# Representation Varieties of the Fundamental Groups of Compact Orientable Surfaces

A.S.Rapinchuk* V.V.Benyash-Krivetz
V.I.Chernousov

Institute of Mathematics
Academy of Sciences of Belarus
ul. Surganova, 11
Minsk, 220072, Belarus

**Abstract**

We show that the representation variety for the surface group in characteristic zero is (absolutely) irreducible and rational over **Q**.

*Visiting the University of Michigan (Ann Arbor, MI 48109, USA) in 1992-94.

# Introduction

Let $\Gamma$ be a finitely generated group. For any algebraic group $G$ the set $\mathbf{R}(\Gamma, G)$ of all representations ( = homomorphisms) $\rho : \Gamma \to G$ is known to have a natural structure of an algebraic variety, and endowed with this structure is called *the variety representations of* $\Gamma$ *in* $G$ (cf. [Lu-M], [Pl-R]). In the case $G = \mathbf{GL}_n$ which is analyzed by the classical representation theory, $\mathbf{R}(\Gamma, \mathbf{GL}_n)$ is denoted simply by $\mathbf{R}_n(\Gamma)$ and called *the variety of* $n$*-dimensional representations* of $\Gamma$. Since $\mathbf{R}_n(\Gamma)$ is defined by the equations arising from the relations for the generators of $\Gamma$, a special role in this theory is played by the one-relator groups

$$\Gamma =< x_1, \ldots, x_n \mid r = 1 > .$$

The methods of this paper allow to consider in full the case : $n \geq 4$, $r = r_1[x_{n-3}, x_{n-2}]$ $[x_{n-1}, x_n]$ where $[x, y] = xyx^{-1}y^{-1}$ is the commutator of $x$ and $y$, and $r_1$ is an arbitrary word in the derived subgroup of the free group $F(x_1, \ldots, x_{n-4})$. The most notable groups of this kind are the fundamental groups $\Gamma_g$ of compact orientable surfaces of genus $g > 1$, and that is why we formulate our results (which remain valid also for $g = 1$) for these groups. So, let $\Gamma_g (g \geq 1)$ be the group with $2g$ generators $x_1, y_1, \ldots, x_g, y_g$ and a single defining relation

$$[x_1, y_1] \ldots [x_g, y_g] = 1.$$

Then a description of $\mathbf{R}_n(\Gamma_g)$ for the ground field of characteristic 0 is given by

**Theorem 1** $\mathbf{R}_n(\Gamma_g)$ *is an (absolutely) irreducible* $\mathbf{Q}$*-rational variety of dimension*

$$\dim \mathbf{R}_n(\Gamma_g) = \begin{cases} (2g-1)n^2 + 1 & \text{if } g > 1, \\ n^2 + n & \text{if } g = 1. \end{cases}$$

Informally speaking, Theorem 1 means that "almost all" $n$-dimensional representations of $\Gamma_g$ can be parametrized by some rational functions thus yielding a nice description of the totality of representations of $\Gamma_g$. However to complete in a sense the representation theory for $\Gamma_g$ one should supplement the latter with a description of the equivalence classes of representations. In geometric terms this amounts to the analysis of the corresponding variety $\mathbf{X}_n(\Gamma_g)$ of $n$*-dimensional characters*. Recall that $\mathbf{X}_n(\Gamma)$ can be defined as a categorical quotient of $\mathbf{R}_n(\Gamma)$ modulo the action of $\mathbf{GL}_n$ by conjugation and that the points of $\mathbf{X}_n(\Gamma)$ are in one-to-one correspondence with the equivalence classes of fully reducible representations of $\Gamma$ (cf.[Lu-M]). (Another realization of $\mathbf{X}_n(\Gamma)$ is given in [Pl].)

**Theorem 2** *The character variety* $\mathbf{X}_n(\Gamma_g)$ *is irreducible and* $\mathbf{Q}$*-unirational, of dimension* $(2g-2)n^2 + 2$ *(resp.,* $2n$ *) for* $g > 1$ *(resp.,* $g = 1$*). Moreover,* $\mathbf{X}_n(\Gamma_g)$ *is* $\mathbf{Q}$*-rational if* $g > 1$ *and* $n \leq 3$.