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PARTIAL DIFFERENTIAL EQUATIONS

Boundary Value Problems for Complete Quasi-Hyperbolic Differential Equations with Variable Domains of Smooth Operator Coefficients: I

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Complete quasi-hyperbolic operator-differential equations of even order with constant domains were considered in [1, 2]. Quasi-hyperbolic operator-differential equations of even order with variable domains in the case of a two-term leading part were analyzed in [3]. Complete hyperbolic operator-differential equations of the second order with variable domains were investigated in [4, 5]. In the present paper, we generalize and improve the results of all above-mentioned papers and consider complete quasi-hyperbolic operator-differential equations of even order with variable domains. In applications, such equations include hyperbolic equations such that the coefficients in the equations and in the boundary conditions [3] smoothly depend on time, singular hyperbolic equations [4], and "hyperbolic" equations of higher-order in the space variables, represented in the second part of the present paper.

1. STATEMENT OF THE PROBLEMS

In a Hilbert space H with inner product $(\cdot\,,\cdot)$ and norm $|\cdot|$, we consider boundary value problems

$$L_m(\lambda_m)\,u \equiv (-1)^{m-1}\frac{d^{2m}u}{dt^{2m}} + \sum_{k=0}^{m-1}\frac{d^k}{dt^k}A_{2k+1}(t)\frac{d^{k+1}u}{dt^{k+1}} + \sum_{k=1}^{m-1}\frac{d^k}{dt^k}A_{2k}(t)\frac{d^k u}{dt^k} + \lambda_m A_0(t)u = f, \qquad (1)$$

$$d^i u/dt^i\big|_{t=0} = d^j u/dt^j\big|_{t=T} = 0, \qquad 0 \le i \le m, \qquad 0 \le j \le m-2, \qquad m = 1,2,\ldots, \qquad (2)$$

on a bounded interval $]0,T[$, where u and f are functions of the variable t ranging in H and $\lambda_m \ge 1$ is a numerical parameter. The linear unbounded closed operators $A_s(t)$ in H with t-dependent domains $D(A_s(t))$, $s = 0,\ldots,2m-1$, are subjected to the following conditions.

I. For all $t \in [0,T]$, the operators $A_0(t)$ are self-adjoint in H and satisfy the inequality

$$(A_0(t)u,u) \ge c_0(t)|u|^2$$

for all $u \in D(A_0(t))$, $c_0(t) > 0$, and their inverses are $A_0^{-1}(t) \in \mathscr{B}([0,T],\mathscr{L}(H))$ (where $\mathscr{B}([0,T],\mathscr{L}(H))$ is the set of linear operators in $\mathscr{L}(H)$, which are bounded with respect to $t \in [0,T]$ and in the norm) and have the strong t-derivative [6, p. 22] $dA_0^{-1}(t)/dt \in \mathscr{B}([0,T],\mathscr{L}(H))$ in H satisfying the inequality

$$-\left(\left(dA_0^{-1}(t)/dt\right)g,g\right) \le c_0^{(1)}\left(A_0^{-1}(t)g,g\right) \qquad \forall g \in H. \qquad (3)$$

To state constraints for $A_s(t)$, $s > 0$, we introduce appropriate spaces. By [6], for $A_0(t)$ with each $t \in [0,T]$, we introduce the fractional powers $A_0^\gamma(t)$, $|\gamma| \le 1$, with domains $D(A_0^\gamma(t))$. If we equip $D\left(A_0^{\alpha/(2m)}(t)\right)$ with the Hermitian norms $|v|_{\alpha,t} = |A_0^{\alpha/(2m)}(t)v|$, then we obtain Hilbert spaces $W^\alpha(t)$, $|\alpha| \le 2m$, $W^0(t) = H$.

II. For each $t \in [0,T]$, the operators $dA_0^{-1}(t)/dt$ have the strong derivatives

$$d^j A_0^{-1}(t)/dt^j \in \mathscr{B}([0,T],\mathscr{L}(H)), \qquad 2 \le j \le m+1,$$

with respect to t in H, which satisfy the inequalities

$$\left|\left(\left(d^j A_0^{-1}(t)/dt^j\right) g, v\right)\right| \le c_0^{(j)} |g|_{-(m+1-j),t} |v|_{-m,t} \qquad \forall g, v \in H. \tag{4}$$

III. For each $t \in [0,T]$, the operators $A_s(t) \in \mathscr{B}\left([0,T], \mathscr{L}\left(W^{2m-s}(t), H\right)\right)$, $s > 0$, satisfy the inequalities

$$|(A_s(t)u, v)| \le c_s |u|_{m-[(s+1)/2],t} |v|_{m-[s/2],t} \qquad \forall u, v \in D\left(A_0(t)\right), \tag{5}$$

where $[\cdot]$ is the integer part of a number, and have the strong t-derivatives [7]

$$d^i A_s(t)/dt^i \in \mathscr{B}\left([0,T], \mathscr{L}\left(W^{2m-s+\tau}(t), H\right)\right), \qquad \tau > 0, \qquad 1 \le i \le [s/2], \qquad s > 0.$$

For each $t \in [0,T]$, the operators $A_{2k}(t)$, $k = 1, \ldots, m-1$, are symmetric on $D\left(A_0(t)\right)$ in H, and the $A_s(t)$, $s > 0$, satisfy the inequalities

$$(-1)^{[(s+1)/2]} \operatorname{Re}\left(\left(d^i A_s(t)/dt^i\right) u, u\right) \le c_s^{(i)} |u|^2_{m-[(s+1)/2],t} \qquad \forall u \in D\left(A_0(t)\right), \tag{6}$$

where $i = 1$ for $s = 2k$, $k = 1, \ldots, m-1$ and $i = 0$ for $s = 2k+1$, $k = 0, \ldots, m-1$.

We use the following notion of strong derivative of variable unbounded operators $A(t)$ [with variable domains $D(A(t))$ in H] with respect to the parameter t [7].

Definition 1. The operators $A(t)$ are said to be *strongly differentiable* with respect to t at $t_0 \in [0,T]$ on $u(t_0) \in D\left(A(t_0)\right)$ if there exist $u(t) \in D(A(t))$ and $t \ne t_0$ such that there exist derivatives

$$u'(t_0) = \lim_{\Delta t \to 0} \left\{\left(u(t_0 + \Delta t) - u(t_0)\right)/\Delta t\right\} \in D\left(A(t_0)\right),$$
$$h'(t_0) = \lim_{\Delta t \to 0} \left\{\left(A(t_0 + \Delta t) u(t_0 + \Delta t) - A(t_0) u(t_0)\right)/\Delta t\right\} \in H$$

in the strong sense in H and the limits $u'(t_0)$ and $h'(t_0)$ are independent of the choice of $u(t)$. The value of the strong derivative $A'(t_0)$ of the operators $A(t)$ on $u(t_0)$ at t_0 is defined as

$$A'(t_0) u(t_0) = h'(t_0) - A(t_0) u'(t_0).$$

The set of all such $u(t_0)$ forms the domain $D\left(A'(t_0)\right) \subset D\left(A(t_0)\right)$ of the derivative operator $A'(t_0)$. The operators $A(t)$ are said to be *strongly differentiable* with respect to t in $[0,T]$ on $D\left(A'(t)\right)$ if they are strongly differentiable with respect to t at each $t_0 \in [0,T]$ and on each $u(t_0) \in D\left(A'(t_0)\right)$.

IV. For each $t \in [0,T]$, all operators $A_s(t)$, $s > 0$, satisfy the inequality

$$\left|\left(A_s(t) A_0^{-1}(t) g, v\right)\right| \le \tilde{c}_s^{(0)} |g|_{-[(s+1)/2],t} |v|_{-[s/2],t} \qquad \forall g, v \in H, \tag{7}$$

the inclusion $A_s(t)\left(d^j A_0^{-1}(t)/dt^j\right) \in \mathscr{B}([0,T], \mathscr{L}(H))$, $1 \le j \le [(s+1)/2]$, the inequalities

$$\begin{aligned} &\left|\left(A_s(t)\left(d^j A_0^{-1}(t)/dt^j\right) g, v\right)\right| \le \tilde{c}_s^{(j)} |g|_{-[(s+1)/2]+j,t} |v|_{-[s/2]-1,t} \\ &\forall g, v \in H, \qquad 2 \le j \le [(s+1)/2], \end{aligned} \tag{8}$$

and either inequality (8) with $j = 1$ or the inequality

$$\left|\left(A_s(t)\left(dA_0^{-1}(t)/dt\right) g, v\right)\right| \le \tilde{c}_s^{(1)} |g|_{-[(s-1)/2],t} |v|_{-[s/2],t} \qquad \forall g, v \in H, \tag{9}$$

inequality (8) with $j = 2$ and with $\tilde{c}_s^{(j)} |g|_{-[(s+1)/2]+1,t} |v|_{-[s/2],t}$ on the right-hand side, the inclusion

$$\left(dA_s(t)/dt\right)\left(dA_0^{-1}(t)/dt\right) \in \mathscr{B}([0,T], \mathscr{L}(H)),$$

the inequality

$$\left|\left(\left(dA_s(t)/dt\right)\left(dA_0^{-1}(t)/dt\right) g, v\right)\right| \le \bar{c}_s^{(1)} |g|_{-[(s-1)/2],t} |v|_{-[s/2],t} \qquad \forall g, v \in H, \tag{10}$$

and the inequality

$$(-1)^k \operatorname{Re}\left(A_{2k}(t)\left(dA_0^{-1}(t)/dt\right)g,g\right) \le \bar{c}_{2k}^{(0)}|g|_{-k,t}^2 \qquad \forall g \in H \tag{11}$$

for $s = 2k > 0$; and if $s = 2k+1 > 0$, then the operators $A_{2k+1}(t)\left(dA_0^{-1}(t)/dt\right)$ are symmetric on $D\left(A_{2k+1}(t)\right)$ in H. For each $t \in [0,T]$, the operators $A_{2k}(t)$, $k = 1,\dots,m-1$, satisfy the inequalities

$$\begin{aligned}(-1)^k \operatorname{Re}\left[(A_{2k}(t)u, A_0(t)v) - (A_0(t)u, A_{2k}(t)v)\right]&\\ \le \tilde{c}_{2k}|u|_{2m-k-1,t}|v|_{2m-k,t} \qquad \forall u,v \in D\left(A_0(t)\right)&\end{aligned} \tag{12}$$

and the operators $A_{2k+1}(t)$, $k = 0,\dots,m-1$, satisfy the inequalities

$$(-1)^{k+1} \operatorname{Re}\left(A_{2k+1}(t)u, A_0(t)u\right) \le \tilde{c}_{2k+1}|u|_{2m-k-1,t}^2 \qquad \forall u \in D\left(A_0(t)\right). \tag{13}$$

In inequalities (3)–(13), all constants $c_0^{(j)}$, c_s, $c_s^{(i)}$, $\tilde{c}_s^{(j)}$, $\bar{c}_s^{(i)}$, $\tilde{c}_s \ge 0$ are independent of g, u, v, and t.

In the present paper, we prove the well-posed solvability of boundary value problems (1), (2) in the strong sense for $\lambda_m \ge \tilde{\lambda}_m$ in the case of higher-order complete equations. The proof is performed by the modification and generalization of the well-known method of energy inequalities in [8], unlike which we first derive *a priori* estimates for strong solutions of problems (1), (2) with the use of smoothing operators $A_\varepsilon^{-1}(t)$ and then, by using Lemma 8 in [9], show that the range of the operators of problems (1), (2) is dense in the space of right-hand sides. Equations (1) contain only leading terms, and the well-posedness of boundary value problems for equations with additional lower terms is to be discussed in Remark 2.

2. INTERPOLATION INEQUALITIES

Throughout the sequel, we need the following auxiliary assertions. The following assertion deals with the continuity of the derivative $dA^{-1}(t)/dt$ in the corresponding pair of Hilbert scales of spaces $\{W^q(t)\}$, $|q| \le 2m$.

Lemma 1. *Let $A(t)$ be linear self-adjoint positive operators in the Hilbert space H with t-dependent domains $D(A(t))$. If their inverses $A^{-1}(t) \in \mathscr{B}([0,T],\mathscr{L}(H))$ have the strong derivative*

$$dA^{-1}(t)/dt \in \mathscr{B}\left([0,T],\mathscr{L}\left(H, W^{2m(1-\beta)}(t)\right)\right) \cap \mathscr{B}\left([0,T],\mathscr{L}\left(W^{-2m}(t), W^{-2m\beta}(t)\right)\right),$$

$0 \le \beta \le 1$, *in H for all t, then*

$$\left|A^{1-\beta-\alpha}(t)\left(dA^{-1}(t)/dt\right)A^\alpha(t)x\right| \le \mathscr{M}|x| \qquad \forall x \in D\left(A^\alpha(t)\right), \qquad 0 \le \alpha \le 1, \tag{14}$$

where

$$\mathscr{M} = \operatorname*{ess\,sup}_{0<t<T}\left\{\left\|A^{1-\beta}(t)\left(dA^{-1}(t)/dt\right)\right\|_{\mathscr{L}(H)}, \left\|\overline{A^{-\beta}(t)\left(dA^{-1}(t)/dt\right)A(t)}\right\|_{\mathscr{L}(H)}\right\},$$

and the bar stands for the closure of operators by continuity in H.

Proof. The operators $\mathscr{A} = \mathscr{B} = A^{\beta-1}(t)$ and $\mathscr{T} = A^{1-\beta}(t)\left(dA^{-1}(t)/dt\right)$ satisfy Remark 7.1 in [6, pp. 177–179] in $H_1 = H$ for all t; in particular, $|\mathscr{B}\mathscr{T}x| \le \mathscr{M}|\mathscr{A}x|$ for all $x \in H$. By applying the Heinz inequality (7.6) in [6, p. 178] to them, we obtain the estimate (14) for all t and all $0 \le \alpha \le 1-\beta$. The operators $\mathscr{A} = \mathscr{B} = A^\beta(t)$ and $\mathscr{T} = A^{-\beta}(t)\left(dA^{-1}(t)/dt\right)$ also satisfy the above-mentioned Remark 7.1; likewise, by applying the Heinz inequality (7.6) from [6, p. 178] to them, we obtain the estimates (14) for all t and all $1-\beta \le \alpha \le 1$. The proof of Lemma 1 is complete.

The following assertion is a generalization of the Daletskii theorem [6, p. 231] to self-adjoint operators with variable domains.

Lemma 2. *Under the assumptions of Lemma* 1, *the operators* $A^{-\gamma}(t)$, $\beta \le \gamma < 1$, *have the strong derivative*

$$\frac{dA^{-\gamma}(t)}{dt} = \frac{\sin \pi\gamma}{\pi} \int\limits_0^{+\infty} s^{-\gamma} A(t) R(-s) \frac{dA^{-1}(t)}{dt} A(t) R(-s) ds, \quad \beta \le \gamma < 1,$$
$$R(-s) = (A(t) + s)^{-1},$$

for each $t \in [0, T]$ *in* H, *and*

$$\left| A^{\gamma-\beta}(t) \left(dA^{-\gamma}(t)/dt \right) x \right| \le \mathscr{M}_\gamma |x| \qquad \forall x \in H, \qquad \beta \le \gamma < 1, \tag{15}$$
$$\left| A^{-\beta}(t) \left(dA^{-\gamma}(t)/dt \right) A^{\gamma}(t) x \right| \le \mathscr{M}_\gamma |x| \qquad \forall x \in D\left(A^{\gamma}(t)\right), \qquad \beta \le \gamma < 1. \tag{16}$$

Proof. The integral representation of the derivative $dA^{-\gamma}(t)/dt$ is obtained by the differentiation of the integral representation of negative fractional powers $A^{-\gamma}(t)$ [6, p. 137] for $\gamma > \beta$ by virtue of the relation

$$dR(-s)/dt = A(t)R(-s) \left(dA^{-1}(t)/dt \right) A(t) R(-s),$$

the estimates $\left\| A^{\beta}(t) R(-s) \right\|_{\mathscr{L}(H)} \le N_\beta/(1+s)^{1-\beta}$, $s > 0$, $0 \le \beta < 1$, and the boundedness of the operators $A^{1-\beta}(t) \left(dA^{-1}(t)/dt \right)$ [and, for $\gamma = \beta$, by virtue of the estimates (15) proved below].

If $Q = A^{\gamma}(t) R(-s) A^{1-\beta}(t) \left(dA^{-1}(t)/dt \right) A(t) R(-s)$ and $x, y \in D(A(t))$, then we have

$$\begin{aligned} |(Qx, y)| &= \left| \left(A^{1-\beta-(1-\gamma)/2}(t) \left(dA^{-1}(t)/dt \right) A^{(1-\gamma)/2}(t) A^{(1+\gamma)/2}(t) R(-s) x, A^{(1+\gamma)/2}(t) R(-s) y \right) \right| \\ &\le \left\| \overline{A^{1-\beta-(1-\gamma)/2}(t) \left(dA^{-1}(t)/dt \right) A^{(1-\gamma)/2}(t)} \right\|_{\mathscr{L}(H)} \left| A^{(1+\gamma)/2}(t) R(-s) x \right| \\ &\quad \times \left| A^{(1+\gamma)/2}(t) R(-s) y \right| \end{aligned}$$

for all t, where the bar stands for the closure of operators by continuity in H. By using inequalities (14) for $\alpha \le 1 - \beta$ and the spectral expansion of the operators $A(t)$ and by following [6, p. 230], we obtain the estimates (15) with constants $\mathscr{M}_\gamma = \mathscr{M} c_\gamma$, where $c_\gamma = (1/3) \int_0^{+\infty} \sigma^{-\gamma} (1+\sigma)^{-2} d\sigma$.

If $Q = A(t) R(-s) A^{-\beta}(t) \left(dA^{-1}(t)/dt \right) A^{1+\gamma}(t) R(-s)$ and $x, y \in D(A(t))$, then

$$\begin{aligned} |(Qx, y)| &= \left| \left(A^{-\beta+(1-\gamma)/2}(t) \left(dA^{-1}(t)/dt \right) A^{(1+\gamma)/2}(t) A^{(1+\gamma)/2}(t) R(-s) x, A^{(1+\gamma)/2}(t) R(-s) y \right) \right| \\ &\le \left\| \overline{A^{-\beta+(1-\gamma)/2}(t) \left(dA^{-1}(t)/dt \right) A^{(1+\gamma)/2}(t)} \right\|_{\mathscr{L}(H)} \left| A^{(1+\gamma)/2}(t) R(-s) x \right| \\ &\quad \times \left| A^{(1+\gamma)/2}(t) R(-s) y \right| \end{aligned}$$

for all t, where the bar stands for the closure of operators by continuity in H. Here we use inequality (14) for $0 \le \alpha \le 1$ and the spectral expansion of the operators $A(t)$; following [6, p. 230], we obtain the estimate (16). The proof of Lemma 2 is complete.

The derivative $dA^{-\gamma}(t)/dt$ is also continuous in an appropriatepair of Hilbert space scales $\{W^q(t)\}$, $|q| \le 2m$.

Lemma 3. *Under the assertions of Lemma* 1, *the estimates*

$$\begin{gathered} \left| A^{\gamma-\beta-\alpha}(t) \left(dA^{-\gamma}(t)/dt \right) A^{\alpha}(t) x \right| \le \mathscr{M}_\gamma |x| \\ \forall x \in D\left(A^{\alpha}(t)\right), \qquad \beta \le \gamma < 1, \qquad 0 \le \alpha \le \gamma, \end{gathered} \tag{17}$$

are valid for each $t \in [0, T]$.

Proof. By the estimates (15) and (16), it suffices to apply Lemma 1 to the operators $A^{\gamma}(t)$ instead of the operators $A(t)$. The proof of Lemma 3 is complete.

We need interpolation inequalities in the negative Hilbert space scale $\{\mathscr{H}^q\}$, $-m \le q \le 0$, where $\mathscr{H}^q = L_2(]0, T[, W^q(t))$, with Hermitian norms $\|\cdot\|_q$. The space $\mathscr{H}^q$ is the set of all functions $u : [0, T] \ni t \to u(t) \in H$ for which $u(t) \in D\left(A^{q/(2m)}(t)\right)$, $t \in [0, T]$, and the functions $h(t) = A^{q/(2m)}(t) u(t) \in \mathscr{H}^0 = \mathscr{H} = L_2(]0, T[, H)$.

Lemma 4. *Under the assumptions of Lemma* 1, *the inequalities*

$$\left\|\frac{d^i w}{dt^i}\right\|_{-i}^2 \le \tau \left\|\frac{d^m w}{dt^m}\right\|_{-m}^2 + c_{2m}^{(i)}(\tau)\|w\|_0^2, \qquad \tau > 0, \qquad 0 < i < m, \tag{18}$$

$$\int_0^T (T-t)\left|\frac{d^i w}{dt^i}\right|_{-i,t}^2 dt \le \tau \int_0^T (T-t)\left|\frac{d^m w}{dt^m}\right|_{-m,t}^2 dt + \tilde{c}_{2m}^{(i)}(\tau)\int_0^T (T-t)|w|^2 dt, \tag{19}$$
$$\tau > 0, \qquad 0 < i < m,$$

are valid for $\beta = 1/(2m)$ *and for all* $w \in \mathscr{W}^m$, $\mathscr{W}^m = \Big\{w \in \mathscr{H} : d^k w/dt^k \in \mathscr{H},\ 1 \le k \le m;$ $(d^i w/dt^i)|_{t=0} = (d^j w/dt^j)|_{t=T} = 0,\ 0 \le i \le m-1,\ 0 \le j \le m-2\Big\}$, *where* $c_{2m}^{(i)}(\tau), \tilde{c}_{2m}^{(i)}(\tau) > 0$ *are constants independent of* w *and* t.

Proof. The estimates (18) and (19) can be proved by induction over i with the use of integration by parts, the estimates (17), and the δ-inequality $2ab \le \delta a^2 + \delta^{-1}b^2$ for all $\delta > 0$.

In the derivation of *a priori* estimates for solutions of the boundary value problems (1), (2), we need interpolation inequalities in the positive Hilbert space scale $\{\mathscr{H}^q\}$, $0 \le q \le m$, induced by self-adjoint operators with variable domains.

Lemma 5. *Under the assumptions of Lemma* 1, *the inequalities*

$$\left\|\frac{d^i u}{dt^i}\right\|_{m-i}^2 \le \tau \left\|\frac{d^m u}{dt^m}\right\|_0^2 + c_{2m+1}^{(i)}(\tau)\|u\|_m^2, \qquad \tau > 0, \qquad 0 < i < m, \tag{20}$$

$$\int_0^T (T-t)\left|\frac{d^i u}{dt^i}\right|_{m-i,t}^2 dt \le \tau \int_0^T (T-t)\left|\frac{d^m u}{dt^m}\right|^2 dt + \tilde{c}_{2m+1}^{(i)}(\tau)\int_0^T (T-t)|u|_{m,t}^2 dt, \tag{21}$$
$$\tau > 0, \qquad 0 < i < m,$$

are valid for $\beta = 1/(2m)$ *and for all* $u \in E^m$ (*the spaces* E^m *are to be defined below, at the beginning of Section* 3), *where the* $c_{2m+1}^{(i)}(\tau), \tilde{c}_{2m+1}^{(i)}(\tau) > 0$ *are constants independent of* u *and* t.

Proof. The operators $\mathscr{A}_\varepsilon(t) = A(t)(I + \varepsilon A(t))^{-1}$ with domains $D(\mathscr{A}_\varepsilon(t)) = H$ are bounded, self-adjoint, and positive in H for all $\varepsilon > 0$. One can readily show that

$$\begin{aligned} &\left\|\mathscr{A}_\varepsilon^{-\beta}(t)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-1}(t)\right\|_{\mathscr{L}(H)} \le \mathscr{M}, \\ &\left\|\mathscr{A}_\varepsilon^{-1}(t)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-\beta}(t)\right\|_{\mathscr{L}(H)} \le \mathscr{M} \end{aligned} \tag{22}$$

for all t.

By applying the Heinz inequality (7.6) in [6, p. 178] to the operators $\mathscr{A} = \mathscr{B} = \mathscr{A}_\varepsilon^{\beta-1}(t)$ and $\mathscr{T} = \mathscr{A}_\varepsilon^{-\beta}(t)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-1}(t)$, we obtain

$$\left|\mathscr{A}_\varepsilon^{-\beta-\alpha}(t)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-1+\alpha}(t)x\right| \le \mathscr{M}|x| \qquad \forall x \in H, \qquad 0 \le \alpha \le 1-\beta, \tag{23}$$

for all t.

By integrating the integral representation of positive fractional powers $\mathscr{A}_\varepsilon^{\gamma}(t)$ [6, p. 140], we obtain

$$\frac{d\mathscr{A}_\varepsilon^{\gamma}(t)}{dt}x = \frac{\sin \pi\gamma}{\pi}\int_0^{+\infty} s^{\gamma} R_\varepsilon(-s)\frac{d\mathscr{A}_\varepsilon(t)}{dt}R_\varepsilon(-s)x\,ds$$
$$\forall x \in D(A(t)), \qquad 0 < \gamma < 1-\beta,$$

for all t, where $R_\varepsilon(-s) = (\mathscr{A}_\varepsilon(t) + s)^{-1}$. If $Q = \mathscr{A}_\varepsilon^{-\beta}(t) R_\varepsilon(-s)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-\gamma}(t) R_\varepsilon(-s)$ and $x, y \in D(A(t))$, then

$$\begin{aligned}|(Qx,y)| &= \Big|\Big(\mathscr{A}_\varepsilon^{-\beta-(1-\gamma)/2}(t)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-1+(1-\gamma)/2}(t)\mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)\\ &\qquad \times R_\varepsilon(-s)x, \mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)y\Big)\Big|\\ &\le \left\|\mathscr{A}_\varepsilon^{-\beta-(1-\gamma)/2}(t)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-1+(1-\gamma)/2}(t)\right\|_{\mathscr{L}(H)}\\ &\qquad \times \left|\mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)x\right|\left|\mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)y\right|\end{aligned}$$

for all t. By virtue of (23) and the spectral expansions of the operators $\mathscr{A}_\varepsilon(t)$, we obtain

$$\left|\mathscr{A}_\varepsilon^{-\beta}(t)\,(d\mathscr{A}_\varepsilon^{\gamma}(t)/dt)\,\mathscr{A}_\varepsilon^{-\gamma}(t)x\right| \le \mathscr{M}_{-\gamma}|x| \qquad \forall x \in H, \qquad \beta \le \gamma \le 1-\beta, \tag{24}$$

for all t. If $Q = \mathscr{A}_\varepsilon^{-\gamma}(t)R_\varepsilon(-s)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-\beta}(t)R_\varepsilon(-s)$ and $x, y \in D(A(t))$, then

$$\begin{aligned}|(Qx,y)| &= \Big|\Big(\mathscr{A}_\varepsilon^{-1+(1-\gamma)/2}(t)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-\beta-(1-\gamma)/2}(t)\mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)\\ &\qquad \times R_\varepsilon(-s)x, \mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)y\Big)\Big|\\ &\le \left\|\mathscr{A}_\varepsilon^{-1+(1-\gamma)/2}(t)\,(d\mathscr{A}_\varepsilon(t)/dt)\,\mathscr{A}_\varepsilon^{-\beta-(1-\gamma)/2}(t)\right\|_{\mathscr{L}(H)}\\ &\qquad \times \left|\mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)x\right|\left|\mathscr{A}_\varepsilon^{(1-\gamma)/2}(t)R_\varepsilon(-s)y\right|\end{aligned}$$

for all t.

By inequalities (23) and the spectral expansions of the operators $\mathscr{A}_\varepsilon(t)$, we have

$$\left|\mathscr{A}_\varepsilon^{-\gamma}(t)\,(d\mathscr{A}_\varepsilon^{\gamma}(t)/dt)\,\mathscr{A}_\varepsilon^{-\beta}(t)x\right| \le \mathscr{M}_{-\gamma}|x| \qquad \forall x \in H, \qquad \beta \le \gamma \le 1-\beta, \tag{25}$$

for all t. By (24) and (25), the operators $\mathscr{A}_\varepsilon^{\gamma}(t)$ satisfy inequalities (22) with constants $\mathscr{M}_{-\gamma}$ instead of $\mathscr{M}_\gamma$, and inequalities (23) with $\mathscr{A}_\varepsilon^{\gamma}(t)$ instead of $\mathscr{A}_\varepsilon(t)$ are valid for all t:

$$\begin{gathered}\left|\mathscr{A}_\varepsilon^{-\beta-\alpha}(t)\,(d\mathscr{A}_\varepsilon^{\gamma}(t)/dt)\,\mathscr{A}_\varepsilon^{-\gamma+\alpha}(t)x\right| \le \mathscr{M}_{-\gamma}|x|\\ \forall x \in H, \qquad \beta \le \gamma \le 1-\beta, \qquad 0 \le \alpha \le \gamma - \beta.\end{gathered} \tag{26}$$

The operators $\mathscr{A}_\varepsilon(t)$ satisfy half of the assumptions of Lemma 1; i.e.,

$$\left\|\mathscr{A}_\varepsilon^{1-\beta}(t)\left(d\mathscr{A}_\varepsilon^{-1}(t)/dt\right)\right\|_{\mathscr{L}(H)} \le \mathscr{M}$$

for all t. Consequently, by Lemmas 2 and 3,

$$\frac{d\mathscr{A}_\varepsilon^{-\gamma}(t)}{dt} = \frac{\sin \pi\gamma}{\pi}\int_0^{+\infty} s^{-\gamma}\mathscr{A}_\varepsilon(t)R_\varepsilon(-s)\frac{dA^{-1}(t)}{dt}\mathscr{A}_\varepsilon(t)R_\varepsilon(-s)ds, \qquad \beta \le \gamma < 1,$$

and $\mathscr{A}_\varepsilon(t)$ satisfies inequalities (15) and the corresponding part of inequalities (17):

$$\begin{gathered}\left|\mathscr{A}_\varepsilon^{\gamma-\beta-\alpha}(t)\left(d\mathscr{A}_\varepsilon^{-\gamma}(t)/dt\right)\mathscr{A}_\varepsilon^{\alpha}(t)x\right| \le \mathscr{M}_\gamma|x|\\ \forall x \in H, \qquad \beta \le \gamma < 1, \qquad 0 \le \alpha \le \gamma - \beta.\end{gathered} \tag{27}$$

By using integration by parts, the estimates (26) and (27), the Schwarz inequality, and the δ-inequality, by induction on i we obtain

$$\left\|\mathscr{A}_\varepsilon^{(m-i)/(2m)}(t)\frac{d^i u}{dt^i}\right\|_0^2 \le \tau\left\|\frac{d^m u}{dt^m}\right\|_0^2 + c_{2m+1}^{(i)}(\tau)\left\|\mathscr{A}_\varepsilon^{1/2}(t)u\right\|_0^2, \qquad \tau > 0, \qquad 0 < i < m,$$

for all $u \in D(L_m)$, where the $c^{(i)}_{2m+1}(\tau) > 0$ are constants independent of u, ε, and t. By letting ε tend to zero and by using the well-known property (29) in [9], we obtain inequalities (17), which can be generalized by the passage to the limit from the sets $D(L_m)$ (see the beginning of Section 3 below) to E^m.

Inequalities (21) can be proved in a similar way. The proof of Lemma 5 is complete.

3. UNIQUENESS OF STRONG SOLUTIONS

We first introduce function spaces and define strong solutions. As spaces of strong solutions of the boundary value problems (1), (2), we take the Hilbert spaces E^m that are the completions of the sets

$$D(L_m) = \Bigg\{ u \in \mathscr{H} : \ u(t) \in D(A_0(t)), \ t \in \,]0,T[\,; \\ \frac{d^{2m}u}{dt^{2m}}, \frac{d^s u}{dt^s}, \frac{d^{[s/2]}}{dt^{[s/2]}} A_s(t) \frac{d^{[(s+1)/2]}u}{dt^{[(s+1)/2]}} \in \mathscr{H}, \ s = 0, \dots, 2m-1; \\ \frac{d^k u}{dt^k} \in \mathscr{H}^{m-k}, \ k = 1, \dots, m-1; \\ \left.\frac{d^i u}{dt^i}\right|_{t=0} = \left.\frac{d^j u}{dt^j}\right|_{t=T} = 0, \ i = 0, \dots, m, \ j = 0, \dots, m-2 \Bigg\}$$

in the Hermitian norms $|||u|||_m = \left(\|d^m u/dt^m\|_0^2 + \|u\|_m^2\right)^{1/2}$. As spaces of right-hand sides of Eq. (1), we take the Banach spaces $\hat{F}^{-(m-1)}$ that are the completions of the set $\mathscr{H}$ in the norms

$$\langle\|f\|\rangle_{-(m-1)} = \sup_{v \in \hat{E}^{m-1}} \left\{ \left| \int_0^T (f, v)dt \right| / \langle\|v\|\rangle_{m-1} \right\},$$

where the Hilbert spaces $\hat{E}^{m-1}$ are the completions of the sets

$$\hat{\mathscr{D}}^m = \Bigg\{ v \in \mathscr{H} : \ d^k v/dt^k \in \mathscr{H}^{m-k}, \ 0 \le k \le m, \\ \left(d^i v/dt^i\right)\big|_{t=0} = \left(d^i v/dt^i\right)\big|_{t=T} = 0, \ 0 \le i \le m-1 \Bigg\}$$

in the Hermitian norms $\langle\|v\|\rangle_{m-1} = \left(\sum_{k=0}^{m-1} \left\|(T-t)^{-1} d^k v/dt^k\right\|^2_{m-1-k}\right)^{1/2}$. The boundary value problems (1), (2) correspond to linear unbounded operators $L_m(\lambda_m)$: $E^m \supset D(L_m) \to \hat{F}^{-(m-1)}$ with dense domains $D(L_m)$. Throughout the following, we assume that the sets $\hat{\mathscr{D}}^m$ are dense in $\mathscr{H}$, and here we restrict our considerations to one of sufficient conditions under which this assumption is satisfied.

Lemma 6. *If the inverse operators $A_0^{-1}(t) \in \mathscr{B}([0,T], \mathscr{L}(H))$ of positive self-adjoint operators $A_0(t)$ have the strong derivatives $d^j A_0^{-1}(t)/dt^j \in \mathscr{B}([0,T], \mathscr{L}(H, W^{m-j}(t)))$, $1 \le j \le m$, in H for all t, then the sets $\hat{\mathscr{D}}^m$ are dense in $\mathscr{H}$.*

The proof is similar to that of Lemma 1 in [9].

Remark 1. The density of $\hat{\mathscr{D}}^m$ in $\mathscr{H}$ is necessary for the construction of a meaningful dual pair $\hat{E}^{m-1} \subset \mathscr{H} \subset \hat{F}^{-(m-1)}$ in the sense of the representation of values of functionals from $\hat{F}^{-(m-1)}$ via the inner product in $\mathscr{H}$. In applications to boundary value problems, the density of $\hat{\mathscr{D}}^m$ in $\mathscr{H}$ almost always takes place without any additional smoothness requirement for $A_0^{-1}(t)$, since, as a rule, the set of all infinitely differentiable compactly supported functions lies in $\hat{\mathscr{D}}^m$ and is obviously dense in $\mathscr{H}$.

Throughout the following, we assume that the operators $L_m(\lambda_m)$ can be closed; i.e., it follows from the fact that $u_n \to 0$ in E^m and $L_m(\lambda_m)\,u_n \to f$ in $\hat{F}^{-(m-1)}$ as $n \to \infty$ for $u_n \in D(L_m)$ that $f = 0$, and here we restrict our considerations to one of sufficient conditions for them to be closed.

Lemma 7. *Let the assumptions of Lemma* 5 *with* $m > 1$ *and Lemma* 6 *be satisfied, let the operators* $A_s(t)$ *satisfy inequalities* (5), *and let*

$$d^j A_0^{-1}(t)/dt^j \in \mathscr{B}\left([0,T], \mathscr{L}\left(H, W^{2m-j}(t)\right)\right), \qquad 1 \le j \le m-1.$$

Then each operator $L_m(\lambda_m)$ *is closable.*

Proof. After integration by parts, we find that, by virtue of (5) and (20), the values of the antilinear continuous functional $f \in \left(\hat{E}^{m-1}\right)'$ on

$$v \in \mathscr{D}^m = \left\{\hat{v} \in \hat{\mathscr{D}}^m : \ \hat{v}(t) \in D\left(A_0(t)\right), \ t \in [0,T]; \ A_0(t)\hat{v} \in \mathscr{H}\right\}$$

are

$$\begin{aligned} f(v) &= \lim_{n\to\infty} \int\limits_0^T (L_m(\lambda)u_n, v)\,dt \\ &= \lim_{n\to\infty} \left\{ -\int\limits_0^T \left(\frac{d^m u}{dt^m}, \frac{d^m v}{dt^m}\right) dt + \sum_{k=0}^{m-1} \int\limits_0^T \left(A_{2k+1}(t)\frac{d^{k+1}u_n}{dt^{k+1}}, \frac{d^k v}{dt^k}\right) dt \right. \\ &\quad \left. + \sum_{k=1}^{m-1} \int\limits_0^T \left(A_{2k}(t)\frac{d^k u_n}{dt^k}, \frac{d^k v}{dt^k}\right) dt + \lambda_m \int\limits_0^T (u_n, A_0(t)v)\,dt \right\} = 0, \qquad u_n \in D\left(L_m\right). \end{aligned}$$

Let us show that $\mathscr{D}^m$ is dense in $\hat{E}^{m-1}$. Let

$$\sum_{k=0}^{m-1} \int\limits_0^T (T-t)^{-2} \left(A_0^{(m-1-k)/(2m)}(t)\frac{d^k v}{dt^k}, A_0^{(m-1-k)/(2m)}(t)\frac{d^k w}{dt^k}\right) dt = 0 \qquad \forall v \in \mathscr{D}^m$$

for some function $w \in \hat{E}^{m-1}$. Here we set $v = A_\varepsilon^{-1}(t)h = \left(I + \varepsilon A_0(t)\right)^{-1} h \in \mathscr{D}^m$, $\varepsilon > 0$, where $d^k h/dt^k \in \mathscr{H}$, $0 \le k \le m$, and $\left.(d^i h/dt^i)\right|_{t=0} = \left.(d^i h/dt^i)\right|_{t=T} = 0$, $0 \le i \le m-1$, generalize the resulting relation by the passage to the limit on all $h \in \mathscr{H}$ such that $(T-t)^{-1} d^k h/dt^k \in \mathscr{H}$, $0 \le k \le m-1$, and $\left.(d^i h/dt^i)\right|_{t=0} = \left.(d^i h/dt^i)\right|_{t=T} = 0$, $0 \le i \le m-2$, set $h = w$, and obtain the relations

$$\begin{aligned} &\sum_{k=0}^{m-1} \int\limits_0^T (T-t)^{-2} \left(A_\varepsilon^{-1}(t)A_0^{(m-1-k)/(2m)}(t)\frac{d^k w}{dt^k}, A_0^{(m-1-k)/(2m)}(t)\frac{d^k w}{dt^k}\right) dt \\ &\quad = -\sum_{k=1}^{m-1}\sum_{j=1}^{k} C_k^j \int\limits_0^T (T-t)^{-2} \left(A_0^{(m-1-k)/(2m)}(t)\frac{d^j A_\varepsilon^{-1}(t)}{dt^j}\frac{d^{k-j}w}{dt^{k-j}}, A_0^{(m-1-k)/(2m)}(t)\frac{d^k w}{dt^k}\right) dt. \end{aligned}$$

Here and throughout the following, C_p^j is the binomial coefficient. In these relations, the passage to the limit as $\varepsilon \to 0$ in view of (29) implies that $\langle\|w\|\rangle_{m-1}^2 = 0$, i.e., $w = 0$. Since $\mathscr{D}^m$ is dense in $\hat{E}^{m-1}$, we have $f = 0$. The proof of Lemma 7 is complete.

Then we construct the closures $\overline{L}_m(\lambda_m) : E^m \supset D(\overline{L}_m) \to \hat{F}^{-(m-1)}$ of the operators $L_m(\lambda_m)$. The domains $D(\overline{L}_m)$ of the operators $\overline{L}_m(\lambda_m)$ contain all functions $u \in E^m$ for each of which there exist a sequence $u_n \in D(L_m)$ and a functional $f \in \hat{F}^{-(m-1)}$ such that $|||u_n - u|||_m \to 0$ and $\langle \| L_m(\lambda_m) u_n - f \| \rangle_{-(m-1)} \to 0$ as $n \to \infty$. In addition, we assume that

$$\overline{L}_m(\lambda_m) u = \lim_{n\to\infty} L_m(\lambda_m) u_n = f.$$

Definition 2. The solutions of the operator equation $\overline{L}_m(\lambda_m) u = f$, $f \in \hat{F}^{-(m-1)}$, $m = 1, 2, \ldots$, are referred to as *strong solutions* of the boundary value problems (1), (2).

Let us derive *a priori* estimates that imply the uniqueness and stability of solutions.

Theorem 1. *If conditions* I *and* III *are satisfied,*

$$dA_0^{-1}(t)/dt \in \mathscr{B}\left([0,T], \mathscr{L}\left(H, W^{2m-1}(t)\right)\right) \cap \mathscr{B}\left([0,T], \mathscr{L}\left(W^{-2m}(t), W^{-1}(t)\right)\right)$$

for $m > 1$, $\hat{\mathscr{D}}^m$ *are dense in* $\mathscr{H}$, *and the operators* $L_m(\lambda_m)$ *admit the closures* $\overline{L}_m(\lambda_m)$, *then there exist constants* $c_0(m) > 0$ (*independent of* u) *and sets* $\hat{\Lambda}_1 = [1, +\infty[$ *for* $m = 1$ *and* $\hat{\Lambda}_m = [\hat{\lambda}_m, +\infty[$ *for* $m > 1$ *such that*

$$|||u|||_m \le c_0(m) \left\langle \left\| \overline{L}_m(\lambda_m) u \right\| \right\rangle_{-(m-1)} \qquad \forall u \in D(\overline{L}_m), \qquad \forall \lambda_m \in \hat{\Lambda}_m, \qquad m = 1, 2, \ldots \tag{28}$$

Proof. In the space H, we consider the smoothing operators $A_\varepsilon^{-1}(t) = (I + \varepsilon A_0(t))^{-1}$, $\varepsilon > 0$, with values in $D(A_0(t))$. They have the following properties [9]:

(1) the norms $\|A_\varepsilon^{-\alpha}(t)\|_{\mathscr{L}(H)} \le 1$, $0 \le \alpha \le 1$, are bounded uniformly with respect to ε and t, and

$$\left\| A_\varepsilon^{-\alpha}(t) v - v \right\|_0 \to 0 \qquad \forall v \in \mathscr{H}, \qquad 0 \le \alpha \le 1, \tag{29}$$

as $\varepsilon \to 0$;

(2) the operators $A_\varepsilon^{-1}(t)$ have the strong derivative $dA_\varepsilon^{-1}(t)/dt \in \mathscr{B}([0,T], \mathscr{L}(H))$ in H.

By integrating only the term containing $A_0(t)$ by parts once in $L_m(\lambda_m)$, we obtain

$$\begin{aligned}
&2\operatorname{Re}\int_0^T e^{c(T-t)}(-1)^{m-1}\left(\frac{d^{2m}u}{dt^{2m}}, A_\varepsilon^{-1}(t)J(t)u\right)dt \\
&+ 2\operatorname{Re}\sum_{k=1}^{m-1}\int_0^T e^{c(T-t)}\left(\left[\frac{d^k}{dt^k}A_{2k+1}(t)\frac{d}{dt} + \frac{d^k}{dt^k}A_{2k}(t)\right]\frac{d^k u}{dt^k}, A_\varepsilon^{-1}(t)J(t)u\right)dt \\
&+ 2\operatorname{Re}\int_0^T e^{c(T-t)}\left(A_1(t)\frac{du}{dt}, A_\varepsilon^{-1}(t)J(t)u\right)dt \\
&+ (2m-1)\lambda_m\int_0^T e^{c(T-t)}\left(A_0(t)u, A_\varepsilon^{-1}(t)u\right)dt \\
= {}& 2\operatorname{Re}\int_0^T e^{c(T-t)}\left(L_m(\lambda)u, A_\varepsilon^{-1}(t)J(t)u\right)dt \\
&+ \lambda_m\int_0^T e^{c(T-t)}(T-t)\Phi_\varepsilon(u,u)dt \qquad \forall u \in D(L_m), \qquad m = 1, 2, \ldots,
\end{aligned} \tag{30}$$

where $J(t)=(T-t)(d/dt)+(m-1)$ and $\Phi_\varepsilon(u,u)=((d\,(A_0(t)A_\varepsilon^{-1}(t))\,/dt)\,u,u)-c\,(A_0(t)A_\varepsilon^{-1}(t)u,u)$. In the form $\Phi_\varepsilon(u,u)$, we use the formula

$$d\left(A_0(t)A_\varepsilon^{-1}(t)\right)/dt=-A_0(t)A_\varepsilon^{-1}(t)\left(dA_0^{-1}(t)/dt\right)A_0(t)A_\varepsilon^{-1}(t)$$

in [9], inequalities (3), and the first property of the operators $A_\varepsilon^{-1}(t)$ and obtain

$$\Phi_\varepsilon(u,u)\le\left(c_0^{(1)}-c\right)\left|A_0^{1/2}(t)A_\varepsilon^{-1/2}(t)u\right|^2. \tag{31}$$

If, in (30), we use the estimate (31) and let ε tend to zero in the resulting inequality with regard to property (29), then we obtain the inequalities

$$\begin{aligned}&2\operatorname{Re}\int_0^T e^{c(T-t)}(-1)^{m-1}\left(\frac{d^{2m}u}{dt^{2m}},J(t)u\right)dt+2\operatorname{Re}\int_0^T e^{c(T-t)}\left(A_1(t)\frac{du}{dt},J(t)u\right)dt\\&+2\operatorname{Re}\sum_{k=1}^{m-1}\int_0^T e^{c(T-t)}\left(\left[\frac{d^k}{dt^k}A_{2k+1}(t)\frac{d}{dt}+\frac{d^k}{dt^k}A_{2k}(t)\right]\frac{d^ku}{dt^k},J(t)u\right)dt\\&+(2m-1)\lambda_m\int_0^T e^{c(T-t)}\left(A_0(t)u,u\right)dt\le 2\operatorname{Re}\int_0^T e^{c(T-t)}\left(L_m\left(\lambda_m\right)u,J(t)u\right)dt\end{aligned}$$

for all $c\ge c_0^{(1)}$. Hence, by integrating by parts m times in the first integral and k times in the third integral and by using the symmetry of the operators $A_{2k}(t)$, $k=1,\dots,m-1$, we obtain the inequalities

$$\begin{aligned}&\int_0^T e^{c(T-t)}\left[\left|\frac{d^mu}{dt^m}\right|^2+(A_0(t)u,u)\right]dt\\&\le 2\operatorname{Re}\int_0^T e^{c(T-t)}\left(L_m(\lambda)u,J(t)u\right)dt\\&+2\operatorname{Re}\sum_{i=0}^{m-3}C_{m-1}^i\int_0^T\left(\frac{d^mu}{dt^m},\frac{d}{dt}\left[\frac{d^{m-1-i}e^{c(T-t)}}{dt^{m-1-i}}\frac{d^iJ(t)u}{dt^i}\right]\right)dt\\&+(2m-2)\operatorname{Re}\int_0^T\frac{d^2e^{c(T-t)}}{dt^2}\left(\frac{d^mu}{dt^m},\frac{d^{m-2}J(t)u}{dt^{m-2}}\right)dt\\&-\sum_{k=1}^{m-1}(-1)^k\left\{2\operatorname{Re}\sum_{i=0}^{k-1}C_k^i\int_0^T\frac{d^{k-i}e^{c(T-t)}}{dt^{k-i}}\left(\left[A_{2k+1}(t)\frac{d}{dt}+A_{2k}(t)\right]\frac{d^ku}{dt^k},\frac{d^iJ(t)u}{dt^i}\right)dt\right.\\&+(2m-2-2k)\operatorname{Re}\int_0^T e^{c(T-t)}\left(\left[A_{2k+1}(t)\frac{d}{dt}+A_{2k}(t)\right]\frac{d^ku}{dt^k},\frac{d^ku}{dt^k}\right)dt\\&\left.-\int_0^T\frac{de^{c(T-t)}(T-t)}{dt}\left(A_{2k}(t)\frac{d^ku}{dt^k},\frac{d^ku}{dt^k}\right)dt-\int_0^T e^{c(T-t)}(T-t)\left(\frac{dA_{2k}(t)}{dt}\frac{d^ku}{dt^k},\frac{d^ku}{dt^k}\right)dt\right\}\\&-\sum_{k=0}^{m-1}(-1)^k\times 2\operatorname{Re}\int_0^T e^{c(T-t)}(T-t)\left(A_{2k+1}(t)\frac{d^{k+1}u}{dt^{k+1}},\frac{d^{k+1}u}{dt^{k+1}}\right)dt\end{aligned}$$

$$- (2m-2)\,\mathrm{Re}\int\limits_0^T e^{c(T-t)}\left(A_1(t)\frac{du}{dt},u\right)dt - (2m-1)c\int\limits_0^T e^{c(T-t)}(T-t)\left|\frac{d^m u}{dt^m}\right|^2 dt$$

$$+ [1-(2m-1)\lambda_m]\int\limits_0^T e^{c(T-t)}\left(A_0(t)u,u\right)dt. \qquad (32)$$

If $m = 1$, then the right-hand side of inequality (32) without the first integral can be estimated above in view of inequalities (6) for $s = 1$ and $i = 0$ applied to the eighth integral on the right-hand side by the quantity

$$(2c_1 - c)\int\limits_0^T e^{c(T-t)}(T-t)\left|\frac{du}{dt}\right|^2 dt + (1-\lambda_1)\int\limits_0^T e^{c(T-t)}\left(A_0(t)u,u\right)dt,$$

which is nonpositive for all $c \ge c_2 = \max\left\{c_0^{(1)}, 2c_1\right\}$ and all $\lambda_1 \ge 1$.

If $m > 1$, then the right-hand side of inequality (32) without the first integral can be estimated above in view of inequality (5) applied to the fourth, fifth, sixth, and ninth integrals and inequality (6) applied to the seventh and eighth integrals of the right-hand side by the quantities

$$\begin{gathered}
\sum_{i=0}^{m-1} c_{2m+2}^{(i)}\int\limits_0^T (T-t)\left|\frac{d^m u}{dt^m}\right|\,\left|\frac{d^i u}{dt^i}\right|dt + \sum_{i=0}^{m-2} c_{2m+3}^{(i)}\int\limits_0^T \left|\frac{d^m u}{dt^m}\right|\,\left|\frac{d^i u}{dt^i}\right|dt \\
+ \sum_{k=1}^{m}\sum_{i=0}^{k-1} c_{2m+4}^{(k,i)}\int\limits_0^T (T-t)\left|\frac{d^k u}{dt^k}\right|_{m-k,t}\left|\frac{d^i u}{dt^i}\right|_{m-i,t}dt \\
+ \sum_{k=1}^{m}\sum_{i=0}^{k-1} c_{2m+5}^{(k,i)}\int\limits_0^T \left|\frac{d^k u}{dt^k}\right|_{m-k,t}\left|\frac{d^i u}{dt^i}\right|_{m-i,t}dt \\
+ \sum_{k=1}^{m-1} c_{2m+6}^{(k)}\int\limits_0^T (T-t)\left|\frac{d^k u}{dt^k}\right|^2_{m-k,t}dt + \sum_{k=1}^{m-1} c_{2m+7}^{(k)}\int\limits_0^T \left|\frac{d^k u}{dt^k}\right|^2_{m-k,t}dt \\
+ [2c_{2m-1} - (2m-1)c]\int\limits_0^T e^{c(T-t)}(T-t)\left|\frac{d^m u}{dt^m}\right|^2 dt \\
+ [1-(2m-1)\lambda_m]\int\limits_0^T e^{c(T-t)}|u|^2_{m,t}dt,
\end{gathered} \qquad (33)$$

where $c_{2m+p}^{(i)}, c_{2m+p}^{(k,i)} \ge 0$ are constants depending only on c, T, m, $c_s^{(i)}$, and c_s. In the first two sums of these quantities, we use the δ-inequality and the interpolation inequalities

$$\begin{gathered}
\left\|\frac{d^i u}{dt^i}\right\|_0^2 \le \tau^{1-i/m}\frac{i}{m}\left\|\frac{d^m u}{dt^m}\right\|_0^2 + \tau^{-i/m}\left(1-\frac{i}{m}\right)\|u\|_0^2, \qquad \tau > 0, \qquad 0 < i < m, \\
\int\limits_0^T (T-t)\left|\frac{d^i u}{dt^i}\right|^2 dt \le \tau^{1-i/m}\frac{i}{m}\int\limits_0^T (T-t)\left|\frac{d^m u}{dt^m}\right|^2 dt + \tau^{-i/m}\left(1-\frac{i}{m}\right)\int\limits_0^T (T-t)|u|^2 dt, \\
\tau > 0, \qquad 0 < i < m,
\end{gathered} \qquad (34)$$

in [3], and in the remaining sums we use the δ-inequality, inequalities (20) and (21), and

$$|u| \leq \left\| A_0^{-1}(t) \right\|_{\mathscr{L}(H)}^{\alpha} |A_0^{\alpha}(t)u| \qquad \forall u \in D\left(A_0^{\alpha}(t)\right), \qquad 0 \leq \alpha \leq 1, \tag{35}$$

for $\alpha = 1/2$ and find first the constants $c_{2m+8} \geq \max\{c_0^{(1)}, 2c_{2m-1}/(2m-1)\}$ and then $\hat{\lambda}_m \geq 1$ such that, for $c = c_{2m+8}$ and for all $\lambda_m \geq \hat{\lambda}_m$, the expressions (33) are estimated above by the quantities $(1 - c_{2m+9})\,|||u|||_m^2, c_{2m+9} > 0$.

Thus, by estimating the left-hand sides of inequalities (32) (with $c = c_2$ and $\lambda_1 \geq 1$ for $m = 1$ and with $c = c_{2m+8}$ and $\lambda_m \geq \hat{\lambda}_m$ for $m > 1$) from below via $|||u|||_m^2$ and by collecting similar terms, after obvious estimates, we obtain the inequalities

$$c_{2m+9}|||u|||_m \leq 2 \sup_{v \in E^m} \left\{ \left| \int_0^T \left(L_m\left(\lambda_m\right)u, e^{c_{2m+8}(T-t)}J(t)v\right) dt \right| \Big/ |||v|||_m \right\}.$$

Since, by virtue of the inequalities

$$\left\| (T-t)^{-1} \frac{d^k v}{dt^k} \right\|_{m-1-k}^2 \leq 8 \left\| \frac{d^{k+1} v}{dt^{k+1}} \right\|_{m-k-1}^2 + 8\left(\mathscr{M}_{1/(2m)} + \mathscr{M}_{-(m-k)/(2m)}\right)^2 \left\| \frac{d^k v}{dt^k} \right\|_{m-k}^2,$$
$$0 \leq k \leq m-1,$$

whose proof is similar to the proof of Lemma 5, and inequalities (20), (34), and (35), we have

$$\left\langle \left\| e^{c_{2m+8}(T-t)} J(t) v \right\| \right\rangle_{m-1} \leq c_{2m+10}|||v|||_m,$$

it follows that inequalities (28) are valid with constants $c_0(m) = 2c_{2m+10}/c_{2m+9}$ and can be generalized by the passage to the limit from $D\left(L_m\right)$ to strong solutions of the boundary value problems (1), (2). The proof of the theorem is complete.

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