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SHORT COMMUNICATIONS

Boundary Value Problems for Complete Partial Differential Equations of Variable Order

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Abstract—We prove the well-posedness of new boundary value problems for partially pseudodifferential complete nonclassical equations of variable order in space variables with higher derivatives of odd order in time.

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Boundary value problems for parabolic equations in variable-order partial derivatives (in space variables) were earlier considered for higher derivatives of the first order [1–3] and odd order [4] in time, for hyperbolic equations with higher derivatives of the second order [3] and even order [5] in time, for partially pseudodifferential parabolic equations with higher derivatives of the first order [6] and higher order [7] in time, and for partially pseudodifferential hyperbolic equations with even-order derivatives in time [8]. In the present paper, we prove the strong well-posedness of new boundary value problems for partially pseudodifferential complete nonclassical equations of variable order with odd-order higher derivatives in time. The variable differentiation order depended on the time in [1–5] and on space points at which the differentiation is performed in [6, 7].

1. STATEMENT OF THE PROBLEM

In the domain $G =]0,T[\, \times \mathbb{R}^n$, consider the boundary value problems

$$(-1)^m \frac{\partial^{2m+1}u(t,x)}{\partial t^{2m+1}} + \sum_{s=1}^{2m} \frac{\partial^{[(s+1)/2]}}{\partial t^{[(s+1)/2]}}(-1)^{[s/2]}(I-\Delta_x)^{p_s(t)-\varepsilon_s}\frac{\partial^{[s/2]}u(t,x)}{\partial t^{[s/2]}}$$
$$+ \lambda_m (I-\Delta_x)^{p_0(t)}u(t,x) = f(t,x), \qquad t \in]0,T[, \qquad x=(x_1,\dots,x_n)\in\mathbb{R}^n, \qquad (1)$$

$$\partial^i u(0,x)/\partial t^i = \partial^j u(T,x)/\partial t^j = 0,$$
$$x\in\mathbb{R}^n, \qquad i=0,\dots,m, \qquad j=0,\dots,m-1, \qquad m=0,1,\dots, \qquad (2)$$

where $p_s(t) = p(t)\{1-(s-1)/(2m)\}$, $0<\varepsilon_s<1/(2m)$, $[a]$ is the integer part of the number a, and the fractional partial derivatives with respect to x are defined with the use of the well-known Fourier–Plancherel (F–P) transforms in $L_2(\mathbb{R}^n)$ [9, p. 108]. Here the operators

$$A_s(t)u(x) = (-1)^{[s/2]}F^{-1}[(1+|\xi|^2)^{p_s(t)-\varepsilon_s}F[u](\xi)](x)$$

with domains $D(A_s(t)) = \{u(x)\in L_2(\mathbb{R}^n):\ (1+|\xi|^2)^{p_s(t)-\varepsilon_s}F[u](\xi)\in L_2(\mathbb{R}^n)\}$, $t\in[0,T]$, are pseudodifferential operators of variable order with respect to x [9, pp. 135–136] corresponding to the symbols $a_s(t,x,\xi) = (1+|\xi|^2)^{p_s(t)-\varepsilon_s}$, $s\ge 0$, $\varepsilon_0 = 0$ [10, pp. 94–98 of the Russian translation].

2. EXISTENCE, UNIQUENESS, AND CONTINUOUS DEPENDENCE

The spaces of strong solutions and right-hand sides are the Hilbert spaces $\mathcal{E}^m(G)$ defined as the closures of the sets $D(L_m)$ of all functions $C^{2m+1,p^*}(G)$, where $p^* = [4\max_{t\in[0,T]}p(t)]+1$, satisfying conditions (2) and compactly supported in $\mathbb{R}^n$ in the Hermitian norms

$$|||u|||_m = \left\{ \int_0^T \int_{\mathbb{R}^n} \left(\left| \frac{\partial d^m u(t,x)}{\partial dt^m} \right|^2 + |(I-\Delta_x)^{p_0(t)/2} u(t,x)|^2 \right) dx\, dt \right\}^{1/2}$$

and the Banach spaces $\widehat{\mathcal{F}}^{-m}(G)$ defined as the closures of the set $L_2(G)$ in the norms

$$\langle ||f||\rangle_{-m} = \sup_{v\in\mathcal{E}^m(G)} \left\{ \left| \int_0^T (T-t) \int_{\mathbb{R}^n} f\overline{v}\, dx\, dt \right| (|||v|||_m)^{-1} \right\},$$

respectively. The boundary value problems (1), (2) correspond to linear unbounded operators $L_m(\lambda_m):\ E^m \supset\ D(L_m) \to \widehat{F}^{-m}$ with dense domains $D(L_m)$.

Definition 1. The solutions of the operator equations $\overline{L}_m(\lambda_m)u = f$, $f \in \widehat{F}^{-m}$, $m = 0, 1, \ldots$, are referred to as strong solutions of the boundary value problems (1), (2).

Theorem 1. *If* $n/2 < p(t) \in C^{m+1}[0,T]$, *and the derivative* $p'(t)$, $t \in [0,T]$, *is nonpositive, then there exist* $c_0(m) > 0$, $\widetilde{\Lambda}_0 = [1,+\infty[$ *for* $m = 0$ *and* $\widetilde{\Lambda}_m = [\widetilde{\lambda}_m,+\infty[$, $\widetilde{\lambda}_m \ge 1$, *for* $m > 0$, *such that, for all* $f \in \widehat{\mathcal{F}}^{-m}(G)$ *and* $\lambda_m \in \widetilde{\Lambda}_m$, *the boundary value problems* (1) *and* (2) *have a unique strong solution* $u \in C^{m-1}([0,T], L_2(\mathbb{R}^n)) \cap \mathcal{E}^m(G)$, *and*

$$|||u(t,x)|||_m \le c_0(m)\langle||f(t,x)||\rangle_{-m}, \qquad m = 0,1,\ldots \tag{3}$$

Proof. The proof amounts to verifying the assumptions of Theorem 2 in [11] for the operators $A(t) = (I-\Delta_x)^{p(t)}$ with domains $D(A(t)) = \{u(x) \in L_2(\mathbb{R}^n):\ (1+|\xi|^2)^{p(t)}F[u](\xi) \in L_2(\mathbb{R}^n)\}$ and for the smoothing operators $B_\varepsilon^{-1}(t) = (I-\varepsilon B(t))^{-1}$, $\varepsilon > 0$, $B(t) = -A_0(t)$.

I. Maximal accretivity of the operators $A_0(t)$. The proof of the self-adjointness and positive definiteness of the operators $A_0(t)$ in $L_2(\mathbb{R}^n)$ can be found in [8, p. 533].

II. Existence of self-adjoint operators $A(t)$ *subordinate to the operators* $A_0(t)$. By using two basic properties of the F–P transforms, the Parseval relation $\langle h, g\rangle = (2\pi)^{-n}\langle F[h], F[g]\rangle$, $h, g \in L_2(\mathbb{R}^n)$, for the inner product $\langle\cdot,\cdot\rangle$ in $L_2(\mathbb{R}^n)$, and the inversion formulas $F^{-1}F[g] = FF^{-1}[g] = g$, $g \in L_2(\mathbb{R}^n)$, we find that inequalities (4) in [11] hold for all $t \in [0,T]$ and $u \in D(A_0(t))$:

$$\begin{aligned}\operatorname{Re}\langle A_0(t)u,u\rangle &= \operatorname{Re}\langle F^{-1}[(1+|\xi|^2)^{p_0(t)}F[u]],u\rangle = (2\pi)^{-n}\langle(1+|\xi|^2)^{p_0(t)}F[u],F[u]\rangle \\ &\ge (2\pi)^{-n}\langle(1+|\xi|^2)^{p(t)}F[u],F[u]\rangle = \langle F^{-1}[(1+|\xi|^2)^{p(t)}F[u]],u\rangle = \langle A(t)u,u\rangle.\end{aligned}$$

Let us compute the strong derivative $dA^{-1}(t)/dt$ of the operators $A^{-1}(t)$ inverse to the operators $A(t)$. If $p(t) > n/2$, then, by using the continuity of F^{-1} with respect to t in $L_2(\mathbb{R}^n)$ and the convolution estimate

$$\|h * g\|_{L_2} \le \|h\|_{L_1}\|g\|_{L_2}$$

[9, p. 62] and by performing the differentiation of the operators $A^{-1}(t)g = F^{-1}[(1+|\xi|^2)^{-p(t)}] * g \in L_2(\mathbb{R}^n)$ with respect to t for all $g \in L_2(\mathbb{R}^n)$ [9, pp. 182–185], we prove the existence of a strong derivative $(dA^{-1}(t)/dt)g = -p'(t)(F^{-1}[(1+|\xi|^2)^{-p(t)}\ln(1+|\xi|^2)] * g)$ in $L_2(\mathbb{R}^n)$, and the boundedness of this derivative in $L_2(\mathbb{R}^n)$ follows from the inequalities [8, p. 533] $\|(dA^{-1}(t)/dt)g\| \le (\varrho e)^{-1}|p'(t)|\|g\|$, where $\|\cdot\|$ is the norm in $L_2(\mathbb{R}^n)$, and $\varrho > 0$ is an *arbitrarily small number for* $i = 1$ *in the inequalities* $\ln^i z \le (i/(\varrho e))^i z^\varrho$, $z \ge 1$, $\varrho > 0$, $i = 1, 2, \ldots$ The verification of the corresponding assumption of Theorem 2 for the derivative $dA^{-1}(t)/dt$ can be found in [8, p. 537].

III. Differentiability and symmetry of the operators $A_s(t)$, $s > 0$. The operators $A(t)$ have positive fractional powers $A^{\alpha/(2m)}(t)u = F^{-1}[(1+|\xi|^2)^{p(t)\alpha/(2m)}F[u]]$, $0 \le \alpha \le 2m$. The operators $A_s(t)$ belong to $\mathcal{B}([0,T], \mathcal{L}(W^{2m+1-s}(t), H))$, $s > 0$, since

$$\begin{aligned}\|A_s(t)u\|^2 &= \|F^{-1}[(1+|\xi|^2)^{p_s(t)-\varepsilon_s}F[u]]\|^2 = (2\pi)^{-n}\|(1+|\xi|^2)^{p_s(t)-\varepsilon_s}F[u]\|^2 \\ &\le (2\pi)^{-n}\|(1+|\xi|^2)^{p_s(t)}F[u]\|^2 = \|F^{-1}[(1+|\xi|^2)^{p_s(t)}F[u]]\|^2 = |u|^2_{2m+1-s,t} \quad \forall u \in W^{2m+1-s}(t).\end{aligned}$$

The first strong derivative of the linear unbounded operators $A_s(t)$ with variable domains $D(A_s(t))$ in the sense of the definition in [11] was computed in [8, p. 535] and is equal to

$$(dA_s(t)/dt)u(t) = (-1)^{[s/2]}c_m(s,p)F^{-1}[(1+|\xi|^2)^{p_s(t)-\varepsilon_s}\ln(1+|\xi|^2)F[u(t)]],$$

where $c_m(s,p) = (1-(s-1)/(2m))p'(t)$. In a similar way, by using two basic properties of the F–P transforms, one can show that

$$\begin{aligned}\|(dA_s(t)/dt)u\|^2 &\le c_m^2(s,p)(2\pi)^{-n}\|(1+|\xi|^2)^{p_s(t)-\varepsilon_s+\varrho}F[u]\|^2\\ &\le c_m^2(s,p)(2\pi)^{-n}\|(1+|\xi|^2)^{p_s(t)+p(t)\tau/(2m)}F[u]\|^2 \le c_m^2(s,p)|u|^2_{2m+1-s+\tau,t} \quad \forall u \in W^{2m+1-s+\tau}(t)\end{aligned}$$

for an arbitrarily small $\tau > 0$; i.e., $dA_s(t)/dt \in \mathcal{B}([0,T],\mathcal{L}(W^{2m+1-s+\tau}(t),H))$, $s > 0$, if $0 < \varrho \le \varepsilon_s + n\tau/(4m)$ for $i = 1$. Since $p(t) \in C^{m+1}[0,T]$, it follows that the remaining strong derivatives of the operators $A_s(t)$ can be evaluated recursively in j, and one can show that if $0 < \varrho \le \varepsilon_s + n\tau/(4m)$ for $i = j$, then $d^jA_s(t)/dt^j \in \mathcal{B}([0,T],\mathcal{L}(W^{2m+1-s+\tau}(t),H))$, $j = 2,\dots,[(s+1)/2]$, $s > 0$.

The operators $A_s(t)$ satisfy inequalities (5) in [11] for $j = 0,1$; for example, if $j = 1$, then

$$\begin{aligned}|\langle(dA_s(t)/dt)u,v\rangle| &= \frac{|c_m(s,p)|}{(2\pi)^n}|\langle(1+|\xi|^2)^{p(t)(1/2-[s/2]/(2m)+p_s/(2m))-\varepsilon_s}\\ &\quad\times\ln(1+|\xi|^2)F[u],(1+|\xi|^2)^{p(t)(1/2-[(s-1)/2]/(2m)-p_s/(2m))}F[v]\rangle|\\ &\le |c_m(s,p)||u|_{m-[s/2]+p_s,t}|v|_{m-[(s-1)/2]-p_s,t} \quad \forall u \in W^{m-[s/2]+p_s}(t), \quad \forall v \in W^{m-[(s-1)/2]-p_s}(t)\end{aligned}$$

provided that $0 < \varrho \le \varepsilon_s$, $s > 0$, for $i = 1$. Here we have used two basic properties of F–P transforms. The proof of the symmetry of the operators $A_s(t)$, $s > 0$, can be found in [8, p. 535]. Inequalities (6) in [11] hold for the operators $A_s(t)$, $s > 0$, since

$$\begin{aligned}(-1)^{(s+2)/2}\langle A_s(t)u,u\rangle &= -\langle F^{-1}[(1+|\xi|^2)^{p(t)(1-(s-1)/(2m))-\varepsilon_s}F[u]],u\rangle\\ &= -(2\pi)^{-n}\langle(1+|\xi|^2)^{p(t)(1-(s-1)/(2m))-\varepsilon_s}F[u],F[u]\rangle \le 0 \quad \forall u \in D(A(t)). \quad (4)\end{aligned}$$

IV. Properties of smoothing operators $B_\varepsilon^{-1}(t)$ and the corresponding smoothed operators $A_s(t)$, $s > 0$. 1. In $L_2(\mathbb{R}^n)$, the dissipativity of the operators $B(t) = -A_0(t)$ follows from the positive definiteness of the operators $A_0(t)$, and the proof of the boundedness of the strong derivatives $d^iB^{-1}(t)/dt^i = d^iB^{*-1}(t)/dt^i$, $i = 0,\dots,m+1$, can be found in [8, p. 534]. For example, for $B^{-1}(t)g = -A_0^{-1}(t)g = -(F^{-1}[(1+|\xi|^2)^{-p_0(t)}] * g)$, the strong derivative is

$$(dB^{-1}(t)/dt)g = (1+1/(2m))p'(t)(F^{-1}[(1+|\xi|^2)^{-p_0(t)}\ln(1+|\xi|^2)] * g),$$

and $\|(dB^{-1}/dt)g\| \le (1+1/(2m))|p'(t)|(\varrho e)^{-1}\|g\|$ for all $g \in L_2(\mathbb{R}^n)$. The boundedness of the operators $A_s(t)(dB^{-1}(t)/dt)$ in $L_2(\mathbb{R}^n)$ follows from the relations

$$\begin{aligned}\|A_s(t)(dB^{-1}(t)/dt)g\|^2 &= c_m^2(0,p)\|F^{-1}[(1+|\xi|^2)^{-p(t)s/(2m)-\varepsilon_s}\ln(1+|\xi|^2)F[g]]\|^2\\ &\le c_m^2(0,p)(2\pi)^{-n}(\varrho e)^{-2}\|(1+|\xi|^2)^{\varrho-\varepsilon_s}F[g]\|^2 \le c_m^2(0,p)(\varrho e)^{-2}\|g\|^2,\\ &s = 0,\dots,2m, \qquad \forall g \in L_2(\mathbb{R}^n),\end{aligned}$$

if $0 < \varrho \le \varepsilon_s$ for $i = 1$. Here we have used the F–P transform of the convolution $F[h*g] = F[h]\cdot F[g]$, $h \in L_1(\mathbb{R}^n)$, $g \in L_2(\mathbb{R}^n)$ [9, p. 105]. For the remaining strong derivatives $d^iB^{-1}(t)/dt^i$, the proof of the boundedness of the operators $A_s(t)(d^iB^{-1}(t))/dt^i$, $i = 2,\dots,[s/2]$, $s = 0,\dots,2m$, in $L_2(\mathbb{R}^n)$ can be performed in a similar way with the use of three basic properties of F–P transforms.

2. For $m = 0$, there exists the first limit in (7) in [11]: if $u \in W^{m+1/2}(t)$, then

$$\begin{aligned}|\operatorname{Re}\langle(dB_\varepsilon^{-1}(t)/dt)(B_\varepsilon^{-1}(t))^*u,v\rangle| &= |\operatorname{Re}\langle\varepsilon(dB^{-1}(t)/dt)B(t)B_\varepsilon^{-2}(t)u, B(t)B_\varepsilon^{-1}(t)u\rangle|\\ &\le \|(dB^{-1}(t)/dt)\sqrt{\varepsilon}(-B(t))^{1/2}B_\varepsilon^{-2}(t)(-B(t))^{1/2}u\|\|\sqrt{\varepsilon}(-B(t))^{1/2}B_\varepsilon^{-1}(t)(-B(t))^{1/2}u\| \to 0 \quad (5)\end{aligned}$$

as $\varepsilon \to 0$, since the operators $dB^{-1}(t)/dt$ are bounded in $L_2(\mathbb{R}^n)$, $(-B(t))^{1/2}u \in L_2(\mathbb{R}^n)$, and $\lim_{\varepsilon\to 0} \|\sqrt{\varepsilon}(-B(t))^{1/2}B_\varepsilon^{-1}(t)g\| = 0$ for all $g \in L_2(\mathbb{R}^n)$, where $(-B(t))^{1/2}$ is the square root of the operators $A_0(t)$. The validity of inequalities (7) in [11] follows from (3) in [11].

3(a). For $m > 0$, the limit (8) in [11] does not exist, but inequality (8′) in [11] takes place:

$$\begin{aligned}-\operatorname{Re}\langle (dB_\varepsilon^{-1}(t)/dt)(B_\varepsilon^{-1}(t))^* v, v\rangle &= -\varepsilon \operatorname{Re}\langle B_\varepsilon^{-1}(t)(dB(t)/dt)B_\varepsilon^{-1}(t)(B_\varepsilon^{-1}(t))^* v, v\rangle\\ &\le \varepsilon^2 c_m(0,p)(2\pi)^{-n}\langle (1+|\xi|^2)^{p_0(t)}\ln(1+|\xi|^2)F[B_\varepsilon^{-2}(t)v], (1+|\xi|^2)^{p_0(t)}F[B_\varepsilon^{-2}(t)v]\rangle \le 0\end{aligned}$$

for all $v \in L_2(\mathbb{R}^n)$, since $c_m(0,p) \le 0$, $t \in [0,T]$, by the assumption of Theorem 1.

By analogy with the transformations (5), we have

$$\begin{aligned}\|(d(B_\varepsilon^{-1}(t))^*/dt)v\|^2 &= \|\varepsilon B(t)B_\varepsilon^{-1}(t)B^{-1}(t)(dB(t)/dt)B_\varepsilon^{-1}(t)v\|^2\\ &\le \|\varepsilon B(t)B_\varepsilon^{-1}(t)\|^2\|B^{-1}(t)(dB(t)/dt)B_\varepsilon^{-1}(t)v\|^2 \to 0 \quad \forall v \in W^1(t),\end{aligned} \tag{6}$$

since $\lim_{\varepsilon\to 0}\|\varepsilon(B(t))B_\varepsilon^{-1}(t)\| = 0$ and the values

$$\begin{aligned}&\|B^{-1}(t)(dB(t)/dt)B_\varepsilon^{-1}(t)v\|^2\\ &\quad = \|F^{-1}[(1+|\xi|^2)^{-p_0(t)}] * c_m(0,p)F^{-1}[(1+|\xi|^2)^{p_0(t)}\ln(1+|\xi|^2)F[B_\varepsilon^{-1}(t)v]]\|^2\\ &\quad \le c_m^2(0,p)\|B_\varepsilon^{-1}(t)A^{\delta/(2m)}(t)v\|^2 \le c_m^2(0,p)\|A^{\delta/(2m)}(t)v\|^2 < c_m^2(0,p)|v|_{1,t}^2 \quad \forall v \in W^1(t)\end{aligned} \tag{7}$$

are bounded uniformly with respect to ε provided that $\delta = 2m\varrho/p(t) < 1$ for $0 < \varrho \le n/(4m)$ if $i = 1$. Recurrently in i one can show that $\lim_{\varepsilon\to 0}\|(d^j(B_\varepsilon^{-1}(t))^*/dt^j)v\| = 0$ for all $v \in W^{j-1}(t)$ provided that $0 < \varrho \le n/(4m)$ for $i = j = 2, \dots, m+1$, and therefore,

$$\lim_{\varepsilon\to 0}\|(d^iB_\varepsilon^{-1}(t)/dt^i)(B_\varepsilon^{-1}(t))^* v\| = 0$$

for all $v \in W^{i-1}(t)$ and for $i = 2, \dots, m+1$.

By virtue of the estimates (6), as $\varepsilon \to 0$ we have

$$\begin{aligned}&\|(dB_\varepsilon^{-1}(t)/dt)(d(B_\varepsilon^{-1}(t))^*/dt)v\|^2\\ &\qquad \le \|\varepsilon B(t)B_\varepsilon^{-1}(t)\|^2\|B^{-1}(t)(dB(t)/dt)B_\varepsilon^{-1}(t)(d(B_\varepsilon^{-1}(t))^*/dt)v\|^2 \to 0\end{aligned}$$

for all $v \in W^1(t)$, since, by virtue of the estimates (7) and (6) and the commutativity of $A^{\delta/(2m)(t)}$ with $B(t)$,

$$\|B^{-1}(t)(dB(t)/dt)B_\varepsilon^{-1}(t)(d(B_\varepsilon^{-1}(t))^*/dt)v\|^2 \le c_m^2(p)|(d(B_\varepsilon^{-1}(t))^*/dt)v|_{\delta,t}^2,$$

$$|(d(B_\varepsilon^{-1}(t))^*/dt)v|_{\delta,t}^2 \le \|\varepsilon B(t)B_\varepsilon^{-1}(t)\|^2|B^{-1}(t)(dB(t)/dt)B_\varepsilon^{-1}(t)v|_{\delta,t}^2 \to 0, \tag{8}$$

$$|B^{-1}(t)(dB(t)/dt)B_\varepsilon^{-1}(t)v|_{\delta,t}^2 \le c_m^2(0,p)\|B_\varepsilon^{-1}(t)A^{2\delta/(2m)}(t)v\|^2 < c_m^2(0,p)|v|_{1,t}^2 \quad \forall v \in W^1(t), \tag{9}$$

if $2\delta = 2m\varrho/p(t) < 1$ for $0 < \varrho \le n/(4m)$ and $i = 1$. Recurrently with respect to i' and j, one can show that if $(i'+j)\delta = 2m\varrho/p(t) < i'+j-1$ for $0 < \varrho \le n(i'+j-1)/(4m)$ and for $i = \max\{i', j\}$, then $\lim_{\varepsilon\to 0}\|(d^{i'}B_\varepsilon^{-1}(t)/dt^{i'})(d^jB_\varepsilon^{-1}(t)/dt^j)v\| = 0$ for all $v \in W^{i'+j-1}(t)$, $j = 1, \dots, m+1-i'$, and $i' = 1, \dots, m+1$.

3(b). For all $\tilde v \in W^{m-k}(t)$, there exists a limit (9) in [11] as $\varepsilon \to 0$:

$$\begin{aligned}|\langle A_{2k+1}(t)(dB_\varepsilon^{-1}(t)/dt)(B_\varepsilon^{-1}(t))^*\tilde v, \tilde v\rangle| &= |\langle F^{-1}[(1+|\xi|^2)^{p(t)(1/2-k/(2m))-\varepsilon_{2k+1}}\\ &\quad \times F[(dB_\varepsilon^{-1}(t)/dt)(B_\varepsilon^{-1}(t))^*\tilde v]], F^{-1}[(1+|\xi|^2)^{p(t)(1/2-k/(2m))}F[\tilde v]]\rangle|\\ &\le |(dB_\varepsilon^{-1}(t)/dt)(B_\varepsilon^{-1}(t))^*\tilde v|_{m-k-\delta_k',t}|\tilde v|_{m-k,t} \to 0,\end{aligned}$$

since, by virtue of the commutativity of the operators $A(t)$ with $B(t)$, the estimates (8) hold for $\delta' = m-k-\delta_k'$, $\delta_k' = 2m\varepsilon_{2k+1}/\max_{t\in[0,T]}p(t) \ge 0$, $k = 0, \dots, m-1$, $v = (B_\varepsilon^{-1}(t))^*\tilde v$, and, by virtue of the estimate (9),

$$\begin{aligned}|(dB_\varepsilon^{-1}(t)/dt)(B_\varepsilon^{-1}(t))^*\tilde v|_{m-k-\delta_k',t}^2 &\le c_m^2(0,p)\|B_\varepsilon^{-2}(t)A^{(m-k-\delta_k'+\varrho')/(2m)}(t)\tilde v\|^2\\ &\le c_m^2(0,p)\|A^{(m-k-\delta_k'+\varrho')/(2m)}(t)\tilde v\|^2 \le c_m^2(0,p)|\tilde v|_{m-k}^2\end{aligned}$$

provided that $0 < \varrho' \le \delta'_k$, $k = 0, \dots, m-1$, $\varrho' = 2m\varrho/\max_{t\in[0,T]} p(t) \ge 0$, for $i = 1$. In the same way, one can show that

$$\lim_{\varepsilon\to 0}\langle A_{2k+1}(t)B_\varepsilon^{-1}(t)(d(B_\varepsilon^{-1}(t))^*/dt)v, v\rangle = 0,$$
$$\lim_{\varepsilon\to 0}\langle A_{2k}(t)(dB_\varepsilon^{-1}(t)(B_\varepsilon^{-1}(t))^*/dt)v, w\rangle = 0$$

for all $v \in W^{m-k}(t)$, for all $w \in W^{m-k+1}(t)$, and for $k = 1, \dots, m-1$. If $\varepsilon \to 0$ strongly in $W^{m-k}(t)$, then

$$|(d(B_\varepsilon^{-1}(t))^*/dt)v|^2_{m-k,t} \le \|\varepsilon B(t)B_\varepsilon^{-1}(t)\|^2 |B^{-1}(t)(dB(t)/dt)B_\varepsilon^{-1}(t)v|^2_{m-k,t} \to 0$$

by virtue of the estimates (8) and the inequality [which is a consequence of the estimate (9)]

$$|B^{-1}(t)(dB(t)/dt)B_\varepsilon^{-1}(t)v|^2_{m-k,t} \le c_m^2(0,p)\|B_\varepsilon^{-1}(t)A^{(m-k+\delta)/(2m)}(t)v\|^2 < c_m^2(0,p)|v|^2_{m-k+1,t}$$

for all $v \in W^{m-k+1}(t)$, $k = 1, \dots, m-1$, provided that $\delta = 2m\varrho/p(t) < 1$ for $0 < \varrho \le n/(4m)$ and for $i = 1$. One can readily show that

$$\lim_{\varepsilon\to 0}|(d^j(B_\varepsilon^{-1}(t))^*/dt^j)v|_{m-k,t} = 0 \qquad \forall v \in W^{m-k+1}(t)$$

if $0 < \varrho \le n/(4m)$ for $i = j = 2, \dots, k$ and $k = 2, \dots, m-1$;

$$\lim_{\varepsilon\to 0}|(d^j B_\varepsilon^{-1}(t)(B_\varepsilon^{-1}(t))^*/dt^j)v|_{m-k,t} = 0 \qquad \forall v \in W^{m-k+1}(t)$$

if $j\delta = 2m\varrho/p(t) < m-k+1$ for $0 < \varrho \le n(m-k+1)/(4m)$ and for $j = 1, 2$, $k = 1, \dots, m$, and for all $v \in W^{m-k+j-1}(t)$, if $j\delta = 2m\varrho/p(t) < m-k+j-1$ for $0 < \varrho \le n(m-k+j-1)/(4m)$, $i = j = 2, \dots, k+1$, and $k = 2, \dots, m$. The property $\lim_{\varepsilon\to 0}\|B_\varepsilon^{-1}(t)w - w\| = 0$, $w \in L_2(\mathbb{R}^n)$, and the commutativity of the operators $A(t)$ and $B(t)$ imply the existence of the limits (7) and (8). By virtue of the estimates (4), we have inequalities (12′) from [11]. The limits (13) in [11] exist for all $s = 1, \dots, 2m$ and are zero for all $\varepsilon > 0$. Inequalities (3) correspond to inequalities (14) in [11]. The proof of Theorem 1 is complete.

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