq c_3|u|^2 \qquad \forall u \in D(A(t)), \tag{5}$$

where $c_3 > 0$ is a constant independent of u and t.

Condition B2. $A^{1/2}(t)B(t)\,(dA^{-1}(t)/dt) \in L_\infty(]0,T[,\mathscr{L}(H))$, and

$$-\operatorname{Re}(B(t)u, A(t)u) \le c_4\left|A^{1/2}(t)u\right|^2 \qquad \forall u \in D(A(t)), \tag{6}$$

where $c_4 \ge 0$ is a constant independent of u and t.

Let us prove the strong solvability of the Cauchy problem (1), (2). We also present an example of a mixed problem for a complete second-order hyperbolic partial differential equation whose coefficients contain nonintegrable singularities with respect to the variable t.

2. THE UNIQUENESS THEOREM FOR STRONG SOLUTIONS

First, we introduce spaces and give the definition of strong solutions of the Cauchy problem (1), (2). As the space of strong solutions, we choose the Hilbert space E that is the completion of the set

$$D(L) = \Big\{u \in L_2(]0,T[,H) :\ u(t) \in D(A(t)),\ t \in \Theta;\ d^2u/dt^2,\ A^{1/2}(t)du/dt,$$
$$A(t)u \in L_2(]0,T[,H);\ (A(t)u,u)|_{t=0} = 0,\ |(A(t)u,u)|\Big|_{t=T} < +\infty;\ u(0) = du(0)/dt = 0\Big\}$$

in the norm

$$\|u\|_E = \left(\int_0^T \left|\frac{du}{dt}\right|^2 dt + \int_0^T \left|A^{1/2}(t)u\right|^2 dt\right)^{1/2}.$$

As the space of right-hand sides f of Eq. (1), we choose the Banach space F that is the completion of $L_2(]0,T[,H)$ in the norm

$$\|f\|_F = \sup_{v\in E_0}\left\{\left|\int_0^T (f,\Im(t)v)dt\right| \Big/ \|v\|_0\right\}.$$

Here E_0 is the set of all functions $v \in L_2(]0,T[,H)$ such that $\sqrt{c_1(t)}v \in L_2(]0,T[,H)$, the derivative $dv/dt \in L_2(]0,T[,H)$ exists, and v satisfies the condition $v(0) = 0$, equipped with the Hermitian norm

$$\|v\|_0 = \left(\int_0^T \left|\frac{dv}{dt}\right|^2 dt + \int_0^T c_1(t)|v|^2 dt\right)^{1/2},$$

and $\Im(t)v = (T-t)dv/dt$. Note that F is the so-called negative space corresponding to the positive space $E_\Im$ equipped with the norm $\|w\|_\Im = \|\Im^{-}1(t)w\|_0$, where $\Im^{-1}(t)$ is the inverse of $\Im(t)$.

The Cauchy problem (1), (2) corresponds to a linear unbounded operator $L : E \supset D(L) \to F$ with domain $D(L)$. Let this operator L satisfy the closability criterion for linear operators in Banach spaces: if $u_n \to 0$ in E and $Lu_n \to f$ in F as $n \to \infty$, then $f = 0$. By $\bar L$ we denote the closure of L, and by $D(\bar L)$ we denote its domain. The solutions of the operator equation $\bar L u = f$, $f \in F$, are referred to as *strong solutions* of the Cauchy problem (1), (2).

Now we derive an *a priori* estimate for strong solutions of the Cauchy problem (1), (2).

Theorem 1. *Let the set $D(L)$ be dense in $L_2(]0,T[,H)$, and let the operator L admit the closure $\bar L$. If Conditions* A1, A2, *and* B1 *are satisfied, then there exists a constant $c_0 > 0$ independent of u and such that $\|u\|_E \le c_0\left\|\bar L u\right\|_F$ for all $u \in D(\bar L)$.*

Proof. In H, we consider the abstract smoothing operators $A_\varepsilon^{-1}(t) = [I+\varepsilon A(t)]^{-1}$, $\varepsilon > 0$, whose ranges are contained in $D(A(t))$ and which satisfy the following conditions.

A1. If $\varepsilon \to 0$, then $\int_0^T |A_\varepsilon^{-1}(t)v - v|^2\,dt \to 0$ for all $v \in L_2(]0,T[,H)$.

A2. The operators $A_\varepsilon^{-1}(t)$ are strongly and regularly differentiable with respect to t in H, and their derivatives satisfy $dA_\varepsilon^{-1}(t)/dt \in L_\infty(]0,T[,\mathscr{L}(H))$.

These properties were proved in [3].

Integrating by parts once more, we obtain the identity

$$2\,\mathrm{Re}\int\limits_0^T e^{c(T-t)}\left(\frac{d^2u}{dt^2},\ A_\varepsilon^{-1}(t)\Im(t)u\right)dt+\int\limits_0^T e^{c(T-t)}\left(A(t)u, A_\varepsilon^{-1}(t)u\right)dt$$

$$=2\,\mathrm{Re}\int\limits_0^T e^{c(T-t)}\left(Lu, A_\varepsilon^{-1}(t)\Im(t)u\right)dt-2\,\mathrm{Re}\int\limits_0^T e^{c(T-t)}(T-t)\left(B(t)\frac{du}{dt}, A_\varepsilon^{-1}(t)\frac{du}{dt}\right)dt \qquad (7)$$

$$+\int\limits_0^T e^{c(T-t)}(T-t)\Phi_\varepsilon(u,u)dt,$$

where $\Phi_\varepsilon(u,u)=\left(\left(d\left(A(t)A_\varepsilon^{-1}(t)\right)/dt\right)u,\ u\right)-c\left(A(t)A_\varepsilon^{-1}(t)u,\ u\right)$. Using the formula [3]

$$d\left(A(t)A_\varepsilon^{-1}(t)\right)/dt=-A(t)A_\varepsilon^{-1}(t)\left(dA^{-1}(t)/dt\right)A(t)A_\varepsilon^{-1}(t),$$

inequalities (4), and the relation $|A_\varepsilon^{-1/2}(t)v|\le|v|$ in the operator-differential form $\Phi_\varepsilon(u,u)$, we obtain

$$\Phi_\varepsilon(u,u)\le(c_2-c)\left|A^{1/2}(t)A_\varepsilon^{-1/2}(t)u\right|^2. \qquad (8)$$

If in (7) we use the estimate (8) and let ε tend to zero in the resulting inequality with regard for Property A1, then we obtain the inequality

$$2\,\mathrm{Re}\int\limits_0^T e^{c(T-t)}\left(\frac{d^2u}{dt^2},\ \Im(t)u\right)dt+\int\limits_0^T e^{c(T-t)}(A(t)u,u)dt$$

$$\le 2\,\mathrm{Re}\int\limits_0^T e^{c(T-t)}(Lu,\Im(t)u)dt-2\,\mathrm{Re}\int\limits_0^T e^{c(T-t)}(T-t)\left(B(t)\frac{du}{dt},\frac{du}{dt}\right)dt\equiv S_1+S_2$$

for $c\ge c_2$. Integrating by parts once more, we arrive at the inequality

$$\int\limits_0^T e^{c(T-t)}\left|\frac{du}{dt}\right|^2dt+\int\limits_0^T e^{c(T-t)}(A(t)u,u)dt\le S_1+S_2-c\int\limits_0^T e^{c(T-t)}(T-t)\left|\frac{du}{dt}\right|^2dt,$$

whose right-hand side does not exceed S_1 for $c\ge\max\{c_2,2c_3\}$ by virtue of (5). This, together with simple estimates, yields $\|u\|_E\le c_0\|Lu\|_F$ for all $u\in D(L)$. Performing the passage to the limit, we can generalize this inequality to all strong solutions of the Cauchy problem (1), (2).

It follows from Theorem 1 that if there exists a strong solution of the Cauchy problem (1), (2), then it is unique and continuous with respect to f.

3. THE EXISTENCE THEOREM FOR STRONG SOLUTIONS

The following statement establishes the strong solvability of the Cauchy problem (1), (2).

Theorem 2. *Let the set $D(L)$ be dense in $L_2(]0,T[,H)$, and let the operator L admit the closure $\bar L$. If Conditions* A1, A2, B1, *and* B2 *are satisfied and the operator $dA^{-1}(t)/dt$ has the strong regular derivative $d^2A^{-1}(t)/dt^2$ such that $A^{1/2}(t)\left(d^iA^{-1}(t)/dt^i\right)\in L_\infty(]0,T[,\mathscr{L}(H))$, $i=1,2$, then for each $f\in F$, the strong solution $u\in E$ of the Cauchy problem* (1), (2) *exists, is unique, and satisfies the inequality $\|u\|_E\le c_0\|f\|_F$.*

Proof. By virtue of a corollary of the Hahn–Banach theorem on the continuation of linear continuous functionals, to prove the strong solvability of the Cauchy problem (1), (2), it suffices to show that the range $R(L)$ of the operator L is dense in F, i.e., if

$$\int\limits_0^T(Lu,\Im(t)v)dt=0\qquad\forall u\in D(L) \qquad (9)$$

for some $v\in E_0$, then $v=0$.

In (9), we set $u = A^{-1}(t)h$, where $h \in L_2(]0,T[,H)$ is such that $dh/dt, d^2h/dt^2 \in L_2(]0,T[,H)$ and condition (2) is satisfied; then we obtain

$$\begin{aligned}
&\int_0^T \left(\frac{d^2h}{dt^2},\ A^{-1}(t)e^{c(T-t)}\Im(t)w\right) dt \\
&\qquad = -\int_0^T \left(h, e^{c(T-t)}\Im(t)w\right) dt - \int_0^T e^{c(T-t)}(T-t)\Phi(h,w)dt \\
&\qquad - \int_0^T \left(B(t)\frac{dA^{-1}(t)}{dt}h + B(t)A^{-1}(t)\frac{dh}{dt},\ e^{c(T-t)}\Im(t)w\right) dt,
\end{aligned} \tag{10}$$

where

$$\Phi(h,w) = \left(\frac{d^2A^{-1}(t)}{dt^2}h + 2\frac{dA^{-1}(t)}{dt}\frac{dh}{dt},\ \frac{dw}{dt}\right)$$

and w is found from the Cauchy problem $\Im(t)w = e^{c(t-T)}\Im(t)v$, $w(0) = 0$. We can readily see that $w \in E_0$.

It follows from (10) that the function $A^{-1}(t)\Im(t)w$ has the derivative in $L_2(]0,T[,H)$ and vanishes for $t = T$. In (10), we perform integration by parts, extend the resulting relation to all functions $h \in L_2(]0,T[,H)$, $dh/dt \in L_2(]0,T[,H)$, $h(0) = 0$, with the use of the passage to the limit, set $h = w$, and take the doubled real part; then we obtain

$$\begin{aligned}
&-2\,\mathrm{Re}\int_0^T \left(\frac{dw}{dt},\ \frac{d}{dt}\left[A^{-1}(t)e^{c(T-t)}(T-t)\frac{dw}{dt}\right]\right) dt + 2\,\mathrm{Re}\int_0^T e^{c(T-t)}(T-t)\left(w,\ \frac{dw}{dt}\right) dt \\
&\qquad = -2\,\mathrm{Re}\int_0^T e^{c(T-t)}(T-t)\Phi(w,w)dt \\
&\qquad - 2\,\mathrm{Re}\int_0^T e^{c(T-t)}(T-t)\left(B(t)\frac{dA^{-1}(t)}{dt}w + B(t)A^{-1}(t)\frac{dw}{dt},\ \frac{dw}{dt}\right) dt.
\end{aligned} \tag{11}$$

The following statement is used for integration by parts in the first integral occurring on the left-hand side of (11).

Lemma [3]. *Let E_1, F_1, and G_1 be Banach spaces, $T_1 : E_1 \to F_1$ be a linear bounded operator, and $S_1 : F_1 \to G_1$ be a linear closed operator with dense domain. If the domain of the product $S_1 \circ T_1$ of the operators T_1 and S_1 is dense in E_1, then the adjoint operator $(S_1 \circ T_1)^*$ is equal to the weak closure of the product $T_1^* \circ S_1^*$ of the adjoint operators T_1^* and S_1^*, respectively.*

Applying this lemma in $E_1 = F_1 = L_2(]0,T[,H)$ and $G_1 = L_2(]0,T[,H) \times H$ to $T_1 = A^{-1}(t)e^{c(T-t)}(T-t)$ and $S_1g = \{dg/dt, g(0)\}$ with domain

$$D(S_1) = \{g \in L_2(]0,T[,H) :\ dg/dt \in L_2(]0,T[,H), g(T) = 0\}$$

in the first two terms of the expression

$$-\int_0^T \left(\frac{dw}{dt}, \frac{d}{dt}\left[A^{-1}(t)e^{c(T-t)}(T-t)\frac{dw}{dt}\right]\right) dt - Te^{cT}\left(\frac{dw}{dt}, A^{-1}(t)\frac{dw}{dt}\right)\bigg|_{t=0} + Te^{cT}\left|A^{-1/2}(t)\frac{dw}{dt}\right|^2\bigg|_{t=0}, \tag{12}$$

we find that it is equal to

$$\int_0^T \left(\frac{d}{dt}\left[A^{-1}(t)e^{c(T-t)}(T-t)\frac{dw}{dt}\right],\ \frac{dw}{dt}\right)dt - \int_0^T \left(\left(\frac{d}{dt}\left[A^{-1}(t)e^{c(T-t)}(T-t)\right]\right)\frac{dw}{dt},\ \frac{dw}{dt}\right)dt$$
$$+ Te^{cT}\left|A^{-1/2}(t)\frac{dw}{dt}\right|^2\Bigg|_{t=0}. \tag{13}$$

Note that $T_1^* = T_1$, $S_1^*(\{p,p(0)\}) = -dp/dt$, $D(S_1^*) = \{\{p,p(0)\} \in G_1 :\ dp/dt \in L_2(]0,T[,H)\}$, and $\overline{T_1^* \circ S_1^*}dw/dt = -d\left[A^{-1}(t)e^{c(T-t)}(T-t)dw/dt\right]/dt + \left(d\left[A^{-1}(t)e^{c(T-t)} \times (T-t)\right]/dt\right)dw/dt$.

The fact that $dw/dt \in D\left((S_1 \circ T_1)^*\right)$ follows from (10), since this relation can be rewritten as follows:

$$\int_0^T \left(\frac{d}{dt}\left[A^{-1}(t)e^{c(T-t)}(T-t)\right]\frac{dh}{dt},\ \frac{dw}{dt}\right)dt$$
$$= \int_0^T \left(\left(\frac{d}{dt}\left[A^{-1}(t)e^{c(T-t)}(T-t)\right]\right)\frac{dh}{dt},\ \frac{dw}{dt}\right)dt - \int_0^T \left(h, e^{c(T-t)}\Im(t)w\right)dt$$
$$- \int_0^T e^{c(T-t)}(T-t)\Phi(h,w)dt - \int_0^T \left(B(t)\frac{dA^{-1}(t)}{dt}h + B(t)A^{-1}(t)\frac{dh}{dt},\ e^{c(T-t)}\Im(t)w\right)dt.$$

Therefore, integrating the terms occurring on the left-hand side of (11) by parts with the use of (12) and (13), we obtain

$$\int_0^T e^{c(T-t)}\left|A^{-1/2}(t)\frac{dw}{dt}\right|^2 dt + Te^{cT}\left|A^{-1/2}(t)\frac{dw}{dt}\right|^2\Bigg|_{t=0} + \int_0^T e^{c(T-t)}|w|^2 dt$$
$$= \int_0^T e^{c(T-t)}(T-t)\Phi_1(w,w)dt, \tag{14}$$

where

$$\Phi_1(w,w) = -c\left|A^{-1/2}(t)\frac{dw}{dt}\right|^2 - c|w|^2 - 2\,\mathrm{Re}\left(\frac{d^2A^{-1}(t)}{dt^2}w,\ \frac{dw}{dt}\right) - 3\left(\frac{dA^{-1}(t)}{dt}\frac{dw}{dt},\ \frac{dw}{dt}\right)$$
$$- 2\,\mathrm{Re}\left(B(t)\frac{dA^{-1}(t)}{dt}w,\ \frac{dw}{dt}\right) - 2\,\mathrm{Re}\left(B(t)A^{-1}(t)\frac{dw}{dt},\frac{dw}{dt}\right).$$

In the operator-differential form $\Phi_1(w,w)$, we use inequalities (4) and (6) and some elementary inequalities. Then we find that $\Phi_1(w,w)$ does not exceed

$$-(c - 3c_2 - 2c_4 - 2)\left|A^{-1/2}(t)dw/dt\right|^2 - (c - c_5 - c_6)\,|w|^2, \tag{15}$$

where $c_5 = \left\|A^{1/2}(t)d^2A^{-1}(t)/dt^2\right\|^2_{\mathscr{L}(H)}$ and $c_6 = \left\|A^{1/2}(t)B(t)dA^{-1}(t)/dt\right\|^2_{\mathscr{L}(H)}$.

The quantity (15) with $c \geq c_7 = \max\{3c_2 + 2c_4 + 2,\ c_5 + c_6\}$ is nonpositive. Therefore, if we estimate the left-hand side of (14) with $c \geq c_7$ below by the quantity $\int_0^T |w|^2 dt$ and the right-hand side by zero, then we obtain $w = 0$ and hence $v = 0$. The proof of Theorem 2 is complete.

Remark. We can generalize the assertion of Theorem 1 in the above-described way (possibly, with a larger value of the constant c_0) and the assertion of Theorem 2 by the Schauder–Ladyzhenskaya method of continuation with respect to the parameter to the case of an equation with a leading part, namely,

$$d^2u/dt^2 + B(t)du/dt + A(t)u + \tilde{B}(t)du/dt + \tilde{A}(t)u = f,$$

where $\tilde{B}(t), \tilde{A}(t)A^{-1/2}(t) \in L_\infty(]0,T[, \mathscr{L}(H))$.

4. EXAMPLE

In the bounded domain $G =]0,T[\times]0,l[$ of the variables t and x, we consider the equation

$$\partial^2 u(t,x)/\partial t^2 - t^{-\beta}\partial^2 u(t,x)/\partial x\,\partial t - t^{-\alpha}\partial^2 u(t,x)/\partial x^2 = f(t,x), \tag{16}$$

where $\alpha \geq 4$ and $\beta \leq \alpha/2 - 1$, with the boundary conditions

$$\partial u(t,0)/\partial x = u(t,l) = 0 \tag{17}$$

and the homogeneous initial conditions

$$u(0,x) = 0, \qquad \partial u(0,x)/\partial t = 0. \tag{18}$$

In this example, as the Hilbert space $\mathscr{E}$ of strong solutions of problem (16)–(18), we choose the completion of the set

$$\begin{aligned} D(L) = \Bigg\{ & u \in L_2(G):\ u(t,x) \in W_2^2(0,l),\ \partial u(t,0)/\partial x = u(t,l) = 0 \ \ \forall t \in]0,T[\,; \\ & \partial^2 u(t,x)/\partial t^2,\ t^{-\alpha/2}\partial^2 u(t,x)/\partial x\,\partial t,\ t^{-\alpha} \times \partial^2 u(t,x)/\partial x^2 \in L_2(G); \\ & t^{-\alpha}\int_0^l |\partial u(t,x)/\partial x|^2 dx\Bigg|_{t=0} = 0,\ t^{-\alpha}\int_0^l |\partial u(t,x)/\partial x|^2 dx\Bigg|_{t=T} < +\infty; \\ & u(0,x) = \partial u(0,x)/\partial t = 0 \Bigg\} \end{aligned}$$

in the Hermitian norm

$$\|u(t,x)\|_{\mathscr{E}} = \left(\int_0^T \int_0^l |\partial u(t,x)/\partial t|^2 dx\,dt + \int_0^T t^{-\alpha} \int_0^l |\partial u(t,x)/\partial x|^2 dx\,dt \right)^{1/2}.$$

As the Banach space Φ of right-hand sides $f(t,x)$ of Eq. (16), we choose the completion of $L_2(G)$ in the norm

$$\|f(t,x)\|_\Phi = \sup_{v \in \mathscr{E}_0} \left\{ \left| \int_0^T \int_0^l f(t,x)(T-t)(\partial v(t,x)/\partial t) dx\,dt \right| \Big/ \|v(t,x)\|_0 \right\},$$

where $\mathscr{E}_0$ is the set of all functions $v \in L_2(G)$ such that $t^{-\alpha/2}v \in L_2(G)$, the derivative $\partial v/\partial t \in L_2(G)$ exists, and $v(0,x) = 0$; this set is equipped with the norm

$$\|v(t,x)\|_0 = \left(\int_0^T \int_0^l |\partial v(t,x)/\partial t|^2 dx\,dt + \int_0^T t^{-\alpha} \int_0^l |v(t,x)|^2 dx\,dt \right)^{1/2}.$$

The following theorem claims that the mixed problem (16)–(18) is well posed in the sense of strong solvability.

Theorem 3. *For each $f \in \Phi$, the mixed problem* (16)–(18) *has a unique strong solution $u \in \mathscr{E}$ satisfying the inequality $\|u\|_{\mathscr{E}} \le c_0 \|f\|_{\Phi}$.*

Proof. It suffices to show that all assumptions of the abstract Theorem 2 are valid. First, we show that the operators $A(t)$, $t \in]0,T[$, induced by the differential expression $A(t)u(t,x) = -t^{-\alpha}\partial^2 u(t,x)/\partial x^2$, $t \in]0,T[$, and the boundary conditions (17) satisfy all assumptions of the abstract Theorem 2 in the Hilbert space $H = L_2(0,l)$. The operators $A(t)$, $t \in]0,T[$, with domains

$$D(A(t)) = \Big\{ u \in L_2(G) : \ u(t,x) \in W_2^2(0,l) \ \ \forall t \in]0,T[; \ t^{-\alpha}\partial^2 u(t,x)/\partial x^2 \in L_2(G); \\ \partial u(t,0)/\partial x = u(t,l) = 0 \Big\}$$

are self-adjoint in $L_2(0,l)$, since they are obviously symmetric in $L_2(0,l)$ and have bounded inverses

$$A^{-1}(t)g = t^{\alpha} \int\limits_x^l \left(\int\limits_0^s g(t,\tau)d\tau \right) ds$$

on $L_2(0,l)$. Their boundedness follows from the inequalities $\|A^{-1}(t)g\|_{0,\Omega}^2 \le (1/3)l^4 T^{2\alpha} \|g\|_{0,\Omega}^2$ for all $g \in L_2(0,l)$, where $\|\cdot\|_{0,\Omega}$ is the norm in the Hilbert space $L_2(0,l)$. The operators $A(t)$ satisfy inequalities (3) with $c_1(t) = 2l^{-2}t^{-\alpha}$.

The operators $A^{-1}(t)$ have the first strong derivative

$$\frac{dA^{-1}(t)}{dt} g = \alpha t^{\alpha-1} \int\limits_x^l \left(\int\limits_0^s g(t,\tau)d\tau \right) ds,$$

which is a bounded operator in $L_2(0,l)$, namely, $\|(dA^{-1}(t)/dt)\, g\|_{0,\Omega}^2 \le (1/3)\alpha^2 l^4 T^{2\alpha-2} \|g\|_{0,\Omega}^2$ for all $g \in L_2(0,l)$, and satisfies inequalities (4), since

$$-\left(\frac{dA^{-1}(t)}{dt} g, \ g \right)_{0,\Omega} = -\alpha t^{\alpha-1} \int\limits_0^l \left(\int\limits_0^x g(t,\tau)d\tau \right)^2 dx \le 0,$$

where $(\cdot\,,\cdot)_{0,\Omega}$ is the inner product in $L_2(0,l)$. The operators $dA^{-1}(t)/dt$ have the strong derivative

$$\frac{d^2A^{-1}(t)}{dt^2} g = \alpha(\alpha-1) t^{\alpha-2} \int\limits_x^l \left(\int\limits_0^s g(t,\tau)d\tau \right) ds$$

such that $A^{1/2}(t)\,(d^i A^{-1}(t)/dt^i) \in L_\infty\left(]0,T[, \mathscr{L}\left(L_2(0,l)\right)\right)$, $i = 1,2$, since

$$\left\| A^{1/2}(t)\,(dA^{-1}(t)/dt)\, g \right\|_{0,\Omega}^2 \le (1/2)\alpha^2 l^2 T^{\alpha-2} \|g\|_{0,\Omega}^2 \qquad \forall g \in L_2(0,l),$$

$$\left\| A^{1/2}(t)\,(d^2A^{-1}(t)/dt^2)\, g \right\|_{0,\Omega}^2 \le (1/2)\alpha^2(\alpha-1)^2 l^2 T^{\alpha-4} \|g\|_{0,\Omega}^2 \qquad \forall g \in L_2(0,l).$$

Second, we show that the operators $B(t)$, $t \in]0,T[$, induced by the differential expression $B(t)u(t,x) = -t^{-\beta}\partial u(t,x)/\partial x$, $t \in]0,T[$, also satisfy all assumptions of the abstract Theorem 2 in the Hilbert space $H = L_2(0,l)$. The operators $B(t)$ are subordinate to $A^{1/2}(t)$: $\|B(t)u\|_{0,\Omega}^2 \le T^{\alpha-2\beta} \left\|A^{1/2}(t)u\right\|_{0,\Omega}^2$ for all $u \in D\left(A^{1/2}(t)\right)$, where $D\left(A^{1/2}(t)\right)$ are the domains of the operators $A^{1/2}(t)$, and satisfy inequalities (5), namely, $-2\operatorname{Re}(B(t)u,u)_{0,\Omega} = -t^{-\beta}|u(t,0)|^2 \le 0$ for all $u(t,x) \in D(A(t))$. Since

$$\left\| A^{1/2}(t)B(t)\,(dA^{-1}(t)/dt)\, g \right\|_{0,\Omega}^2 \le \alpha^2 T^{\alpha-2\beta-2} \|g\|_{0,\Omega}^2 \qquad \forall g \in L_2(0,l),$$

it follows that $A^{1/2}(t)B(t)dA^{-1}(t)/dt \in L_\infty(]0,T[,\mathscr{L}(L_2(0,l)))$, and from the inequalities

$$-2\operatorname{Re}(B(t)u, A(t)u)_{0,\Omega} = -t^{-\alpha-\beta}|\partial u(t,l)/\partial x|^2 \leq 0 \qquad \forall u(t,x) \in D(A(t))$$

we find that inequalities (6) hold.

Finally, it is obvious that the set $D(L)$ is dense in $L_2(G)$; in a standard way, we can show that the linear operator $L : \mathscr{E} \supset D(L) \to \Phi$ admits the closure. The proof of Theorem 3 is complete.

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