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ON GENERALIZED TRIANGLE GROUPS OF TYPE $(2, 15, 2)$

Tits [1] proved that if G is a finitely generated linear group then G contains either a non abelian free subgroup or a solvable subgroup of finite index. Let Γ be an arbitrary finitely generated group. One says that the Tits alternative holds for Γ if Γ contains either a non abelian free subgroup or a solvable subgroup of finite index.

A generalized triangle group $T(k, l, m, R)$ has a presentation

$$T(k, l, m, R) = \langle a, b; a^k = b^l = R^m(a, b) = 1\rangle,$$

where $m \geq 2$ and $R(a, b)$ is a cyclically reduced word in the free group on a, b which is not a proper power. If $R(a, b) = ab$ then we obtain an ordinary triangle group.

Rosenberger [2] stated the following conjecture.

C o n j e c t u r e 1. *The Tits alternative holds for generalized triangle groups.*

Fine, Levin, and Rosenberger [3] proved this conjecture in the following cases: 1) $l = 0$ or $k = 0$; 2) $m \geq 3$. Now suppose that $k, l, m \geq 2$. Let $\Gamma = T(k, l, m, R)$ and $s(\Gamma) = 1/k + 1/l + 1/m$. If $s(\Gamma) < 1$ then Baumslag, Morgan and Shalen [4] proved that the group Γ contains a non abelian free subgroup. Using some new methods, Howie [5] proved Conjecture 1 in the case when $s(\Gamma) = 1$ and up to equivalence $R \neq ab$. If $s(\Gamma) = 1$ and $R = ab$ then Γ is an ordinary triangle group. The classical result says that Γ contains $\mathbb{Z}$ as a subgroup of finite index.

Now consider groups of the form

$$\Gamma = T(2, l, 2, R) = \langle a, b; a^2 = b^l = R^2(a, b) = 1\rangle, \quad (1)$$

where $l > 2$, $R = ab^{v_1} \dots ab^{v_s}$, $0 < v_i < l$. In the following cases Conjecture 1 holds for Γ: 1) $s \leq 4$ (Levin and Rosenberger [6]); 2) $l > 5$ and $l \neq 6, 12, 15, 30, 60$ (Beniash-Kryvets [7]–[9]); 3) $l = 6, 12, 30, 60$ (Beniash-Kryvets, Barkovich [10]). In this paper we prove the following theorem.

T h e o r e m 1. *Let $\Gamma = T(2, 15, 2, R)$ be a group of the form* (1) *with $s \geq 5$. Then Γ contains a free subgroup of rank* 2.

Thus, Conjecture 1 is still open for groups $T(k, l, 2, R)$ with $k = 2, 3$ and $l = 3, 4, 5$.

We shall prove several auxiliary results used in the proof of Theorem 1. Throughout we shall denote the ring of algebraic integers in $\mathbb{C}$ by $\mathcal{O}$, the group of units in $\mathcal{O}$ by $\mathcal{O}^*$, the free group of rank 2 with generators g and h by $F_2 =< g, h >$, the greatest common divisor of integers a and b by (a, b), the image of a matrix $A \in \mathrm{SL}_2(\mathbb{C})$ in $\mathrm{PSL}_2(\mathbb{C})$ by $[A]$, the trace of a matrix A by $\operatorname{tr} A$, and the identity matrix in $\mathrm{SL}_2(\mathbb{C})$ by E. The following lemma characterizes elements of finite order in $\mathrm{PSL}_2(\mathbb{C})$.

L e m m a 1. *Let $2 \leq m \in \mathbb{Z}$ and $\pm E \neq X \in \mathrm{SL}_2(\mathbb{C})$. Then $[X]^m = 1$ in $\mathrm{PSL}_2(\mathbb{C})$ if and only if $\operatorname{tr} X = 2\cos\frac{r\pi}{m}$ for some $r \in \{1, \dots, m-1\}$.*

We shall use standard facts from geometric representation theory (see [11], [12]). Here we recall some notations. Let $F_= \langle g, h\rangle$ be a free group, $R(F_2) = \mathrm{SL}_2(\mathbb{C})^2$ be a representation variety of F_2 in $\mathrm{SL}_2(\mathbb{C})$. For an arbitrary element $u \in F_n$ one can consider the regular function

$$\tau_u : R(F_2) \to \mathbb{C}, \qquad \tau_u(\rho) = \operatorname{tr} \rho(u).$$

Usually τ_u is called *a Fricke character* of the element u.

Lemma 2. *For all* $\alpha, \beta, \gamma \in \mathbb{C}$ *there exist matrices* $A, B \in \mathrm{SL}_2(\mathbb{C})$ *such that*

$$\tau_g(A,B) = \mathrm{tr}\, A = \alpha, \quad \tau_h(A,B) = \mathrm{tr}\, B = \beta, \quad \tau_{gh}(A,B) = \mathrm{tr}\, AB = \gamma.$$

This lemma can be easily proved by straightforward computations.

For every $u \in F_2$ we have

$$\tau_u = Q_u(\tau_g, \tau_h, \tau_{gh}),$$

where $Q_u \in \mathbb{Z}[x,y,z]$ is a uniquely determined polynomial with integer coefficients (see [11]). The polynomial Q_u is usually called the Fricke polynomial of the element u.

Consider polynomials $P_n(\lambda)$ satisfying the initial conditions $P_{-1}(\lambda) = 0$, $P_0(\lambda) = 1$ and the recurrence relation

$$P_n(\lambda) = \lambda P_{n-1}(\lambda) - P_{n-2}(\lambda).$$

If $n < 0$ then we set $P_n(\lambda) = -P_{|n|-2}(\lambda)$. The degree of the polynomial $P_n(\lambda)$ is equal to n if $n > 0$ and to $|n| - 2$ if $n < 0$. It is easy to verify by induction on n that

$$P_n(2\cos\varphi) = \frac{\sin(n+1)\varphi}{\sin\varphi}. \tag{2}$$

Lemma 3. *Let* $k, l \in \mathbb{Z}$ *and assume that* $(k,l) = 1$ *and* $l \geq 2$ *is not a power of a prime. Then* $2\sin\frac{k\pi}{l} \in \mathcal{O}^*$.

We require more detailed information on the Fricke polynomials. Let $w = g^{\alpha_1}h^{\beta_1}\ldots g^{\alpha_s}h^{\beta_s} \in F_2$ and let $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$. Let us treat the Fricke polynomial $Q_w(x,y,z)$ as a polynomial in z. Set

$$Q_w(x,y,z) = M_n(x,y)z^n + M_{n-1}(x,y)z^{n-1} + \ldots + M_0(x,y).$$

Lemma 4. ([14]) *The degree of the Fricke polynomial* $Q_w(x,y,z)$ *with respect to* z *is equal to* s *and its leading coefficient* $M_s(x,y)$ *has the form*

$$M_s(x,y) = \prod_{i=1}^{s} P_{\alpha_i - 1}(x) P_{\beta_i - 1}(y). \tag{3}$$

A subgroup $H \in \mathrm{PSL}_2(\mathbb{C})$ is called *non-elementary* if H is infinite, irreducible and non-isomorphic to a dihedral group.

Lemma 5. ([15]) *Let* $H \in \mathrm{PSL}_2(\mathbb{C})$ *be a non-elementary subgroup. Then* H *contains a non-abelian free subgroup.*

Lemma 6. ([11]) *Let* $A, B \in \mathrm{SL}_2(\mathbb{C})$ *and* $\mathrm{tr}\, A = x$, $\mathrm{tr}\, B = y$, *and* $\mathrm{tr}\, AB = z$. *A subgroup* $< A, B >$ *is irreducible if and only if*

$$\mathrm{tr}\, ABA^{-1}B^{-1} = x^2 + y^2 + z^2 - xyz - 2 \neq 2.$$

Let Γ be a group from Theorem 1, that is,

$$\Gamma = T(2,15,2,R) = \langle a, b; a^2 = b^{15} = R^2(a,b) = 1 \rangle, \tag{4}$$

where $R = ab^{v_1}\ldots ab^{v_s}$, $0 < v_i < 15$, $s > 4$. Set $V = \sum_{i=1}^{s} v_i$. If $(V, 15) \neq 1$ then Γ contains a non-abelian free subgroup (see [8]). So we shall assume that $(V, 15) = 1$. Without loss of generality we may assume that

$$V \equiv 1 \pmod{15}.$$

If $V \not\equiv 1 \pmod{15}$ then one can apply an automorphism of the free product $\langle a; a^2 = 1\rangle * \langle b; b^{15} = 1\rangle$, $a \mapsto a$, $b \mapsto b^p$, were $(p, 15) = 1$ and $pV \equiv 1 \pmod{15}$, to the word $R(a, b)$. To prove Theorem 1, we construct a representation $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ such that $\rho(\Gamma)$ contains a non-abelian free subgroup. Set

$$\beta = 2\cos\frac{\pi}{15}, \qquad f_R(z) = Q_R(0, \beta, z), \tag{5}$$

where Q_R is the Fricke polynomial of R.

Let z_0 be a root of a polynomial $f_R(z)$. By Lemma 2 there exist matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that $\mathrm{tr}\, A = 0$, $\mathrm{tr}\, B = \beta$, and $\mathrm{tr}\, AB = z_0$. Let $G(z_0)$ be a subgroup of $\mathrm{PSL}_2(\mathbb{C})$, generated by $[A], [B]$.

The group $G(z_0)$ is an epimorphic image of Γ since by Lemma 1

$$[A]^2 = [B]^l = R^2([A], [B]) = 1.$$

Lemma 7. *Numbers* $\pm 2\sin\frac{\pi}{15}$ *are not roots of the polynomial* $f_R(z)$.

Proof. Suppose that $f_R(-2\sin\frac{\pi}{15}) = 0$. Let ε be a primitive root of unity of degree 60. Consider a representation $\rho_k : F_2 \to \mathrm{SL}_2(\mathbb{C})$ defined by

$$\rho_k(g) = A = \begin{pmatrix} \varepsilon^{15} & 0 \\ 1 & \varepsilon^{-15} \end{pmatrix}, \qquad \rho_k(h) = B_k = \begin{pmatrix} \varepsilon^{2k} & x \\ 0 & \varepsilon^{-2k} \end{pmatrix}. \tag{6}$$

Then we have $\mathrm{tr}\, A = 0$, $\mathrm{tr}\, B_1 = \beta$, and $\mathrm{tr}\, AB_1 = x - 2\sin\frac{\pi}{15}$. So we obtain

$$f_R(z)(\rho_1) = f_R(x - 2\sin\frac{\pi}{15}) = g(x) = \mathrm{tr}\, R(A, B_1).$$

Since $-2\sin\frac{\pi}{15}$ is a root of $f_R(z)$, 0 is a root of $g(x)$. This means that a constant term of $g(x)$ is equal to 0. On the other hand, a constant term of $\mathrm{tr}\, R(A, B_1)$ is equal to

$$\varepsilon^{15s+2V} + \varepsilon^{-15s-2V} = 2\cos(\frac{15s + 2V}{30}\pi) \neq 0,$$

since $(V, 15) = 1$ by assumption. This contradiction proves that $-2\sin\frac{\pi}{l}$ is not a root of $f_R(z)$. Analogously, considering a representation ρ_{-1} we obtain that $2\sin\frac{\pi}{15}$ is not a root of $f_R(z)$. Lemma is proved.

Lemma 8. *Assume that the polynomial* $f_R(z)$ *has a root* $z_0 \neq 0$. *Then* Γ *contains a non-abelian free subgroup.*

Proof. By Lemma 7 we have $z_0 \neq \pm 2\sin\frac{\pi}{15}$. Let us show that $G(z_0)$ is a non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. By Lemma 6 $G(z_0)$ is irreducible since

$$\mathrm{tr}\, ABA^{-1}B^{-1} - 2 = z_0^2 - 4\sin^2\frac{k\pi}{15} \neq 0.$$

Second, $G(z_0)$ is not a dihedral group since two of three numbers $\mathrm{tr}\, A$, $\mathrm{tr}\, B$, $\mathrm{tr}\, AB$ are not equal to 0 (see [16]). Third, it follows from classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ in [16] that $G(z_0)$ is infinite because it is irreducible and contains an element $[B]$ of order greater than 5. Thus, $G(z_0)$ (and consequently Γ) contains a non-abelian free subgroup. Lemma is proved.

Bearing in mind lemmas 7 and 8, we shall assume in what follows that

$$f_R(z) = M_R z^s, \tag{7}$$

where by Lemma 4 and (3)

$$M_R = \prod_{i=1}^{s} P_{v_i - 1}(2\cos\frac{\pi}{15}) = (2\sin\frac{\pi}{15})^{-s} \prod_{i=1}^{s} 2\sin\frac{v_i\pi}{15}. \tag{8}$$

Let A, B_t be matrices defined in (6), $W(A,B_t)=AB_t^{u_1}\dots AB_t^{u_s}$, where $0<u_i<15$. A set $(u_1,\dots,u_s)$ will be considered as cyclically ordered. Let

$$l_i=|\{j\mid u_j=i\}|,\qquad f_{i,j}=|\{r\mid u_r=i,u_{r+1}=j\}|. \tag{9}$$

We have the following equations:

$$\sum_{i=1}^{14} l_i=s,\quad \sum_{i=1}^{14} f_{ij}=l_j,\quad \sum_{j=1}^{14} f_{ij}=l_i,\quad i,j=1,\dots,14. \tag{10}$$

The following lemma will be heavily used.

L e m m a 9. *Let $g(x)=\operatorname{tr}W(A,B_t)=a_0x^s+\dots+a_s$, $h_i=P_{i-1}(\varepsilon^{2t}+\varepsilon^{-2t})$. Then we have $a_0=\prod_{j=1}^s h_{u_j}$ and*

$$\begin{aligned}a_2=&a_0\sum_{i=1}^{14}\frac{f_{ii}}{h_i}\left(\frac{l_i-2}{h_i}+\sum_{j\neq i}\frac{l_j\varepsilon^{2ti-2tj}}{h_j}\right)+\\ &a_0\sum_{i\neq j}\frac{f_{ij}}{h_i}\left(\frac{l_i-1}{h_i}+\frac{(l_j-1)\varepsilon^{2ti-2tj}}{h_j}+\sum_{k\neq i,k\neq j}\frac{l_k\varepsilon^{2ti-2tk}}{h_k}\right)-\\ &a_0\left(\sum_{i=1}^{14}\frac{l_i(l_i-1)}{2h_i^2}(\varepsilon^{4ti}+\varepsilon^{-4ti})+\sum_{i\neq j}\frac{l_il_j}{h_ih_j}(\varepsilon^{2ti+2tj}+\varepsilon^{-2ti-2tj})\right).\end{aligned} \tag{11}$$

This lemma can be proved by direct computations.

L e m m a 10.*Let $f_R(z)=M_Rz^s$.*

1. *If s is odd then*

$$s=4s_1+1,\quad l_1+l_{14}-1=l_2+l_{13}=l_4+l_{11}=l_7+l_8=s_1,\quad l_3=l_5=l_6=l_9=l_{10}=l_{12}=0. \tag{12}$$

2. *If s is even then*

$$s=4s_1,\quad l_1+l_{14}+1=l_2+l_{13}-1=l_4+l_{11}=l_7+l_8=s_1,\quad l_3=l_5=l_6=l_9=l_{10}=l_{12}=0. \tag{13}$$

P r o o f. Let ρ_{-1} be a representation defined by (6). Then

$$g(x)=f_R(x+2\sin\frac{\pi}{15})=M_R(x+2\sin\frac{\pi}{15})^s=\operatorname{tr}R(A,B_{-1}). \tag{14}$$

Comparing constant terms in (14), we obtain

$$\prod_{i=1}^{s}2\sin\frac{v_i\pi}{15}=2\cos\frac{15s-2V}{30}\pi. \tag{15}$$

1. If $s=2s'+1$ then we have $2\cos\frac{30s'+15-2V}{30}\pi=2\sin\frac{V\pi}{15}\in\mathcal{O}^*$ because $(V,15)=1$. Hence $2\sin\frac{v_i\pi}{15}\in\mathcal{O}^*$ for $i=1,\dots,s$. By Lemma 10 $(v_i,15)=1$ for all v_i's. Since $V=15V_1+1$ one can write (15) in the form

$$(2\sin\frac{\pi}{15})^{l_1+l_{14}-1}(2\sin\frac{2\pi}{15})^{l_2+l_{13}}(2\sin\frac{4\pi}{15})^{l_4+l_{11}}(2\sin\frac{7\pi}{15})^{l_7+l_8}=1 \tag{16}$$

It follows from(16) that $l_1+l_{14}-1=l_2+l_{13}=l_4+l_{11}=l_7+l_8=s_1$, where $s=4s_1+1$ as required.

Case 2 can be proved analogously. Lemma is proved.

Let

$$\Gamma_1 = T(2,5,2,R) = \langle c, d; c^2 = d^5 = S^2(c,d) = 1\rangle, \tag{17}$$

where $S = cd^{u_1} \dots cd^{u_s}$, $0 < u_i < 5$, $s > 4$. Set $U = \sum_{i=1}^{s} u_i$. If $(U,5) \neq 1$ then Γ contains a non-abelian free subgroup (see [8]). So we shall assume that $(U,5) = 1$. As above, without loss of generality we may assume that

$$U \equiv 1 \pmod 5.$$

Set

$$h_S(z) = Q_S(0, 2\cos\frac{\pi}{5}, z), \tag{18}$$

where Q_S is the Fricke polynomial of S. Let z_0 be a root of a polynomial $h_S(z)$ and $A, B \in \mathrm{SL}_2(\mathbb{C})$ be matrices such that $\operatorname{tr} A = 0$, $\operatorname{tr} B = \beta$, $\operatorname{tr} AB = z_0$. As above we shall denote a subgroup of $\mathrm{PSL}_2(\mathbb{C})$, generated by $[A], [B]$, by $G(z_0)$. The group $G(z_0)$ is an epimorphic image of Γ_1 since by Lemma 1

$$[A]^2 = [B]^5 = S^2([A],[B]) = 1.$$

L e m m a 11. *Numbers $\pm 2\sin\frac{\pi}{5}$ are not roots of the polynomial $h_S(z)$.*

The proof of Lemma 11 is similar to proof of Lemma 7.

L e m m a 12. *Assume that the polynomial $h_S(z)$ has a root $z_0 \notin \{0, \pm 1, \pm 2\cos\frac{2\pi}{5}\}$. Then Γ_1 contains a non-abelian free subgroup.*

P r o o f. By Lemma 11 we have $z_0 \neq \pm 2\sin\frac{\pi}{5}$. Let us show that $G(z_0)$ is a non-elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. First, $G(z_0)$ is irreducible by Lemma 6 since

$$\operatorname{tr} ABA^{-1}B^{-1} - 2 = z_0^2 - 4\sin^2\frac{\pi}{5} \neq 0.$$

Second, $G(z_0)$ is not a dihedral group since two of the three numbers $\operatorname{tr} A$, $\operatorname{tr} B$, $\operatorname{tr} AB$ are not equal to 0 (see [14]). Third, it follows from classification of finite subgroups of $\mathrm{PSL}_2(\mathbb{C})$ (see [15]) that $G(z_0)$ is infinite. Thus, $G(z_0)$ (and consequently Γ_1) contains a non-abelian free subgroup.

Bearing in mind lemmas 11 and 12, we shall assume in what follows that

$$h_S(z) = K_S z^{a_1}(z-1)^{a_2}(z+1)^{a_3}(z - 2\cos\frac{2\pi}{5})^{a_4}(z + 2\cos\frac{2\pi}{5})^{a_5}, \tag{19}$$

where by Lemma 4

$$K_S = \prod_{i=1}^{s} P_{u_i - 1}(2\cos\frac{\pi}{5}) = (2\cos\frac{\pi}{5})^{l_2 + l_3}. \tag{20}$$

and l_2, l_3 defined by (10).

L e m m a 13.*Let $h_S(z)$ has the form* (19). *Then*

1. *If s is even then $a_1 = 0$, $a_2 = a_3$, $a_4 = a_5$ and $l_2 + l_3 = 2a_3 + 1$.*
2. *If s is odd then $a_1 = 1$, $a_2 = a_3$, $a_4 = a_5$ and $l_2 + l_3 = 2a_3$.*

P r o o f. Let ρ_{-3} be a representation defined by (6). Then

$$p(x) = h_S(x + 2\cos\frac{3\pi}{10}) = \operatorname{tr} R(A, B_{-3}). \tag{21}$$

Comparing constant terms in (21),we obtain

$$(2\cos\frac{\pi}{5})^{l_2+l_3}(2\cos\frac{3\pi}{10})^{a_1}(2\cos\frac{3\pi}{10} - 1)^{a_2}(2\cos\frac{3\pi}{10} + 1)^{a_3}(2\cos\frac{3\pi}{10} - 2\cos\frac{2\pi}{5})^{a_4}$$
$$(2\cos\frac{3\pi}{10} + 2\cos\frac{2\pi}{5})^{a_5} = 2\cos\frac{5s + 2U}{10}\pi. \tag{22}$$

1. If s is even then we have $2\cos\frac{5s+2U}{10}\pi = 2\cos\frac{\pi}{5} \in \mathcal{O}^*$ in (22). Hence $a_1 = 0$. Using identities

$$(2\cos\frac{3\pi}{10} - 1)(2\cos\frac{3\pi}{10} + 1) = (2\cos\frac{\pi}{5})^{-2},$$
$$(2\cos\frac{3\pi}{10} - 2\cos\frac{2\pi}{5})(2\cos\frac{3\pi}{10} + 2\cos\frac{2\pi}{5}) = 1,$$

we can write (22) in the form

$$(2\cos\frac{\pi}{5})^{l_2+l_3-2a_3-1}(2\cos\frac{3\pi}{10} - 1)^{a_2-a_3}(2\cos\frac{3\pi}{10} - 2\cos\frac{2\pi}{5})^{a_4-a_5} = 1. \tag{23}$$

It is not difficult to see that (23) implies

$$a_2 = a_3, \quad a_4 = a_5, \quad l_2 + l_3 = 2a_3 + 1,$$

as required.

The case when s is odd can be proved analogously.

Proof of Theorem 1. First consider the case when s is odd. Without loss of generality we may assume that $s = 4s_1 + 1$. Consider a representation ρ_1, defined by (6). Then we have

$$f_1(x) = f_R(z)(\rho) = M_R(x + \varepsilon^{17} + \varepsilon^{-17})^s = \operatorname{tr} R(A, B_1). \tag{24}$$

By (24) the coefficient a_2 of the polynomial $f_1(x)$ is equal to

$$a_2 = M_R(\varepsilon^{17} + \varepsilon^{-17})^2 s(s-1)/2.$$

Taking into account Lemma 9, we obtain following equation

$$\begin{aligned}(\varepsilon^{17} + \varepsilon^{-17})^2 s(s-1)/2 = &\sum_{i=1}^{14} \frac{f_{ii}}{h_i}\left(\frac{l_i - 2}{h_i} + \sum_{j\neq i}\frac{l_j\varepsilon^{2i-2j}}{h_j}\right) + \\ &\sum_{i\neq j}\frac{f_{ij}}{h_i}\left(\frac{l_i - 1}{h_i} + \frac{(l_j-1)\varepsilon^{2i-2j}}{h_j} + \sum_{k\neq i, k\neq j}\frac{l_k\varepsilon^{2i-2k}}{h_k}\right) - \\ &\sum_{i=1}^{14}\frac{l_i(l_i-1)}{2h_i^2}(\varepsilon^{4i} + \varepsilon^{-4i}) + \sum_{i\neq j}\frac{l_i l_j}{h_i h_j}(\varepsilon^{2i+2j} + \varepsilon^{-2i-2j}).\end{aligned} \tag{25}$$

The equation (25) can be written in the form

$$X_0 + X_1\varepsilon^2 + X_2\varepsilon^4 + X_3\varepsilon^6 + X_4\varepsilon^8 + X_5\varepsilon^{10} + X_6\varepsilon^{12} + X_7\varepsilon^{14} = 0, \tag{26}$$

where $X_0, \ldots, X_7$ are polynomials with integer coefficients of l_i, f_{ij}, and s_1. Since $1, \varepsilon^2, \ldots, \varepsilon^{14}$ are linearly independent over $\mathbb{Q}$, one obtains a system

$$X_0 = X_1 = \cdots = X_7 = 0. \tag{27}$$

Now, consider an epimorphic image $\Gamma_1 = \langle c, d; c^2 = d^5 = R^2(c,d) = 1\rangle$ of the group Γ, where $R(c,d) = cd^{v_1}\ldots cd^{v_s}$. If the group Γ_1 has a representation into $\mathrm{PSL}_2(\mathbb{C})$ with non-elementary image, then Γ contains a non-abelian free subgroup. So we shall assume that for any representation $\rho : \Gamma_1 \to \mathrm{PSL}_2(\mathbb{C})$ the image group $\rho(\Gamma_1)$ is either finite or dihedral. We can write the word $R(c,d)$ from the free product $\langle c; c^2 = 1\rangle * \langle d; d^5 = 1\rangle$ in the form $S(c,d) = cd^{u_1}\ldots cd^{u_s}$, where $u_i \equiv V_i \bmod 5$ and $0 < u_i < 5$. Let $U = \sum_{i=1}^{s} u_i$. Since $(V, 15) = 1$, we have $(U, 5) = 1$. By Lemma 10

$$h_S(z) = K_S(z^2 - 1)^{a_2}(z^2 - 4\cos^2(\frac{2\pi}{5}))^{a_4}. \tag{28}$$

Let ρ_3 be a representation defined by (6). Then

$$p(x) = h_S(x - 2\cos\frac{3\pi}{10}) = \operatorname{tr} R(A, B_2). \tag{29}$$

It follows from (28) that the coefficient b_2 of the polynomial $p(x)$ by x^{s-2} is equal to

$$b_2 = K_S(a_4 + a_2(2\cos\frac{2\pi}{5})^2 + 4(2s_1^2 - s_1)(2\cos\frac{3\pi}{10})^2).$$

On the other hand, we can apply Lemma 9 to compute b_2. First,we set

$$l_1' = l_1 + l_{11}, \qquad l_2' = l_2 + l_7, \qquad l_3' = l_3 + l_{13}, \qquad l_4' = l_4 + l_{14},$$

and

$$f_{11}' = f_{11} + f_{1,11} + f_{11,1} + f_{11,11}, \qquad f_{12}' = f_{12} + f_{17} + f_{11,2} + f_{11,7},$$
$$\vdots$$
$$f_{43}' = f_{43} + f_{14,3} + f_{4,13} + f_{14,13}, \qquad f_{44}' = f_{44} + f_{14,4} + f_{4,14} + f_{14,14}.$$

Let $h_i' = 1/P_{i-1}(\varepsilon^6 + \varepsilon^{-6})$.Then we have an equation

$$\begin{aligned} &a_4 + a_2(2\cos\frac{2\pi}{5})^2 + 4(2s_1^2 - s_1)(2\cos\frac{3\pi}{10})^2 = \\ &\sum_{i=1}^{4} \frac{f_{ii}'}{h_i'}\left(\frac{l_i'-2}{h_i'} + \sum_{j\neq i}\frac{l_j'\varepsilon^{6i-6j}}{h_j'}\right) + \\ &\sum_{i\neq j}\frac{f_{ij}'}{h_i'}\left(\frac{l_i'-1}{h_i'} + \frac{(l_j'-1)\varepsilon^{6i-6j}}{h_j'} + \sum_{k\neq i, k\neq j}\frac{l_k'\varepsilon^{6i-6k}}{h_k'}\right) - \\ &\sum_{i=1}^{l-1}\frac{l_i'(l_i'-1)}{2h_i'^2}(\varepsilon^{12i} + \varepsilon^{-12i}) + \sum_{i\neq j}\frac{l_i'l_j'}{h_i'h_j'}(\varepsilon^{6i+6j} + \varepsilon^{-6i-6j}). \end{aligned} \tag{30}$$

The equation (30) can be written in the form

$$Y_0 + Y_1\varepsilon^2 + Y_2\varepsilon^4 + Y_3\varepsilon^6 + Y_4\varepsilon^8 + Y_5\varepsilon^{10} + Y_6\varepsilon^{12} + Y_7\varepsilon^{14} = 0, \tag{31}$$

where $Y_0, \ldots, Y_7$ are polynomials with integer coefficients of l_i, f_{ij}, and s_1. As above one obtains a system

$$Y_0 = Y_1 = \cdots = Y_7 = 0. \tag{32}$$

Thus, we have the equations (10), (12), (27), (32). We solve this system by computer with Maple and obtain, in particular, that

$$\begin{aligned} 0 = &-71s_1^2 + 1 + 5l_2 + 17l_7 + 15l_8 - 5(l_{11} - f_{11,1} - f_{11,2} - f_{11,2} - f_{11,4} - f_{11,8} - f_{11,11} - f_{11,14}) - \\ &f_{11,4} - 5f_{14,13} - 6f_{8,7} - 5f_{14,7} - 2f_{8,8} - 4f_{8,4} - 4f_{7,7} - 9f_{7,13} - 5f_{2,1} - 4f_{8,1} - 5f_{2,4} - f_{2,7} - 8f_{2,8} - 5f_{2,13} - \\ &4f_{8,2} - f_{4,7} + (f_{11,8} + 5f_{13,8} + f_{14,8} + f_{4,8} + f_{1,8}) + (f_{11,11} + 4f_{1,11}) + (5f_{13,13} + 9f_{13,14} + 10f_{13,4} + \\ &10f_{13,1} + 10f_{13,2} + 10f_{13,8} + 9f_{13,11}) + (4f_{4,2} + 5f_{11,2} + 5f_{14,2} + 4f_{1,2} + 4f_{13,2}) + (f_{4,4} + 4f_{4,14} + 3f_{4,11}) \leq \\ &-71s_1^2 + 1 + (10l_2 + 10l_{13}) + (4l_4 + 4l_{11}) + (17l_7 + 17l_8) + 3l_8 \leq -71s_1^2 + 34s_1 + 1. \end{aligned} \tag{33}$$

It follows from (34) that $s_1 < 1$ which is a contradiction.

Now consider the case when s is even. We may assume that $s = 4s_1$.As above we obtain the following equation:

$$0 = 9s_1^2 - l_1 - 2l_2 - 4l_8 - l_{11} - l_{13} + f_{2,8} + f_{2,13} + f_{8,2} + f_{13,2} + f_{8,7} + f_{13,7} - (f_{2,2} + f_{2,4} + f_{2,7} + f_{2,14}) - (f_{2,2} + f_{4,2} + f_{7,2} + f_{14,2}) - (f_{4,4} + f_{4,14}) - 2(f_{7,2} + f_{7,4} + f_{7,7} + f_{7,11} + f_{7,2}) - (f_{2,7} + f_{4,7} + f_{7,7} + f_{14,7}) - (f_{14,4} + f_{14,14}) - (f_{8,8} + f_{13,8}) - f_{13,13} > 9s_1^2 - (l_1 + l_{14}) - 2(l_2 + l_{13}) - (l_4 + l_1 1) - 4(l_7 + l_8) - l_8 + 1 = 9s_1^2 - 9s_1. \quad (34)$$

It follows from (35) that $s_1 < 1$ which is a contradiction. Theorem 1 is proved.

Summary

It is proved that a generalized triangle group of type $(2, 15, 2)$ contains a non-abelian free subgroups. Hence such a group satisfies the Tits alternative and Rosenberger's conjecture is true for it.

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Доказано, что обобщенные треугольные группы типа $(2, 15, 2)$ содержат неабелеву свободную подгруппу и, следовательно, удовлетворяют альтернативе Титса.

Табл. 0. Ил. 0. Библиогр. — 15 назв.