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Decomposing one-relator products of cyclic groups into free products with amalgamation

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Abstract. The problem of the decomposition of one-relator products of cyclics into non-trivial free products with amalgamation is considered. Two theorems are proved, one of which is as follows.

Let $G = \langle a, b \mid a^{2n} = R^m(a,b) = 1\rangle$, *where* $n \geqslant 0$, $m \geqslant 2$, *and* $R(a,b)$ *is a cyclically reduced word containing* b *in the free group on* a *and* b. *Then* G *is a non-trivial free product with amalgamation.*

One consequence of this theorem is a proof of the conjecture of Fine, Levin, and Rosenberger that each two-generator one-relator group with torsion is a non-trivial free product with amalgamation.

Bibliography: 13 titles.

Introduction

A one-relator free product of a family of groups $\{G_i\}$, $i \in I$, is the group $G = (*G_i)/N(S)$, where S is a cyclically reduced word in the free product $*G_i$ and $N(S)$ is its normal closure. We call S a *relator*. One-relator free products share many properties with one-relator groups [1]. We consider the case when the G_i are cyclic groups and the relator is a proper power, that is, $S = R^m$, where R is a cyclically reduced word in $*G_i$ and $m \geqslant 2$.

Definition 1. A group G having a presentation

$$G = \langle a_1, \dots, a_n \mid a_1^{l_1} = \dots = a_n^{l_n} = R^m(a_1, \dots, a_n) = 1\rangle,$$

where $n \geqslant 2$, $m \geqslant 2$, $l_i = 0$ or $l_i \geqslant 2$ for all i, and $R(a_1, \dots, a_n)$ is a cyclically reduced word in the free group on $a_1, \dots, a_n$, is called a *one-relator product of* n *cyclics*.

This paper considers the problem of the decomposition of one-relator products of cyclics into non-trivial free products with amalgamation. The first general results on the decomposition of such groups were obtained in [2]. Theorem 3 in [2] says

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that if G is a free one-relator product of n cyclics ($n \geqslant 3$), then G is a non-trivial free product with amalgamation.

The case of free one-relator products of 2 cyclics, that is, of groups

$$G = \langle a, b \mid a^m = b^n = R^l(a, b) = 1 \rangle,$$

where $l \geqslant 2$ and $R(a, b)$ is a cyclically reduced word in the free group on a and b, is considerably more complicated. Such groups are called *generalized triangle groups.* Sometimes these groups admit a decomposition into a non-trivial free product with amalgamation and sometimes they do not. For example, it is known (cf. [3]) that the *ordinary triangle groups*

$$T(m, n, p) = \langle a, b \mid a^m = b^n = (ab)^p = 1 \rangle,$$

where $m, n, p \geqslant 2$, are non-trivial free products with amalgamation. Zieschang [3] has studied the problem of the decomposition into a non-trivial free product with amalgamation for planar discontinuous groups. He has given a complete answer to the question when such a group G is a non-trivial free product with amalgamation in all cases except for the groups

$$H_1 = \langle a, b \mid [a, b]^n = 1 \rangle \text{ and } H_2 = \langle c, d \mid c^2 = [c, d]^n = 1 \rangle, n \geqslant 2.$$

Rosenberger [4] has proved that the groups H_1 and H_2 are non-trivial free products with amalgamation if n is not a power of 2. In recent papers [5] and [6] this property was established for arbitrary n. This result is also an immediate consequence of Theorem 1. The following conjecture was stated in [2].

Conjecture 1. *A two-generator one-relator group with torsion is a non-trivial free product with amalgamation.*

Note that if a group G has more that 2 generators, then the assertion of Conjecture 1 holds by Theorem 3 in [2] mentioned above. Moreover, this conjecture has been proved in [2] for groups G having a presentation $G = \langle a, b \mid R^m(a, b) = 1 \rangle$, where $m \geqslant 2$ and $R(a, b)$ is a cyclically reduced word not belonging to the derived subgroup of the free group on a and b.

In the present paper we prove the following two results.

Theorem 1. *Let* $G = \langle a, b \mid a^{2n} = R^m(a, b) = 1 \rangle$, *where* $n \geqslant 0$, $m \geqslant 2$, *and* $R(a, b)$ *is a cyclically reduced word containing* b *in the free group on* a *and* b. *Then* G *is a non-trivial free product with amalgamation.*

Theorem 2. *Let* $G = \langle a, b \mid a^n = R^m(a, b) = 1 \rangle$, *where* $n = 0$ *or* $n \geqslant 2$, $m \geqslant 3$, *and* $R(a, b)$ *is a cyclically reduced word containing* b *in the free group on* a *and* b. *If* $R(a, b) = a^{u_1} b^{v_1} \cdots a^{u_s} b^{v_s}$, *where* $0 < u_i < n$, $v_i \neq 0$ *for* $i = 1, \ldots, s$, *and* $\prod_{i=1}^{s} |v_i| \geqslant 3$, *then* G *is a non-trivial free product with amalgamation.*

Corollary 1. *Let G be a two-generator one-relator group with torsion. Then G is a non-trivial free product with amalgamation.*

Proof of Corollary 1. It is well known (see, for example, [7]) that if G is a finitely generated one-relator group with torsion then G has a presentation of the form $G = \langle a_1, \dots, a_n \mid R^m(a_1, \dots, a_n) = 1\rangle$, where $m \geqslant 2$ and $R(a_1, \dots, a_n)$ is a cyclically reduced word in the free group on $a_1, \dots, a_n$. In our case $G = \langle a, b \mid R^m(a, b) = 1\rangle$, so that G is a non-trivial free product with amalgamation by Theorem 1.

Corollary 2. *The groups $H_1 = \langle a, b \mid [a, b]^n = 1\rangle$ and $H_2 = \langle c, d \mid c^2 = [c, d]^n = 1\rangle$ are non-trivial free products with amalgamation for each $n \geqslant 2$.*

§ 1. Some auxiliary results

Here we shall prove several auxiliary results. Throughout we shall denote the identity matrix in $\mathrm{SL}_2(\mathbb{C})$ by E, the ring of algebraic integers in $\mathbb{C}$ by $\mathcal{O}$, the free group of rank 2 with generators g and h by $F_2 = \langle g, h\rangle$, and the trace of a matrix X by $\operatorname{tr} X$. The following result of Bass [8] plays a key role in the proof of Theorems 1 and 2.

Proposition 1 (see [8]). *Let H be a finitely generated subgroup of $\mathrm{GL}_2(\mathbb{C})$. Then one of the following cases must occur*:

(1) *there exists an epimorphism $f\colon H \to \mathbb{Z}$ such that $f(u) = 0$ for all unipotent elements $u \in H$;*
(2) *H is conjugate to a subgroup of the group of triangular matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with roots of unity a and c;*
(3) *H is conjugate to a subgroup of the group $\mathrm{GL}_2(\mathcal{O})$;*
(4) *H is a non-trivial free product with amalgamation.*

The following observation is useful in the construction of matrices of finite order in $\mathrm{SL}_2(\mathbb{C})$.

Lemma 1. *Let $m > 1$ and let $X \in \mathrm{SL}_2(\mathbb{C})$. If $\operatorname{tr} X = \pm 2\cos(r\pi/m)$, where $r \in \{1, \dots, m-1\}$, then $X^{2m} = E$. In particular, if $\operatorname{tr} X = 0$, then $X^2 = -E$.*

Proof. Let, for example, $\operatorname{tr} X = 2\cos(r\pi/m)$. Then the characteristic polynomial of X has the roots $\alpha = \cos(r\pi/m) + i\sin(r\pi/m)$ and $\alpha^{-1} = \cos(r\pi/m) - i\sin(r\pi/m)$, where α is some $2m$th root of unity. Consequently, X is conjugate to the matrix $X_1 = \operatorname{diag}(\alpha, \alpha^{-1})$ and therefore $X^{2m} = E$.

In what follows we shall require so-called 'Fricke characters' (see [9]–[11]). For each $w = w(g, h) \in F_2$ one can consider the following regular function:

$$\tau_w\colon \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}, \qquad \tau_w(A, B) = \operatorname{tr}(w(A, B)).$$

It has been proved in [9] (see also [12]) that for each $w \in F_2$ we have

$$\tau_w = Q_w(\tau_g, \tau_h, \tau_{gh}),$$

where $Q_w \in \mathbb{Z}[x,y,z]$ is a polynomial with integral coefficients. The function τ_w is called a *Fricke character* and the polynomial Q_w is the *Fricke polynomial* of the element $w \in F_2$. Let u and v be arbitrary elements of F_2. The following relations for Fricke characters follow from the corresponding relations between the traces of arbitrary matrices in $\mathrm{SL}_2(\mathbb{C})$ and can be readily verified:

$$(1)\ \tau_{u^{-1}} = \tau_u; \quad (2)\ \tau_{uv} = \tau_{vu}; \quad (3)\ \tau_{vuv^{-1}} = \tau_u; \quad (4)\ \tau_{uv} = \tau_u\tau_v - \tau_{uv^{-1}}. \qquad (1)$$

The following assertion is well known and can be easily proved by straightforward computations, although it is difficult to give an explicit reference.

Lemma 2. *For all $\alpha, \beta, \gamma \in \mathbb{C}$ there exist matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that*

$$\tau_g(A,B) = \operatorname{tr} A = \alpha, \qquad \tau_h(A,B) = trB = \beta, \quad \text{and} \quad \tau_{gh}(A,B) = \operatorname{tr} AB = \gamma.$$

In particular, this lemma means that the Fricke characters τ_g, τ_h, and τ_{gh} are algebraically independent over $\mathbb{C}$, and therefore the Fricke polynomial Q_w of an element w is well defined. Next, we require an explicit formula for the Fricke polynomial obtained in [13]. To formulate this result we consider polynomials $P_n(\lambda)$ satisfying the recurrence relations

$$P_n(\lambda) = \lambda P_{n-1}(\lambda) - P_{n-2}(\lambda)$$

and the initial conditions

$$P_0(\lambda) = 1, \qquad P_{-1}(\lambda) = 0.$$

If $n < 0$, then we set

$$P_n(\lambda) = -P_{|n|-2}(\lambda)$$

The degree of the polynomial $P_n(\lambda)$ is equal to n if $n > 0$ and to $|n| - 2$ if $n < 0$.

Lemma 3. (1) *The polynomial $P_n(\lambda)$, $n \geqslant 1$, has n zeros, described by the formula*

$$\lambda_{n,k} = 2\cos\frac{k\pi}{n+1}, \qquad k = 1, 2, \ldots, n.$$

(2) *$P_{n-1}(2) = n$ for each $n \in \mathbb{Z}$.*

(3) *Let $x = \tau_g$, let $y = \tau_h$, and let $z = \tau_{gh}$. Then*

$$Q_{gh^n}(x,y,z) = P_{n-1}(y)z - P_{n-2}(y)x. \qquad (2)$$

Proof. (1) It is easy to verify by induction on n that

$$P_n(2\cos\varphi) = \frac{\sin(n+1)\varphi}{\sin\varphi}.$$

Assertion (1) can now be obtained by a straightforward computation.

Assertions (2) and (3) can be proved by induction on n (as regards (2), see also [12], formula (6)).

Further, let $w = g^{\alpha_1}h^{\beta_1}\cdots g^{\alpha_s}h^{\beta_s} \in F_2$ be a cyclically reduced word in F_2, and let $x = \tau_g$, $y = \tau_h$, and $z = \tau_{gh}$. We treat the Fricke polynomial $Q_w(x,y,z)$ as a polynomial in z. Let

$$Q_w(x,y,z) = M_n(x,y)z^n + M_{n-1}(x,y)z^{n-1} + \cdots + M_0(x,y).$$

Proposition 2 (see [13]). *The degree of the Fricke polynomial $Q_w(x,y,z)$ with respect to z is equal to s, that is, the number of blocks of the form $g^{\alpha_i}h^{\beta_i}$ in w. The leading coefficient $M_s(x,y)$ of the polynomial $Q_w(x,y,z)$ has the following form*:

$$M_s(x,y) = \prod_{i=1}^{s} P_{\alpha_i-1}(x)P_{\beta_i-1}(y).$$

The following lemma plays an important role in the proof of Theorems 1 and 2.

Lemma 4. *Let $G = \langle a, b \mid a^n = R^m(a,b) = 1\rangle$ and let $A, B \in \mathrm{SL}_2(\mathbb{C})$ be matrices such that $\operatorname{tr} A = \alpha$, where $\alpha = \pm 2\cos(t\pi/n)$ for some $t \in \{1,\dots,n-1\}$, and $Q_R(\alpha,y,z) = c$, where Q_R is the Fricke polynomial of the element $R(g,h) \in F_2$, $c = \pm 2\cos(r\pi/m)$ for some $r \in \{1,\dots,m-1\}$, $y = \operatorname{tr} B$, and $z = \operatorname{tr} AB$. Let $H = \langle A, B\rangle$. Assume that two following conditions are satisfied*:

(1) *there exists a unipotent (or finite-order) element $u = A^{\alpha_1}B^{\beta_1}\cdots A^{\alpha_s}B^{\beta_s}$ in H such that $l = \sum_{i=1}^{s}\beta_i \neq 0$;*

(2) *there exists an element $h \in H$ such that $\operatorname{tr} h \notin \mathcal{O}$.*

Then the group G is a non-trivial free product with amalgamation.

Proof. We claim that the group H fails conditions (1)–(3) of Proposition 1. Let $f\colon H \to \mathbb{Z}$ be an epimorphism such that $f(z) = 0$ for all unipotent elements $z \in H$. Then $f(A) = 0$ because $A^{2n} = E$ by Lemma 1. Further, $f(u) = lf(B) = 0$, therefore $f(B) = 0$ because, by assumption, u is either unipotent or of finite order. Hence $f(H) = \{0\}$, which is a contradiction. Next, there exists by assumption an element $h \in H$ such that $\operatorname{tr} h \notin \mathcal{O}$. Hence H does not satisfy conditions (2) and (3) of Proposition 1 and H is a non-trivial free product with amalgamation, that is, $H = H_1 *_F H_2$ with $H_1 \neq F \neq H_2$. Since $-E \in Z(H)$, it follows that $-E \in F$. Let $\overline{A}$, $\overline{B}$, $\overline{H}$, $\overline{H}_1$, $\overline{H}_2$, and $\overline{F}$ be the images of A, B, H, H_1, H_2, and F in $\mathrm{PSL}_2(\mathbb{C})$, respectively. Then $\overline{H}_1 \neq \overline{F} \neq \overline{H}_2$, and therefore $\overline{H} = \overline{H}_1 *_{\overline{F}} \overline{H}_2$ is a non-trivial free product with amalgamation. The condition $Q_R(\alpha,y,z) = c$ means that $\operatorname{tr} R(A,B) = c$, so that $R^{2m}(A,B) = E$ by Lemma 1. Thus, $\overline{A}^n = R^m(\overline{A},\overline{B}) = 1$ in $\mathrm{PSL}_2(\mathbb{C})$, so that $\overline{H}$ is an epimorphic image of G and we obtain the assertion of Lemma 4. (It is well known that if $\varphi\colon \Gamma_1 \to \Gamma_2$ is an epimorphism of groups and Γ_2 is a non-trivial free product with amalgamation, then so also is Γ_1.)

The following lemma will be repeatedly used in what follows.

Lemma 5. (1) *For $s, m, M \in \mathbb{Z}$ satisfying the conditions $m \geqslant 2$ and $|M| \geqslant 3$ there exist $\varepsilon \in \{-1,1\}$ and $r \in \{1,\dots,m-1\}$ such that $((-1)^s 2 - c)/M \notin \mathcal{O}$, where $c = 2\varepsilon\cos(r\pi/m)$.*

(2) *For $m \in \mathbb{Z}$, $m \geqslant 2$, there exists an integer $r \in \{1,\dots,m-1\}$ such that $(2 + 2\cos(r\pi/m))^{-1} \notin \mathcal{O}$.*

(3) *If $m \in \mathbb{Z}$, $m \geqslant 3$, then $\cos(\pi/m) \notin \mathcal{O}$.*

(4) *For all $m, M \in \mathbb{Z}$, $m \geqslant 3$ and $|M| \geqslant 3$, there exists $r \in \{1,\dots,m-1\}$ such that $(4/M)\cos(r\pi/m) \notin \mathcal{O}$.*

Proof. (1) Without loss of generality we can assume that s is even. Assume also that for all $\varepsilon \in \{-1,1\}$ and $r \in \{1,\dots,m-1\}$ the ratio $(2-c)/M$ is an algebraic integer. Then $(2-c)/M + (2+c)/M = 4/M$ is an algebraic integer too.

Since $|M| \geqslant 3$, this is possible only if $|M| = 4$. For definiteness, let $M = 4$. If m is even, then we set $r = m/2$. In this case $c = 0$ and $2/M = \frac{1}{2} \notin \mathcal{O}$ which is a contradiction. Assume that m is odd and $F_1 = \{2\cos(2r\pi/m) \mid r = 1, \ldots, m-1\}$. Then

$$\sum_{c \in F_1} \frac{2-c}{4} = \frac{2(m-1) - \sum_{c \in F_1} c}{4} = \frac{2(m-1)+1}{4},$$

because $1 + \sum_{c \in F_1} c = 0$ as the sum of the mth roots of unity. Obviously, $(2(m-1)+1)/4$ is not an algebraic integer; this is a contradiction proving assertion (1).

(2) Since

$$\frac{1}{2 + 2\cos(r\pi/m)} = \frac{1}{4\cos^2(r\pi/(2m))},$$

it is sufficient to prove that $(2\cos(r\pi/(2m)))^{-1}$ does not belong to $\mathcal{O}$ for some $r \in \{1, \ldots, m-1\}$. By virtue of Lemma 3 the polynomial $P_{2m-1}(\lambda)$ has zeros at 0 and $\pm 2\cos(r\pi/(2m))$, $r = 1, \ldots, m-1$. It is easy to verify by induction on m that

$$P_{2m-1}(\lambda) = \lambda\bigl(\lambda^{2m-2} + a_1\lambda^{2m-4} + \cdots + (-1)^{m+1}m\bigr). \tag{3}$$

Hence the numbers $(\pm 2\cos(r\pi/(2m)))^{-1}$ are the zeros of $P_{2m-1}(1/\lambda)$ or, equivalently, of the polynomial

$$g_1(\lambda) = \lambda^{2m-2} + \cdots + (-1)^{m+1}\frac{1}{m}.$$

Since $1/m \notin \mathcal{O}$, at least one of the zeros of $g_1(\lambda)$ is not an algebraic integer and (2) is proved.

(3) It is sufficient to prove that there exists $r \in \{1, \ldots, m-1\}$ such that $\cos(r\pi/m) \notin \mathcal{O}$ because if $\cos(\pi/m) \in \mathcal{O}$, then $\cos(r\pi/m) \in \mathcal{O}$ for each $r \in \mathbb{Z}$. By Lemma 3 the numbers $2\cos(r\pi/m)$, $r = 1, \ldots, m-1$, are the zeros of the polynomial $P_{m-1}(\lambda)$. If $m = 2k$, then it follows from (3) that the numbers $\cos(r\pi/(2k))$, $r = 1, \ldots, 2k-1$, are the zeros of $g_1(2\lambda)$ or, equivalently, the roots of the polynomial equation

$$\lambda^{2k-1} + \cdots + (-1)^{k-1}\frac{k}{2^{2k-2}}\lambda = 0.$$

Since $k \geqslant 2$, it follows that $k < 2^{2k-2}$, so that $k/2^{2k-2}$ is not an algebraic integer. Hence there exists r such that $\cos(r\pi/(2k)) \notin \mathcal{O}$.

Let $m = 2k+1$. Then it is easy to verify by induction that

$$P_{2k}(\lambda) = \lambda^{2k} + \cdots + (-1)^k. \tag{4}$$

By Lemma 3 the numbers $\cos(r\pi/(2k+1))$, $r = 1, \ldots, 2k$, are the zeros of $P_{2k}(2\lambda)$ or, equivalently, the roots of the equation

$$\lambda^{2k} + \cdots + \frac{(-1)^k}{2^{2k}} = 0.$$

Since $(-1)^k/2^{2k} \notin \mathcal{O}$, there exists r such that $\cos(r\pi/(2k+1)) \notin \mathcal{O}$.

(4) It follows from Lemma 3 that the numbers $(4/M)\cos(r\pi/m)$, $r=1,\dots,m-1$, are the zeros of $P_{m-1}(M\lambda/2)$. We consider now two cases.

Let $m=2k+1$. Then the numbers $(4/M)\cos(r\pi/(2k+1))$, $r=1,\dots,2k$, are by (4) the roots of the equation

$$\lambda^{2k}+\dots+(-1)^k\Big(\frac{2}{M}\Big)^{2k}=0.$$

Since $|M|\geqslant 3$, it follows that $(-1)^k(2/M)^{2k}\notin\mathcal{O}$, and therefore the last equation also has a root not belonging to $\mathcal{O}$.

Let $m=2k$. Then the numbers $(4/M)\cos(r\pi/m)$, $r=1,\dots,m-1$, are by (3) the roots of the equation

$$\lambda^{2k-1}+\dots+(-1)^{k+1}k\Big(\frac{2}{M}\Big)^{2k-2}\lambda=0.$$

It is easy to see that $(-1)^{k+1}k(2/M)^{2k-2}\notin\mathcal{O}$. This proves assertion (4) and, at the same time, Lemma 5.

§ 2. Proof of Theorem 1

Let G be a group satisfying the assumptions of Theorem 1. If $n=0$ or $n>1$, then we consider the group $G_1=\langle a,b\mid a^2=R^m(a,b)=1\rangle$, which is an epimorphic image of G. Thus, we can assume without loss of generality that $n=1$. Our aim is to construct a representation $\varphi\colon G\to \mathrm{PSL}_2(\mathbb{C})$ such that the group $\varphi(G)$ is a non-trivial free product with amalgamation. Three cases can occur; we shall consider them separately:

(1) $R(a,b)=ab^{n_1}\cdots ab^{n_s}$ is a cyclically reduced word lying in the subgroup of the free group on a and b generated by a^2 and the derived subgroup; moreover, there exists i such that $|n_i|>1$;

(2) $R(a,b)$ is as in case 1, but $|n_i|=1$ for all $i=1,\dots,s$;

(3) $R(a,b)$ is a cyclically reduced word that does not belong to the subgroup of the free group on a and b generated by a^2 and the derived subgroup.

Case 1. Let $R(a,b)=ab^{n_1}\cdots ab^{n_s}$. It follows from our assumptions about the element $R(a,b)$ that s is even and $\sum_{i=1}^{s}n_i=0$. To apply Lemma 4 we shall show that there exist matrices $A,B\in\mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr}A=0,\qquad \operatorname{tr}B=2,\quad\text{and}\quad \operatorname{tr}AB=z,$$

where z is a complex number not belonging to $\mathcal{O}$ and satisfying the equation

$$Q_R(0,2,z)=c, \tag{5}$$

where $c=\pm2\cos(r\pi/m)$ for some $r\in\{1,\dots,m-1\}$.

By Proposition 2 we can write (5) as follows:

$$M_s(0,2)z^s+M_{s-1}(0,2)z^{s-1}+\dots+M_0(0,2)-c=0, \tag{6}$$

where $M_s(0,2)=\prod_{i=1}^{s}P_0(0)P_{n_i-1}(2)=\prod_{i=1}^{s}n_i$ by Lemma 3. Since s is even, $\sum_{i=1}^{s}n_i=0$ and there exists i such that $|n_i|>1$, it follows that $\prod_{i=1}^{s}|n_i|\geqslant 3$. Thus, $|M_s(0,2)|\geqslant 3$. Furthermore, we have the following result.

Lemma 6. *Let $Q_R(x,y,z) = M_s(x,y)z^s + \cdots + M_0(x,y)$ be the Fricke polynomial of $R = gh^{n_1}\cdots gh^{n_s} \in F_2$, where s is even, $x = \tau_g$, $y = \tau_h$, and $z = \tau_{gh}$. Then $M_0(0,2) = (-1)^{s/2}2$.*

Proof. First, we make the following observations.

(1) $Q_{h^i}(0,2,z) = 2$, for if $A, B \in \mathrm{SL}_2(\mathbb{C})$ and B is a unipotent matrix, then B^i is unipotent for any i. Hence $\tau_{h^i}(A,B) = \operatorname{tr} B^i = 2$.

(2) It follows from Lemma 3 that $Q_{gh^i}(0,2,z) = P_{i-1}(2)z = iz$.

We shall prove the lemma by induction on s. If $s = 2$, then, using relations (1), we obtain

$$Q_{gh^{n_1}gh^{n_2}}(0,2,z) = Q_{gh^{n_1}}(0,2,z)Q_{gh^{n_2}}(0,2,z) - Q_{h^{n_2-n_1}}(0,2,z) = n_1n_2z^2 - 2,$$

that is, $M_0(0,2) = -2$. For arbitrary $s > 2$ we have

$$\begin{aligned} Q_R(0,2,z) = Q_{gh^{n_1}}(0,2,z)&Q_{gh^{n_2}\cdots gh^{n_s}}(0,2,z) \\ &- Q_{gh^{n_3}\cdots gh^{n_s-n_1+n_2}}(0,2,z) = n_1zf(z) - g(z), \end{aligned}$$

where

$$\begin{aligned} f(z) &= Q_{gh^{n_2}\cdots gh^{n_s}}(0,2,z), \\ g(z) &= Q_{gh^{n_3}\cdots gh^{n_s-n_1+n_2}}(0,2,z). \end{aligned}$$

We see that the polynomial $Q_R(0,2,z)$ has the same constant term as $g(z)$ and, by induction,

$$M_0(0,2) = -(-1)^{(s-2)/2}2 = (-1)^{s/2}2.$$

This proves Lemma 6.

Bearing in mind Lemma 6 we can write (6) in the following form:

$$z^s + \frac{M_{s-1}(0,2)}{M_s(0,2)}z^{s-1} + \cdots + \frac{(-1)^{s/2}2 - c}{M_s(0,2)} = 0. \tag{7}$$

It follows from Lemma 5(1) that one can choose $c = \pm 2\cos(r\pi/m)$ in the equation (7) such that the constant term $((-1)^{s/2} - c)/M_s(0,2)$ does not belong to $\mathcal{O}$. Then (7) has a root z_0 that is not an algebraic integer. By Lemma 2 there exist matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr} A = 0, \qquad \operatorname{tr} B = 2, \qquad \operatorname{tr} AB = z_0.$$

Lemma 4 completes the proof of Theorem 1 in the first case.

Case 2. Let $R(a,b) = ab^{\varepsilon_1}ab^{\varepsilon_2}\cdots ab^{\varepsilon_s}$. It follows from our assumptions that s is even, $\varepsilon_i \in \{-1,1\}$ for $i = 1,\dots,s$, and $\sum_{i=1}^s \varepsilon_i = 0$.

Assume for the moment that $\varepsilon_j = \varepsilon_{j+1}$ for some $j < s$. Let $c = a$ and $d = ab^{\varepsilon_j}$ be new generators of the group G. Then it is not hard to verify that G has a presentation of the following form:

$$G = \langle c, d \mid c^2 = R_1^m(c,d) = 1\rangle,$$

where $R_1(c,d) = cd^{l_1}\cdots cd^{l_t}$ is a cyclically reduced word lying in the subgroup of the free group on c and d generated by c^2 and the derived subgroup and there exists l_i such that $|l_i| \geqslant 2$. This case has just been considered above.

Thus, let $R(a,b) = abab^{-1}\cdots abab^{-1} = (abab^{-1})^l$ for some $l > 0$. Then the group G has a presentation of the following form:

$$G = \langle a, b \mid a^2 = (abab^{-1})^t = 1\rangle,$$

where $t \geqslant 2$. We claim that there exist matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ satisfying the conditions

$$(1)\ \operatorname{tr} A = 0; \quad (2)\ \operatorname{tr} AB^{-2}AB^3 = 2; \quad (3)\ \operatorname{tr} AB \notin \mathcal{O}; \quad (4)\ \operatorname{tr} ABAB^{-1} = c;$$

where $c = \pm 2\cos(r\pi/m)$ for some $r \in \{1,\dots,m-1\}$. Using Fricke characters one can write conditions (2) and (4) as the system

$$\begin{cases} \tau_{ghgh^{-1}}(A,B) = c, \\ \tau_{gh^{-2}gh^3}(A,B) = 2. \end{cases} \tag{8}$$

Using relations (1) for Fricke characters it is easy to obtain

$$\tau_{ghgh^{-1}} = -\tau_{gh}^2 + \tau_g\tau_h\tau_{gh} - \tau_h^2 + 2$$

and

$$\tau_{gh^{-2}gh^3} = (\tau_g\tau_h^2 - \tau_h\tau_{gh} - \tau_g)(\tau_h^2\tau_{gh} - \tau_g\tau_h - \tau_{gh}) - \tau_h^5 + 5\tau_h^3 - 5\tau_h.$$

We set $y = \tau_h(A,B) = \operatorname{tr} B$ and $z = \tau_{gh}(A,B) = \operatorname{tr} AB$. Since $\tau_g(A,B) = \operatorname{tr} A = 0$, it follows that

$$\tau_{ghgh^{-1}}(A,B) = -z^2 - y^2 + 2, \qquad \tau_{gh^{-2}gh^3}(A,B) = -(y^3-y)z^2 - (y^5 - 5y^3 + 5y).$$

Thus, one can write (8) in the following form:

$$\begin{cases} z^2 + y^2 - 2 + c = 0, \\ (y^3-y)z^2 + y^5 - 5y^3 + 5y + 2 = 0. \end{cases} \tag{9}$$

It follows by (9) that

$$y^3 - \Bigl(1 + \frac{1}{c+2}\Bigr)y - \frac{2}{c+2} = 0. \tag{10}$$

By Lemma 5(2) there exists $r \in \{1,\dots,m-1\}$ such that $1/(c+2)$ does not belong to $\mathcal{O}$. In this case equation (10) has a root $y_0 \notin \mathcal{O}$. Let (y_0, z_0) be some solution of (9). By Lemma 2 there exist matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr} A = 0, \qquad \operatorname{tr} B = y_0, \quad \text{and} \quad \operatorname{tr} AB = z_0.$$

Lemma 4 completes the proof of Theorem 2 in the second case.

Case 3. Let $R(a,b) = ab^{n_1}\cdots ab^{n_s}$ and assume that $R(a,b)$ does not belong to the subgroup of the free group on a and b generated by a^2 and the derived subgroup. The case $\sum_{i=1}^{s} n_i \neq 0$ has been considered in [2], Theorem 5. Hence we can assume that s is odd and $\sum_{i=1}^{s} n_i = 0$.

First, we consider the case $m = 2$. Let $G_1 = \langle a, b \mid a^2 = b^2 = R^2(a,b) = 1\rangle$. We claim that $G_1 = \langle a, b \mid a^2 = b^2 = 1\rangle$ is the free product of two cyclic groups of order two. It is sufficient to prove that $R^2(a,b) = 1$ in G. We shall prove a more general fact: if $w(a,b) = ab^{l_1}\cdots ab^{l_s}$, where $s = 2k+1$ and $\sum_{i=1}^{s} l_s$ is even, then $w^2(a,b) = 1$ in G_1. We use induction on s. For $s = 1$ the claim is obvious. For arbitrary s there exists an even number among the exponents $l_1,\dots,l_s$. Assume, for example, that l_1 is even. Then $b^{l_1} = 1$ in G_1 and

$$w(a,b) = b^{l_2}ab^{l_3}\cdots ab^{l_s},$$

that is, $w(a,b)$ is conjugate to

$$w_1(a,b) = ab^{l_3}\cdots ab^{l_2+l_s}.$$

Since the sum $l_3 + \cdots + (l_2 + l_s)$ is even as before, we can use induction and obtain that $w_1^2(a,b) = 1$ in G_1. Hence $w^2(a,b) = 1$ in G_1. Since the group G_1 is an epimorphic image of G, it follows that G is a non-trivial free product with amalgamation.

Next, we consider the case of $m \geqslant 3$. We claim that there exist matrices A, B in $\mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr} A = 0, \qquad \operatorname{tr} B = y_0, \qquad \operatorname{tr} AB = 2, \quad \text{and} \quad Q_R(0, y_0, 2) = 2\cos\frac{\pi}{m},$$

where y_0 is a complex number that is not an algebraic integer and $Q_R(x,y,z)$ is the Fricke polynomial of the word $R(g,h) \in F_2$. We have the following result.

Lemma 7. (1) $Q_R(0,y,z) = zf(y,z)$, *where* $f(y,z) \in \mathbb{Z}[y,z]$.

(2) *The polynomial* $f_1(y) = f(y,2)$ *is not constant.*

Proof. (1) It follows from Lemma 3 that $Q_{gh^n}(0,y,z) = P_{n-1}(y)z$, therefore the assertion is proved for $s = 1$. Assume that $s > 1$. Then, using relations (1) one obtains

$$\tau_{gh^{n_1}\cdots gh^{n_s}} = \tau_{gh^{n_1}}\tau_{gh^{n_2}\cdots gh^{n_s}} - \tau_{gh^{n_3}\cdots gh^{n_s - n_1 + n_2}},$$

and by induction we have

$$Q_R(0,y,z) = P_{n_1-1}(y)f_1(y,z)z - zf_2(y,z) = zf(y,z)$$

for some polynomial $f(y,z) \in \mathbb{Z}[y,z]$.

(2) Consider the $\mathbb{Z}$-algebra T generated by all Fricke characters τ_w, $w \in F_2$. Each automorphism $\sigma \in \mathrm{Aut}(F_2)$ induces an automorphism

$$\sigma' \colon T \to T, \qquad \tau_w \mapsto \tau_{\sigma(w)}.$$

We consider an automorphism $\sigma \in \mathrm{Aut}(F_2)$ such that

$$\sigma(g) = g, \qquad \sigma(h) = gh.$$

Then $\sigma' \in \mathrm{Aut}(T)$ is an automorphism such that

$$\sigma'(x) = x_1 = x, \qquad \sigma'(y) = y_1 = z, \qquad \sigma'(z) = z_1 = xz - y.$$

Furthermore,

$$\sigma'(\tau_{R(g,h)}) = \tau_{\sigma(R(g,h))} = \tau_{R(\sigma(g),\sigma(h))},$$

therefore

$$\sigma'(Q_R(x,y,z)) = Q_R(x_1,y_1,z_1) = Q_R(x,z,xz-y).$$

Thus, $\sigma'(Q_R(0,y,2)) = Q_R(0,2,-y)$. It follows from Proposition 2 that the polynomial $Q_R(0,2,-y) \in \mathbb{Z}[y]$ has degree s because its leading coefficient M_s satisfies the inequality $|M_s| = \prod_{i=1}^{s} |n_i| \neq 0$. Hence $Q_R(0,2,-y)$ is not constant, so that $\sigma'^{-1}(Q_R(0,2,-y)) = Q_R(0,y,2) = 2f_1(y)$ is not constant either, which completes the proof.

We can now complete the proof of Theorem 1. By Lemma 7 the equality $Q_R(0,y_0,2) = 2\cos(\pi/m)$ is equivalent to the relation

$$f_1(y_0) = \cos\frac{\pi}{m}\,. \tag{11}$$

By Lemma 5(3), the number $\cos(\pi/m)$ is not an algebraic integer, therefore there exists a root y_0 of equation (11) that is not an algebraic integer either. By Lemma 2 there exist matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr} A = 0, \qquad \operatorname{tr} B = y_0, \quad \text{and} \quad \operatorname{tr} AB = 2.$$

Using Lemma 4 we complete the proof of Theorem 1.

§ 3. Proof of Theorem 2

Bearing in mind Theorem 1 we can assume that $n \geqslant 3$. Let $R(g,h) \in F_2$ and let Q_R be the Fricke polynomial of the word R. We claim that there exist matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr} A = \alpha, \qquad \operatorname{tr} B = 2, \quad \text{and} \quad \operatorname{tr} AB = z,$$

where $\alpha = 2\cos(\pi/n)$, z is a complex number that is not an algebraic integer and satisfies the equation

$$Q_R(\alpha, 2, z) = c, \tag{12}$$

where $c = \pm 2\cos(r\pi/m)$ for some $r \in \{1,\dots,m-1\}$. By Proposition 2 we can rewrite (12) in the following form:

$$M_s(\alpha,2)z^s + M_{s-1}(\alpha,2)z^{s-1} + \dots + M_0(\alpha,2) - c = 0, \tag{13}$$

where

$$M_s(\alpha,2)=\prod_{i=1}^{s}P_{u_i-1}(\alpha)P_{v_i-1}(2).$$

By Lemma 3, $P_{u_i-1}(\alpha)\neq 0$ and $P_{v_i-1}(2)=v_i$, therefore $M_s(\alpha,2)\neq 0$. We write (13) in the following form:

$$z^s+\frac{M_{s-1}(\alpha,2)}{M_s(\alpha,2)}+\cdots+\frac{M_0(\alpha,2)-c}{M_s(\alpha,2)}=0. \tag{14}$$

We claim that we can choose c such that $(M_0(\alpha,2)-c)/M_s(\alpha,2)\notin\mathcal{O}$. Assume that both quantities

$$\frac{M_0(\alpha,2)+2\cos(r\pi/m)}{M_s(\alpha,2)}\qquad\text{and}\qquad\frac{M_0(\alpha,2)-2\cos(r\pi/m)}{M_s(\alpha,2)}$$

belong to $\mathcal{O}$ for each $r\in\{1,\dots,m-1\}$. Then, for their difference we have

$$\frac{4\cos(r\pi/m)}{M_s(\alpha,2)}\in\mathcal{O}.$$

Since $P_r(\lambda)$ has integer coefficients for each r, it follows that $P_{u_i-1}(\alpha)$ belongs to $\mathcal{O}$. Hence

$$\frac{4\cos(r\pi/m)}{M_s(\alpha,2)}P_{u_i-1}(\alpha)=\frac{4\cos(r\pi/m)}{M}\in\mathcal{O}$$

for each $r\in\{1,\dots,m-1\}$, where $M=\prod_{i=1}^{s}v_i\in\mathbb{Z}$ and $|M|\geqslant 3$ by the assumptions of the theorem. We obtain a contradiction with Lemma 5(4). This means that there exists a root z_0 of the equation (14) that does not belong to $\mathcal{O}$. By Lemma 2 there exist matrices $A,B\in\mathrm{SL}_2(\mathbb{C})$ such that

$$\operatorname{tr}A=\alpha,\qquad \operatorname{tr}B=2,\quad\text{and}\quad \operatorname{tr}AB=z_0.$$

Using Lemma 4 we complete the proof of Theorem 2.

In conclusion, we state the following conjecture.

Conjecture 2. *The group $G=\langle a,b\mid a^n=R^m(a,b)=1\rangle$, where $m\geqslant 2$, $n=0$ or $n\geqslant 2$, and $R(a,b)$ is a cyclically reduced word containing b in the free group on a and b, is a non-trivial free product with amalgamation.*

In view of Theorems 1, 2 and [2], Theorem 5, this conjecture has not yet been proved in the following case: n is odd and $R(a,b)=a^lR_1(a,b)$, where $0\leqslant l<n$ and $R_1(a,b)=a^{u_1}b^{v_1}\cdots a^{u_s}b^{v_s}$ is a cyclically reduced word belonging to the derived subgroup of the free group on a and b such that $\prod_{i=1}^{s}|v_i|\in\{1,2\}$.

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