Character classes in the two generator free group

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Let K be a field of characteristic 0. In this paper we consider the problem of determining when two elements in the free two generator group $F_2 =< g, h >$ have the same characters under all representations of F_2 into $\mathrm{SL}_2(K)$.

Definition 1 *The character class $\{w\}$ of an element $w \in F_2$ is the set of all elements in F_2 having the same characters as w under all representations of F_2 into $\mathrm{SL}_2(K)$.*

Clearly, any representation $\rho : F_2 \to \mathrm{SL}_2(K)$ is uniquely determined by the pair of matrices $(\rho(g), \rho(h)) \in \mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$. Therefore, we may (and we will) identify the points of $\mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$ with the corresponding representations of F_2 into $\mathrm{SL}_2(K)$. (In other words, $\mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$ is the representation variety of F_2 into $\mathrm{SL}_2(K)$ (cf. [3])). For arbitrary element $w = w(g, h) \in F_2$ one can consider the function

$$\tau_w : \mathrm{SL}_2(K) \times \mathrm{SL}_2(K) \to K, \qquad (\tau_w)(A, B) = \mathrm{tr}(w(A, B))$$

It was proved by Horowitz [2] (cf. also [1]) that for any $w \in F_2$

$$\tau_w = Q_w(\tau_g, \tau_h, \tau_{gh}),$$

where $Q_w \in \mathbb{Z}[x, y, z]$ is a uniquely defined polynomial with integral coefficients. The function τ_w and the polynomial Q_w are called the Fricke character and the Fricke polynomial of the element $w \in F_2$, respectively. For arbitrary elements $u, v, w \in F_2$, the following relations for the Fricke characters follow from the corresponding relations for the traces of arbitrary matrices in $\mathrm{SL}_2(K)$:

$$1)\tau_{u^{-1}} = \tau_u; \qquad 2)\tau_{uv} = \tau_{vu}; \qquad 3)\tau_{uvu^{-1}} = \tau_v; \qquad 4)\tau_{uv} = \tau_u\tau_v - \tau_{uv^{-1}}.$$

The Fricke characters τ_g, τ_h, τ_{gh} are algebraically independent functions over K in view of the following well known proposition (cf., for example, [2]).

Proposition 1 *For any $\alpha, \beta, \gamma \in K$ there exist matrices $A, B \in \mathrm{SL}_2(K)$ such that*

$$\tau_g(A, B) = \mathrm{tr}\, A = \alpha, \qquad \tau_h(A, B) = \mathrm{tr}\, B = \beta, \qquad \tau_{gh}(A, B) = \mathrm{tr}\, AB = \gamma.$$

Now using the Fricke characters we can rewrite Definition 1 as follows.

Definition $1'$ *The character class $\{w\}$ of an element $w \in F_2$ is the set*

$$\{w\} = \{u \in F_2 \mid \tau_u = \tau_w\} = \{u \in F_2 \mid Q_u = Q_w\}.$$