A GENERALIZATION OF LIONS' THEOREMS TO ACCRETIVE OPERATOR COEFFICIENTS FOR FIRST-ORDER OPERATOR-DIFFERENTIAL EQUATIONS WITH VARIABLE DOMAINS

F. E. Lomovtsev

Аннотация

In the present paper three well-known Theorems of correct solvability in [1, p. 129, 138, 142] are generalized to evolution operator-differential equation with time-variable domains of accretive operators and to new class of the mixed problems for odd-order partial differential equations with time-dependent boundary conditions of general form.

Key Words: *regularization, incorrect problem, non-local condition, differential-operator equation, Hilbert space*

Classification: 35K90, 35L90

1. Statement of the Cauchy problem. In a Hilbert space H with inner product $(\cdot,\cdot)$ and norm $|\cdot|$, on consider the Cauchy problem for operator-differential equation

$$\frac{du}{dt} + A(t)u = f, \quad t \in]0,T[, \tag{1}$$

with the initial condition

$$u(0) = u_0 \in H, \tag{2}$$

where u and f are functions of t ranging in H, and $A(t)$ are linear unbounded closed operators in H with t-depending domains $D(A(t)),\ t\in[0,T]$.

One assume that the operators $A(t)$ satisfy the following conditions.

I. The operators $A(t)$ and their conjugates $A^*(t)$ in H with t-depending domains $D(A^*(t))$ for all $t\in [0,T]$ satisfy the inequalities

$$[u]^2_{(t)} \equiv Re(A(t)u + c_0 u, u) \geq c_1|u|^2 \quad \forall\ u\in D(A(t)), \tag{3}$$

$$\langle v\rangle^2_{(t)} \equiv Re(A^*(t)v + c_0 v, v) \geq c_1|v|^2 \quad \forall\ v\in D(A^*(t)), \tag{4}$$

where $c_0\geq 0$ and $c_1 > 0$ are constants independent of $u,\ v$ and t.

II. The inverses $A_0^{-1}(t)$ of the operators $A_0(t) = A(t) + c_0 I$ in H are strongly continuous with respect to $t\in [0,T]$ in H, [2, p. 22] and for almost all $t\in]0,T[$ they have a weak derivative $dA_0^{-1}(t)/dt\in L_\infty(]0,T[,\mathfrak{L}(H))$, with respect to $t\in [0,T]$ in H [3, p. 172], such that

$$|((dA_0^{-1}(t)/dt)g,h)| \leq c_2[A_0^{-1}(t)g]_{(t)}|h| \quad \forall\ g,\ h\in H, \tag{5}$$

where $c_2\geq 0$ is a constant independent of $g,\ h$ and t.

First, we introduce needing spaces for the considered Cauchy problem. Let H_t^{*-} be the antidual Hilbert spaces of the Hilbert spaces H_t^{*+}, which are obtained as the closure of the sets $D(A^*(t))$ in the Hermitian norms $\langle\cdot\rangle_{(t)}$, $t\in [0,T]$, in (4). Throughout the sequel, all considered abstract functions and operator-valued functions of the variable t are supposed to be strongly measurable in the Lebesgue measure dt on $[0,T]$ in H. We denote the Hilbert spaces $\mathcal{H} = L_2(]0,T[,\, H)$ and $\mathcal{H}^{*-} = L_2(]0,T[,\, H_t^{*-})$.

Second, we define weak solutions of he considered Cauchy problem.

Definition 1. A function $u\in\mathcal{H}$ is called a *weak solution* of the Cauchy problem (1) – (2) for right-hand sides $f\in\mathcal{H}^{*-}$ of the equation (1) and $u_0\in H$ of the initial condition (2), if

$$\int_0^T \Big\{(u(t), A^*(t)\varphi(t)) - \Big(u(t), \frac{d\varphi(t)}{dt}\Big)\Big\}dt = \int_0^T \langle f(t), \varphi(t)\rangle_{(t)} dt + (u_0, \varphi(0))$$

for all $\varphi\in\widetilde{\Phi} \equiv \{\widetilde{\varphi} \in \mathcal{H} : \widetilde{\varphi}(t)\in D(A^*(t))\ \forall t\in[0,T];$ *слабая производная* $d\widetilde{\varphi}/dt,\ A^*(t)\widetilde{\varphi}\in\mathcal{H};$ $\widetilde{\varphi}(T) = 0\}$, where $\langle\cdot,\cdot\rangle_{(t)}$ are the sesquilinear forms of antiduality between the Hilbert spaces H_t^{*+} and H_t^{*-}.

Let us prove existence and uniqueness theorems for weak solutions of the Cauchy problem (1) – (2) in the sense of Definition 1 below and of new mixed problems for odd-order partial differential equations with time-dependent boundary conditions in general form. We first prove existence theorem for weak solutions of this Cauchy problem.

Remark 1. If the operators $A(t)$ satisfy Condition I, then for each $t\in[0,T]$ the norms $[A_0^{-1}(t)g]_{(t)} = \langle A_0^{*-1}(t)g\rangle_{(t)}$ for all $g \in H$, where $A_0^{*-1}(t)$ are the inverse operators of $A_0^*(t) = A^*(t) + c_0 I$.

2. Existence theorem. We formulate and prove the following existence theorem for weak solutions.

Theorem 1. *If Condition I is just, then for each $f\in\mathcal{H}^{*-}$ and $u_0\in H$, there exists a weak solution $u\in\mathcal{H}$ of the Cauchy problem (1) – (2).*

Proof. The proof use the following projection Lions' theorem in [1, p. 37].

Теорема 2. *Let F be a Hilbert space with Hermitian norm $\|\cdot\|_F$ and let Φ be a pre-Hilbert space with Hermitian norm $|||\cdot|||$ and continuously embedded in F, i.e., there exists a constant $c_3 > 0$ such that*

$$\|\varphi\|_F \ \le\ c_3|||\varphi|||\ \ \forall\, \varphi\in\Phi.$$

Let $E(w,\,\varphi)$ be a given sesquilinear form on $F\times\Phi$ continuous with respect to w on F for each $\varphi\in\Phi$, and suppose that there exists a constant $c_4 > 0$ such that

$$|E(\varphi,\varphi)| \ \ge\ c_4|||\varphi|||^2\ \ \forall\, \varphi\in\Phi.$$

If the antilinear functional $\mathsf{L}(\varphi)$ is continuous with respect to φ on Φ, then there exists an element $w\in F$ such that $E(w,\,\varphi) = \mathsf{L}(\varphi)$ for all $\varphi\in\Phi$.

On Hilbert space $F = \mathcal{H}$ and pre-Hilbert space $\Phi = \widetilde{\Phi}$ with the norm

$$|||\varphi||| = \Big(\int_0^T \langle\varphi(t)\rangle_{(t)}^2 dt\, +\, |\varphi(0)|^2\Big)^{1/2},$$

we take respectively the following sesquilinear form and antilinear functional

$$E(w,\,\varphi) = \int_0^T e^{2c_0 t}\Big\{(w(t),\, A^*(t)\varphi(t)) - \Big(w(t),\, \frac{d\varphi(t)}{dt}\Big)\Big\}dt,$$

$$L(\varphi) = \int_0^T \big\langle f(t), \varphi(t) \big\rangle_{(t)} dt \, + \, (u_0, \varphi(0)).$$

For each $\varphi \in \widetilde{\Phi}$, the form $E(w, \varphi)$ is obviously continuous with respect to w on $F = \mathcal{H}$ and, as integration by parts shows, $|E(\varphi, \varphi)| \ \geq \ Re\, E(\varphi, \varphi) \ \geq \ (1/2)|||\varphi|||^2$ for all $\varphi \in \widetilde{\Phi}$. For any $f \in \mathcal{H}^{*-}$ and $u_0 \in H$, the functional $L(\varphi)$ is obviously continuous with respect to φ on $\widetilde{\Phi}$. Thus, by force of Theorem 2 for each $f \in \mathcal{H}^{*-}$ and $u_0 \in H$, there exists a solution $w \in \mathcal{H}$ of the equation $E(w, \varphi) = L(\varphi) \ \forall \varphi \in \widetilde{\Phi}$ and, therefore, a weak solution $u = \exp\{2c_0 t\} w \in \mathcal{H}$ of the Cauchy problem (1) – (2). The proof of this theorem is complete.

3. Uniqueness theorem. We formulate and prove the following uniqueness theorem for weak solutions.

Theorem 3. *If Conditions I–II are satisfied, then for each* $f \in \mathcal{H}^{*-}$ *and* $u_0 \in H$ *the weak solution* $u \in \mathcal{H}$ *of the Cauchy problem (1) – (2) is unique.*

Proof. If $u \in \mathcal{H}$ is a weak solution of the Cauchy problem (1) – (2) for $f = 0$ and $u_0 = 0$, then we have the identity

$$\int_0^T \Big\{ (u(t), \, A^*(t)\varphi(t)) - \Big(u(t), \, \frac{d\varphi(t)}{dt}\Big) \Big\} dt \, = \, 0 \ \ \forall \, \varphi \in \Phi.$$

We may prove that the operators $A_0^{*-1}(t)$ have a weak derivative $dA_0^{*-1}(t)/dt \in L_\infty(]0, T[, \mathfrak{L}(H))$, with respect to t for almost all $t \in]0, T[$ in H. By setting $\varphi = A_0^{*-1}(t) w$, where $w(t) = -\int_t^T e^{-2cs} u(s) ds$ or $u = e^{2ct}(dw/dt)$ and $w(T) = 0$, and by extracting the real part, we obtain the equality

$$Re \int_0^T e^{2ct} \Big\{ \Big(\frac{dw}{dt}, \, w\Big) \, - \, \Big(\frac{dw}{dt}, \, c_0 A_0^{*-1}(t) w \, + \, \frac{dA_0^{*-1}(t)}{dt} w \, + \, A_0^{*-1}(t) \frac{dw}{dt}\Big) \Big\} dt \, = \, 0.$$

After integration by parts in the first inner product, we trove the equality

$$\frac{1}{2}|w(0)|^2 \, + \, c \int_0^T e^{2ct} |w|^2 dt \, + \, \int_0^T e^{2ct} \Big\langle A_0^{*-1}(t) \frac{dw}{dt} \Big\rangle_{(t)}^2 dt \, +$$

$$+ \, Re \int_0^T e^{2ct} \Big(\frac{dw}{dt}, \, c_0 A_0^{*-1}(t) w \, + \, \frac{dA_0^{*-1}(t)}{dt} w\Big) dt \, = \, 0. \tag{6}$$

By estimating from below the left-hand side of equality (6) by means of (3), (5) and by using Remark 1, we have the inequality

$$c \int_0^T e^{2ct} |w|^2 dt \, + \, \int_0^T e^{2ct} \Big\langle A_0^{*-1}(t) \frac{dw}{dt} \Big\rangle_{(t)}^2 dt \, - \, (c_0 c_1^{-1/2} + c_2) \int_0^T e^{2ct} \Big\langle A_0^{*-1}(t) \frac{dw}{dt} \Big\rangle_{(t)} |w| dt \leq 0,$$

which, together with the well-known inequality, implies that

$$\left(c - \frac{(c_0 c_1^{-1/2} + c_2)^2}{4}\right) \int\limits_0^T e^{2ct} |w|^2 dt \le 0.$$

Hence for $c > (c_0 c_1^{-1/2} + c_2)^2/4$, we obtain $w = 0$ and, therefore, $u = 0$ in $\mathcal{H}$. The proof of the theorem is complete.

Remark 2. Theorems 1 and 3 generalize Theorem 1.1 in [1, p. 129] for self-adjoint leading parts $A_1(t)$ of the operator coefficient $A(t)$ to the case of nonsymmetric operators $A_1(t)$ of accretive operators $A(t)$. Under Conditions I–II all weak solutions of the Cauchy problem (1) – (2) satisfy the a priori estimate

$$\int\limits_0^T |u(t)|^2 dt \le \frac{4}{c_1}\Big(\int\limits_0^T \langle f(t)\rangle^2_{(-t)}\, dt + |u_0|^2\Big),$$

where $\langle\cdot\rangle_{(-t)}$ are the norms in the Hilbert spaces H_t^{*-}, $t \in [0,T]$, [1, Remark 1.2, p. 38].

4. Construction of the operators *A(t)*. In a Hilbert space H we indicate one family of operators $A(t)$ with variable domains $D(A(t))$ satisfying Conditions I–II. Let sets $D(A(t))$, $t \in [0,T]$, be closed subspaces of some Hilbert space V continuously embedded in H. Let $P(t)$, $t\in[0,T]$, be the orthogonal projections of the space V onto $D(A(t))$. We assume the following conditions.

I_1. The linear closed operators $A(t) : H \supset D(A(t)) \to H$ satisfy the inequality (3) and their conjugates $A^*(t) : H \supset D(A^*(t)) \to H$, $t\in[0,T]$, satisfy the inequality (4).

II_1. The projection operators $P(t) : V \to V$, $t\in[0,T]$, have the following properties.

(i) For each $u\in V$, the function $P(t)u$ is strongly continuous with respect to t on $[0,T]$ in V and

$$\tau^{-1}(P(t+\tau) - P(t))u \to P'(t)u \quad \text{weakly in } V \quad \text{as} \quad \tau \to 0$$

for almost all t.

(ii) For each $g\in H$, the functions $u(t) = A_0^{-1}(t)g$ are strongly continuous with respect to t on $[0,T]$ in V, and

$$\tau^{-1}P(t+\tau)(u(t+\tau) - u(t)) \to P(t)u'(t) \quad \text{weakly in } V \quad \text{as} \quad \tau \to 0$$

for almost all t.

(iii) The weak derivative $u'(t) = P'(t)u(t) + P(t)u'(t)$ taken in V satisfies the inequalities

$$|(u'(t), h)| \le c_5 [u(t)]_{(t)} |h| \quad \forall\, u(t)\in D(A(t)), \quad \forall\, h\in H \tag{7}$$

for almost all t, where $c_5\geq 0$ is a constant independent of u, h and t.

Теорема 4. *The operators $A(t)$ with properties $I_1 - II_1$ satisfy Conditions $I - II$ respectively.*

Proof. It remains to justify inequality (5). If in the identities

$$\tau^{-1}(u(t+\tau) - u(t)) = \tau^{-1}[P(t+\tau) - P(t)]u(t) + \tau^{-1}P(t+\tau)[u(t+\tau) - u(t)]$$

one passes to the weak limit in V as $\tau \to 0$, then, by using property II_1 one obtains

$$((dA_0^{-1}(t)/dt)g, h) = (u'(t), h) = (P'(t)u(t) + P(t)u'(t), h).$$

for almost all t and for arbitrary $h \in H$. The inequalities (7) are equivalent to the inequalities (5). The proof of the theorem is complete.

Remark 3. Theorem 4 generalizes Theorem 5.1 in [1, p. 138] for self-adjoint leading parts $A_1(t)$ of the operator coefficient $A(t)$ to the case of nonsymmetric operators $A_1(t)$ of accretive operators $A(t)$. In the Theorem 5.1 the operators $P(t)$ are projections not onto the domains $D(A(t))$ as in Theorem 4 but actually onto the domains $D(A_1^{1/2}(t))$ of the square roots $A_1^{1/2}(t)$. Conditions $I_1 - II_1$ generalize the corresponding Lions' conditions. Unlike his conditions imposed on the self-adjoint leading parts $A_1(t)$, which are given by sesquilinear Hermitian forms, our conditions $I_1 - II_1$ are stated for operators $A(t)$ themselves and in purely operator form.

5. Examples of domains *D(A(t))*. Let us give one example of domains $D(A(t))$ satisfying property II_1 in the Hilbert space $H = L_2(\Omega)$, where Ω is a bounded domain with smooth boundary $S \in C^\infty$ in the Euclidian space $\mathbb{R}^n$ of the real variables $x = (x_1, ..., x_n)$.

By [1, Th. 3.2, p. 17], the values of the derivatives

$$\gamma_j u = \partial^j u/\partial\nu^j \in W_2^{2m-j+1/2}(S), \quad j = \overline{0, 2m},$$

along the outward normal ν of S are defined on S for each function u in the Sobolev space $V = W_2^{2m+1}(\Omega)$. If all coefficients $a_{i,j}(t)$ belong to the set $B([0,T], \mathfrak{L}(W_2^{2m-i+1/2}(S)))$, $i \in J_{-m}$, $j \in J_m$, of functions bounded in the norm of linear continuous mappings in $W_2^{2m-i+1/2}(S)$ for each $t \in [0,T]$, then the boundary conditions

$$\begin{cases} \Gamma_j(t)u \equiv \quad \gamma_j u(x') - \sum\limits_{i \in J_{-m}}^{i<j} a_{i,j}(t)\gamma_i u(x') = 0, \quad x' \in S, \quad j \in J_m, \\ \Gamma_{j,k}(t)u \equiv \gamma_j u(x') - \sum\limits_{i \in J_{-m}}^{i<j} a_{i,j}(t)\gamma_i u(x') = 0, \quad x' \in S_k^-, \\ \qquad k = \overline{1,n}, \quad j \in J_m^-, \quad t \in [0,T], \quad m = 0,\ 1, ..., \end{cases} \tag{8}$$

are well defined, where the sets of indices are

$$J_m = \{j_s \in [0, ..., 2m] : s = \overline{1, q}\},$$

$$J_m^- = \{j_s \in ([0, ..., 2m] \setminus J_m) : s = \overline{q+1, m+1}\},$$

$$J_{-m} = [0, ..., 2m] \setminus (J_m \cup J_m^-)$$

and S_k^- – are the sets of all points x' of the boundary S with negative direction cosines of the angles between the outward normal ν to S and the axes Ox_k, $k = \overline{1,n}$. We set

$$D(A(t)) = \{u(t) \in W_2^{2m+1}(\Omega) : \ u(t) \in (8), \ \widetilde{A}(t)u(t) \in L_2(\Omega)\}, \ t \in [0,T],$$

where $\widetilde{A}(t) = \sum\limits_{|\alpha| \le 2m+1} a_\alpha(t,x) D_x^\alpha$ is a differential operator. Consider the operators

$$A(t) : L_2(\Omega) \supset D(A(t)) \ni u \to \widetilde{A}(t)u \in L_2(\Omega), \ t \in [0,T].$$

For each $t\in[0,T]$, the domains $D(A(t))$ are closed in the Sobolev space $W_2^{2m+1}(\Omega)$. Let a sequence $u_p(t) \in D(A(t))$ converge to $u_0(t)$ in the norm of space $W_2^{2m+1}(\Omega)$ as $p \to \infty$. Then, by Theorem 3.2 in [1, p. 17], the values of the boundary operators $\Gamma_j(t)u_p(t) = 0$ and $\Gamma_{j,k}(t)u_p(t) = 0$ converge respectively to the boundary operators $\Gamma_j(t)u_0(t) = 0$ in $W_2^{2m-j+1/2}(S)$, $j \in J_m$, and $\Gamma_{j,k}(t)u_0(t) = 0$ in $W_2^{2m-j+1/2}(S_k^-)$, $j \in J_m^-$, $k = \overline{1,n}$, as $p \to \infty$, i.e., $u_0(t) \in D(A(t))$.

The Sobolev space $W_2^{2m+1}(\Omega)$ can be expanded in the direct sum $W_2^{2m+1}(\Omega) = \overset{\circ}{W}{}_2^{2m+1}(\Omega) \oplus \mathcal{W}_2^{2m+1}(\Omega)$, where $\mathcal{W}_2^{2m+1}(\Omega)$ is the orthogonal subspace to the Sobolev space $\overset{\circ}{W}{}_2^{2m+1}(\Omega)$ that is, to the set of all functions $u \in W_2^{2m+1}(\Omega)$ such that $\gamma_j u|_S = 0$, $j = \overline{0,2m}$. We define the projection operators $P(t) : W_2^{2m+1}(\Omega) \to D(A(t))$ with use of the boundary operators $\Gamma_j(t)$, $j\in J_m$, and $\Gamma_{j,k}(t)$, $j\in J_m^-$, $k = \overline{1,n}$, in a way that differs from that in [1, p. 143] but is equivalent and more general.

Definition 2. For the operators $P(t)$ the projection of a function $u \in W_2^{2m+1}(\Omega)$ are defined by $P(t)u = u(t)$, where $u(t)\in(8)$ and $\inf\limits_{v(t)} \|u - v(t)\|_{2m+1,\Omega} = \|u - u(t)\|_{2m+1,\Omega}$, the infimum being taken for all $v(t)\in D(A(t))$, $t \in [0,T]$.

The following assertion describes the action of the projections $P(t)$.

Lemma 1. *Let the projection operators $P(t)$ be defined on $D(A(t))$ in the space $W_2^{2m+1}(\Omega)$. For each function $u\in W_2^{2m+1}(\Omega)$, there exists a unique function $\tilde{u}(t)\in D(A(t))$ with boundary values*

$$\begin{cases} \gamma_j \tilde{u}(t) = \sum\limits_{i\in J_{-m}}^{i<j} a_{i,j}(t)\gamma_i u, \ j\in J_m, \quad \text{on } S, \\ \gamma_j \tilde{u}(t) = \gamma_j u, \ j\notin J_m \cup J_m^-, \ \text{on } S, \\ \gamma_j \tilde{u}(t) = \sum\limits_{i\in J_{-m}}^{i<j} a_{i,j}(t)\gamma_i u, \ j\in J_m^-, \quad \text{on } S_k^-, \\ \gamma_j \tilde{u}(t) = \gamma_j u, \ j\in J_m^-, \quad \text{on } S_k^+ = S\backslash S_k^-, \ k = \overline{1,n}, \end{cases} \tag{9}$$

in $W_2^{2m-j+1/2}(S)$, $j = \overline{0,2m}$, such that $P(t)u = \tilde{u}(t)$ in $W_2^{2m+1}(\Omega)$ for all $t\in[0,T]$, provided that

$$\gamma_j \tilde{u}(t) \in W_2^{2m-j+1/2}(S) \ \forall j \in J_m^- \tag{10}$$

in the data (9) for all $u \in W_2^{2m+1}(\Omega)$.

Proof. By the above-mentioned Theorem 3.2, the restriction $\gamma = \{\gamma_0, ..., \gamma_{2m}\}$ is an isomorphism of the Hilbert space $\mathcal{W}_2^{2m+1}(\Omega)$ onto the product $\prod\limits_{j=0}^{2m} W_2^{2m-j+1/2}(S)$ of the Hilbert spaces $W_2^{2m-j+1/2}(S)$. Owing to the mapping γ^{-1}, inverse to the restriction $\gamma = \{\gamma_0, ..., \gamma_{2m}\}$, for each $t \in [0,T]$ and for the boundary conditions (9), there exists a unique function $\tilde{u}(t) \in \mathcal{W}_2^{2m+1}(\Omega)$. By construction, this function $\tilde{u}(t) \in D(A(t))$. For all $t \in [0,T]$ and $v(t) \in D(A(t))$, the squared distance $\|u - v(t)\|^2_{2m+1,\Omega}$ is equivalent to

$$\sum_{j=0}^{2m} \|\gamma_j(u - v(t))\|^2_{2m-j+1/2,S} = \sum_{j\in J_m} \|\gamma_j u - \sum_{i\in J_{-m}}^{i<j} a_{i,j}(t)\gamma_i u +$$

$$+ \sum_{i\in J_{-m}}^{i<j} a_{i,j}(t)\gamma_i(u - v(t))\|^2_{2m-j+1/2,S} + \sum_{j\notin J_m\cup J_m^-} \|\gamma_j(u - v(t))\|^2_{2m-j+1/2,S} +$$

$$+\sum_{k=1}^{n}\sum_{j\in J_m^-}\|\gamma_j u-\sum_{i\in J_{-m}}^{i<j}a_{i,j}(t)\gamma_i u+\sum_{i\in J_{-m}}^{i<j}a_{i,j}(t)\gamma_i(u-v(t))\|^2_{2m-j+1/2,S_k^-}+$$

$$+\sum_{k=1}^{n}\sum_{j\in J_m^-}\|\gamma_j(u-v(t))\|^2_{2m-j+1/2,S_k^+},$$

whose minimum is obviously attained for $v(t)=\tilde{u}(t)$. It follows that $P(t)u=\tilde{u}(t)$ in $W_2^{2m+1}(\Omega)$ for $t\in[0,T]$. The proof of the lemma is complete.

For each function $u\in W_2^{2m+1}(\Omega)$, the function $P(t)u=\tilde{u}(t)$ is strongly continuous with respect to t on $[0,T]$ in $W_2^{2m+1}(\Omega)$ provided that the coefficients $a_{i,j}(t)$ are strongly continuous with respect to t in $\mathfrak{L}(W_2^{2m-i+1/2}(S))$, $i\in J_{-m}$, $j\in J_m$. The projections $P(t)$ have a weak derivative with respect to t in $W_2^{2m+1}(\Omega)$, provided that the coefficients $a_{i,j}(t)$ are weakly differentiable with respect to t in $\mathfrak{L}(W_2^{2m-i+1/2}(S))$, $i\in J_{-m}$, $j\in J_m$.

Lemma 2. *Let the coefficients $a_{i,j}(t)$ be strongly continuous with respect to t in $\mathfrak{L}(W_2^{2m-i+1/2}(S))$, $i\in J_{-m}$, $j\in J_m$, and have weak derivatives $a'_{i,j}(t)\in L_\infty(]0,T[,\mathfrak{L}(W_2^{2m-i+1/2}(S))$, $i\in J_{-m}$, $j\in J_m$, with respect to t. For each function $u\in W_2^{2m+1}(\Omega)$, for almost all t, there exists a function $\tilde{w}(t)\in\mathcal{W}_2^{2m+1}(\Omega)$ with boundary values*

$$\begin{cases}\gamma_j\tilde{u}'(t) &=\sum\limits_{i\in J_{-m}}^{i<j}a'_{i,j}(t)\gamma_i u,\ j\in J_m,\quad on\ S,\\ \gamma_j\tilde{u}'(t) &=0,\quad j\notin J_m\cup J_m^-,\quad on\ S,\\ \gamma_j\tilde{u}'(t) &=\sum\limits_{i\in J_{-m}}^{i<j}a'_{i,j}(t)\gamma_i u,\ j\in J_m^-,\quad on\ S_k^-,\\ \gamma_j\tilde{u}'(t) &=0,\quad j\in J_m^-,\quad on\ S_k^+,\ k=\overline{1,n},\end{cases}\tag{11}$$

in $W_2^{2m-j+1/2}(S)$, $j=\overline{0,2m}$, ($\tilde{u}(t)$ is the function in Lemma 1) such that

$$P'(t)u=\tilde{w}(t)\ \textit{for almost all}\ \ t\tag{12}$$

in the space $W_2^{2m+1}(\Omega)$ provided that

$$\gamma_j\tilde{u}'(t)\in W_2^{2m-j+1/2}(S)\ \forall j\in J_m^-\tag{13}$$

in the data (11) for all $u\in W_2^{2m+1}(\Omega)$.

Proof. In the boundary data (9), for almost all t, we pass to weak limits in the quotient $\tau^{-1}(\tilde{u}(t+\tau)-\tilde{u}(t))$ in the spaces $W_2^{2m-j+1/2}(S)$, $j=\overline{0,2m}$, as $\tau\to0$ and use the weak differentiability of the coefficients $a_{i,j}(t)$ with respect to t in $\mathfrak{L}(W_2^{2m-i+1/2}(S))$, $i\in J_{-m}$, $j\in J_m$; then we obtain the boundary values (11), which are taken by the continuation mapping γ^{-1} to a unique function $\tilde{w}(t)\in\mathcal{W}_2^{2m+1}(\Omega)$. Since the isomorphism γ^{-1} takes weak convergent sequences in $\prod\limits_{j=0}^{2m}W_2^{2m-j+1/2}(S)$ to weak convergent sequences in $\mathcal{W}_2^{2m+1}(\Omega)$ and *vice versa*, we have $P'(t)u=\tilde{w}(t)$ in $W_2^{2m+1}(\Omega)$ for almost t. The proof of this lemma and hence property (i) of the projections $P(t)$ are complete.

By using Lemmas 1 and 2 one can prove the following assertion.

Theorem 5. *Let the coefficients $D_x^\beta a_\alpha\in C([0,T]\times\Omega)$, $|\beta|\le|\alpha|$, have the derivative $\partial a_\alpha/\partial t\in L_\infty(]0,T[\times\Omega)$, $|\alpha|\le 2m+1$, for almost all t, let the differential expressions $\widetilde{A}(t)$ with* *the boundary conditions (8) satisfy properties* I_1 *in Theorem 4, and let the differential operator* $\widetilde{A}_0(t) = \widetilde{A}(t) + c_0 I$ *be coercive on* $D(A(t))$ *in* $W_2^{2m+1}(\Omega)$ *(i.e., the inequality*

$$\|u(t)\|_{2m+1,\Omega} \ \leq \ c_6\|\widetilde{A}_0(t)u(t)\|_{0,\Omega} \quad \forall \ u(t){\in}D(A(t)) \tag{14}$$

is valid for all $t{\in}[0,T]$, *where* $c_6 > 0$ *is a constant independent of* u *and* t*). Let their conjugates in* $L_2(\Omega)$ *be given by some differential expressions*

$$\widetilde{A}^*(t) = \sum_{|\alpha|\leq 2m+1} a^*_\alpha(t,x)D^\alpha_x, \ \ \widetilde{A}^*_0(t) = \widetilde{A}^*(t) + c_0 I,$$

with coefficients $D^\beta_x a^*_\alpha{\in}C([0,T]\times\Omega)$, $\partial a^*_\alpha/\partial t \in L_\infty(]0,T[\times\Omega)$, $|\beta| \leq |\alpha| \leq 2m+1$, *and some boundary conditions* $\{\Gamma^*_j(t)\}_{J^*_{2m+1}}$ *satisfying the inequalities*

$$\|(\partial\widetilde{A}^*(t)/\partial t)v(t)\|_{0,\Omega} \ \leq \ c_7\|\widetilde{A}^*_0(t)v(t)\|_{0,\Omega} \ \ \forall \ v(t){\in}D(A^*(t)), \tag{15}$$

where $c_7{\geq}0$ *is a constant independent of* v *and* t*. If all coefficients* $a_{i,j}(t)$ *are strongly continuous with respect to* t *in* $\mathfrak{L}(W_2^{2m-i+1/2}(S)))$ *and have weak derivatives* $a'_{i,j}(t){\in}L_\infty(]0,T[, \ \mathfrak{L}(W_2^{2m-i+1/2}(S)))$ *with respect to* t *in* $W_2^{2m-i+1/2}(S)$, $i{\in}J_{-m}$, $j{\in}J^-_m$, *for almost all* t *such that*

$$\begin{cases} \|a'_{i,j}(t)\gamma_i u\|_{0,S} \ \leq c'_{i,j}[u]_{(t)} & \forall \ u{\in}D(A(t)), \ \ i{\in}J_{-m}, \ \ j{\in}J_m, \\ \|a'_{i,j}(t)\gamma_i u\|_{0,S^-_k} \leq c^{(k)}_{i,j}[u]_{(t)} & \forall \ u{\in}D(A(t)), \ \ i{\in}J_{-m}, \ \ j{\in}J^-_m, \ \ k=\overline{1,n}, \end{cases} \tag{16}$$

where $c'_{i,j}$, $c^{(k)}_{i,j}{\geq}0$ *are constants independent of* u *and* t*, and continuation conditions (10) and (13) are valid, then the projections* $P(t) : W_2^{2m+1}(\Omega) \to D(A(t))$ *satisfy property* II_1 *in Theorem 4.*

Proof. 1. Property (i) of the projections $P(t)$ follows from Lemmas 1 and 2.

2. If the differential operators $A(t)$ satisfy Condition I_1, then there exist their bounded inverse operators $A_0^{-1}(t)$ on $L_2(\Omega)$ and $u(t) = A_0^{-1}(t)g{\in}D(A(t))$ for arbitrary $g{\in}L_2(\Omega)$. Suppose that for an arbitrary given value $t_0{\in}[0,T]$ some sequence $t_p{\to}t_0$ converges to t_0 as $p{\to}\infty$. By virtue of the inequality $\|u(t)\|_{2m+1,\Omega} \ \leq \ c_6\|g\|_{0,\Omega}$, which follows from (14), one can single out a subsequence $t_l \to t_0$ such that $u(t_l) \to w_0$ weakly in $W_2^{2m+1}(\Omega)$ as $l \to \infty$. By passing to weak limits in the boundary operators $\Gamma_j(t_l)u(t_l) = 0$ in $W_2^{2m-j+1/2}(S)$, $j{\in}J_m$, and $\Gamma_{j,k}(t_l)u(t_l) = 0$ in $W_2^{2m-j+1/2}(S^-_k)$, $j \in J^-_m$, $k = \overline{1,n}$, as $l{\to}\infty$ and by using the strong continuity of their coefficients $a_{i,j}(t)$ with respect to t in $W_2^{2m-j+1/2}(S)$, $i \in J_{-m}$, $j \in J_m$, we obtain $\Gamma_j(t_0)w_0 = 0$, $j \in J_m$, and $\Gamma_{j,k}(t_0)w_0 = 0$, $j \in J^-_m$, $k = \overline{1,n}$, i.e., $w_0 \in D(A(t_0))$. Therefore, there exists a function $g_0{\in}L_2(\Omega)$ such that $A_0(t_0)w_0 = g_0$. Since the coefficients a^*_α of the differential operators $\widetilde{A}^*(t)$ are continuous with respect to t for each $v{\in}W_2^{2m+1}(\Omega)$, we trove $\|\widetilde{A}^*_0(t_l)v{-}\widetilde{A}^*_0(t_0)v\|_{0,\Omega}{\to}0$ as $l \to \infty$. So the right-hand sides of the relations

$$(v, g - g_0)_{0,\Omega} = ([\widetilde{A}^*_0(t_l) - \widetilde{A}^*_0(t_0)]v, u(t_l))_{0,\Omega} + (\widetilde{A}^*_0(t_0)v, u(t_l) - w_0)_{0,\Omega}$$

tend to zero as $l{\to}\infty$ for all $v{\in} \overset{\circ}{W}{}_2^{2m+1}(\Omega)$. The symbols $(\cdot,\cdot)_{p,\Omega}$ stand for the inner products in the Sobolev space $W_2^p(\Omega)$. By virtue of density of the set $\overset{\circ}{W}{}_2^{2m}(\Omega)$ in $L_2(\Omega)$, we have $g_0 = g$ in $L_2(\Omega)$, and therefore, $w_0 = A_0^{-1}(t)g = u(t_0)$, i.e., the sequence $u(t_l) \to u(t_0)$ weakly in $W_2^{2m+1}(\Omega)$ as $l \to \infty$. Since the sequence t_p is arbitrary, it follows that $u(t) = A_0^{-1}(t)g$ is weakly continuous functions with respect to t on $[0,T]$ in the Sobolev space $W_2^{2m+1}(\Omega)$.

Let us show that the functions $u(t) = A_0^{-1}(t)g$ are strongly continuous with respect to t in the space $W_2^{2m+1}(\Omega)$. By virtue of the inequalities

$$\|u(t) - u(t_0)\|_{2m+1,\Omega} \;\leq\; c_6\|\tilde{A}_0(t)[u(t) - u(t_0)]\|_{0,\Omega},$$

to this end it suffices to show that their right-hand sides tend to zero as $t \to t_0$. Obviously, $\|\widetilde{A}_0(t)u - \widetilde{A}_0(t_0)u\|_{0,\Omega} \to 0$ for each $u{\in}W_2^{2m+1}(\Omega)$ as $t{\to}t_0$, since the coefficients a_α of the differential operators $\widetilde{A}(t)$ are continuous with respect to t and hence $\|\widetilde{A}_0(t)u\|_{0,\Omega}{\to}\|\widetilde{A}_0(t_0)u\|_{0,\Omega}$ as $t{\to}t_0$. Therefore, the squared norms

$$\|\widetilde{A}_0(t)u(t) - \widetilde{A}_0(t)u(t_0)\|^2_{0,\Omega} = \|g\|^2_{0,\Omega} - 2Re(g, \widetilde{A}_0(t)u(t_0))_{0,\Omega} + \|\widetilde{A}_0(t)u(t_0)\|^2_{0,\Omega}$$

tend to zero as $t \to t_0$ for $u(t) = A_0^{-1}(t)g$.

Now, by using Lemma 2, we prove the weak differentiability of $u(t) = A_0^{-1}(t)g$ with respect to t.

Lemma 3. *Let the assumptions of Theorem 5 [without inequalities (15) u (16)] be valid. Then for the function $u(t) = A_0^{-1}(t)g$ the convergence*

$$\tau^{-1}P(t+\tau)(u(t+\tau) - u(t)) \to w(t) \quad \textit{weakly in} \quad W_2^{2m+1}(\Omega) \quad \textit{as} \quad \tau \to 0,$$

is valid for each $g{\in}L_2(\Omega)$ and for almost all t, and there exists a unique function $\widetilde{w}(t){\in}\mathcal{W}_2^{2m+1}(\Omega)$ with the boundary values (11) for $u = u(t)$ such that

$$u'(t) = -A_0^{-1}(t)(\partial\widetilde{A}(t)/\partial t)u(t) - A_0^{-1}(t)\widetilde{A}_0(t)\widetilde{w}(t) + \widetilde{w}(t), \tag{17}$$

$$P(t)u'(t) = -A_0^{-1}(t)(\partial\widetilde{A}(t)/\partial t)u(t) - A_0^{-1}(t)\widetilde{A}_0(t)\widetilde{w}(t) \tag{18}$$

for almost all t.

Proof. The functions $u(t){\in}D(A(t))$ and $v{\in}W_2^{2m+1}(\Omega)$ satisfy the identities

$$\tau^{-1}(P(t+\tau)[u(t+\tau) - u(t)], v)_{2m+1,\Omega} = \tau^{-1}([P(t+\tau) - P(t)]u(t+\tau), v)_{2m+1,\Omega} -$$

$$-\tau^{-1}([P(t+\tau) - P(t)]u(t), v)_{2m+1,\Omega} + \tau^{-1}(P(t)[u(t+\tau) - u(t)], v)_{2m+1,\Omega} \tag{19}$$

for each $t \in [0, T]$. Since the self-adjoint operators $P(t)$ are weakly differentiable with respect to t for almost all t and the functions $u(t) = A_0^{-1}(t)g$ is strongly continuous with respect to t on $[0, T]$ in $W_2^{2m+1}(\Omega)$, we have

$$\tau^{-1}([P(t+\tau) - P(t)]u(t+\tau),\, v)_{2m+1,\Omega} =$$
$$= \tau^{-1}(u(t+\tau),\, [P(t+\tau) - P(t)]v)_{2m+1,\Omega} \to (u(t),\, P'(t)v)_{2m+1,\Omega}$$

for the first term on the right-hand side in (19) as $\tau \to 0$ for almost all t. By Lemma 2, the her second term has the limit

$$\tau^{-1}([P(t+\tau) - P(t)]u(t),\, v)_{2m+1,\Omega} \to (P'(t)u(t),\, v)_{2m+1,\Omega}$$

as $\tau \to 0$ for almost all t. To compute the limit of the third term on the right-hand side in (19), we need the following assertion.

Lemma 4. [1, p. 144]. *Let the assumptions of Lemma 3 be satisfied, and let $P(t)$: $W_2^{2m+1}(\Omega) \to D(A(t))$ be orthogonal projection operators. Then for each $u,\ v{\in}W_2^{2m+1}(\Omega)$, there exist functions $w_\tau(t)$ [respectively, $v_\tau(t)$]* in $W_2^{2m+1}(\Omega)$ *such that*

$$P(t+\tau)u - w_\tau(t) \in D(A(t)) \;\; [respectively,\; P(t)v - v_\tau(t) \in D(A(t+\tau))]$$

and

$$\tau^{-1} w_\tau(t) \to w'(t) \;\; [respectively,\; \tau^{-1} v_\tau(t) \to v'(t)] \; weakly \; in \; W_2^{2m+1}(\Omega) \; as \; \tau \to 0$$

for almost all t.

Proof. To prove the parenthesized assertions, we supplement the boundary conditions

$$\begin{cases} \gamma_j v_\tau(t) - \sum\limits_{i \in J_{-m}}^{i<j} a_{i,j}(t+\tau)\gamma_i v_\tau(t) = - \sum\limits_{i \in J_{-m}}^{i<j} [a_{i,j}(t+\tau) - a_{i,j}(t)]\gamma_i v(t), \quad j \in J_m, \; on \; S, \\ \gamma_j v_\tau(t) - \sum\limits_{i \in J_{-m}}^{i<j} a_{i,j}(t+\tau)\gamma_i v_\tau(t) = - \sum\limits_{i \in J_{-m}}^{i<j} [a_{i,j}(t+\tau) - a_{i,j}(t)]\gamma_i v(t), \quad j \in J_m^-, \; on \; S_k^-, \\ \qquad k = \overline{1,n}, \end{cases} \tag{20}$$

with the boundary conditions

$$\gamma_j v_\tau(t) = 0, \; j \notin J_m \cup J_m^-, \; on \; S; \; \gamma_j v_\tau(t) = 0, \; j \in J_m^-, \; on \; S_k^+, \; k = \overline{1,n}. \tag{21}$$

By using the mapping γ^{-1}, on the basis of the resulting boundary values $\gamma_j v_\tau(t),\; j = \overline{0,2m}$, we find functions $v_\tau(t) \in \mathcal{W}_2^{2m+1}(\Omega)$ such that $v(t) - v_\tau(t) \in D(A(t+\tau))$ for all small τ. By passing in $\tau^{-1} v_\tau(t)$ to weak limit in $W_2^{2m+1}(\Omega)$ as $\tau \to 0$, with the use of (20), (21) we trove the boundary values

$$\gamma_j v'(t) = - \sum_{i \in J_{-m}}^{i<j} a'_{i,j}(t)\gamma_i v(t), \quad j \in J_m, \; on \; S,$$

$$\gamma_j v'(t) = - \sum_{i \in J_{-m}}^{i<j} a'_{i,j}(t)\gamma_i v(t), \quad j \in J_m^-, \; on \; S_k^-, \; k = \overline{i,n},$$

$$\gamma_j v'(t) = 0, \; j \notin J_m \cup J_m^-, \; on \; S; \; \gamma_j v'(t) = 0, \; j \in J_m^-, \; on \; S_k^+, \; k = \overline{1,n},$$

which, in view of the continuation mapping γ^{-1}, corresponds to some function $v'(t) \in \mathcal{W}_2^{2m+1}(\Omega)$. The proof of the second assertion of Lemma 4 is complete. The first assertion can be proved in a similar way. The proof of Lemma 4 is complete.

Thus, by virtue of the second assertion of Lemma 4 and the main properties of the projections $P(t)$, we have

$$\begin{aligned} (u(t+\tau) - u(t),\, P(t)v)_{2m+1,\Omega} = (P(t+\tau)u,\, P(t)v - v_\tau(t))_{2m+1,\Omega} + \\ +(P(t+\tau)u, v_\tau(t))_{2m+1,\Omega} - (u, P(t)v)_{2m+1,\Omega} = (u,\, P(t)v - v_\tau(t))_{2m+1,\Omega} + \\ +(P(t+\tau)u,\, v_\tau(t))_{2m+1,\Omega} - (u,\, P(t)v)_{2m+1,\Omega} = (P(t+\tau)u - u,\, v_\tau(t))_{2m+1,\Omega}. \end{aligned}$$

Hence it follows that

$$\tau^{-1}(P(t)[u(t+\tau) - u(t)], v)_{2m+1,\Omega} \to (P(t)u - u, v'(t))_{2m+1,\Omega} \;\; \text{as} \;\; \tau \to 0$$

for almost all t and for each $v \in W_2^{2m+1}(\Omega)$.

As a result, by summing the resulting limits for the terms occurring on the right-hand side in identity (19), we obtain

$$\tau^{-1}(P(t+\tau)[u(t+\tau) - u(t),\, v])_{2m,\Omega} \to (P(t)u - u,\, v'(t))_{2m,\Omega} \;\; \text{as} \;\; \tau \to 0$$

for each $v\in W_2^{2m+1}(\Omega)$ and for almost t. Hence the functions $u(t)=A_0^{-1}(t)g$ for all $g\in L_2(\Omega)$ have the weak derivative $u'(t)$ with respect to t in $W_2^{2m+1}(\Omega)$ for almost all t. The prove of property (ii) of the projections $P(t)$ is complete.

Let us compute this weak derivative. The computation of weak derivatives with respect to t in the equation $\tilde{A}_0(t)u(t)=g$ in the space $W_2^{2m+1}(\Omega)$ and in the boundary data (11) in the spaces $W_2^{2m-j+1/2}(S)$, $j=\overline{0,2m}$, necessitates considering the boundary value problems

$$\begin{gathered}\tilde{A}_0(t)u'(t)=-(\partial\tilde{A}(t)/\partial t)u(t),\quad x\in\Omega,\\ \Gamma_j(t)u'(t)=\sum_{i\in J_{-m}}^{i<j}a'_{i,j}(t)\gamma_i u(t),\quad j\in J_m,\ on\ S,\\ \Gamma_{j,k}(t)u'(t)=\sum_{i\in J_{-m}}^{i<j}a'_{i,j}(t)\gamma_i u(t),\quad j\in J_m^-,\ on\ S_k^-,\ k=\overline{1,n},\end{gathered}$$

for almost all t. As was shown above, for almost all t the boundary data (11) for the function $u=u(t)$ corresponds to the function $\widetilde{w}(t)\in\mathcal{W}_2^{2m+1}(\Omega)$. Then the function $w(t)=u'(t)-\widetilde{w}(t)$ is a solution of the boundary value problems

$$\begin{gathered}A_0(t)w(t)=-(\partial\widetilde{A}(t)/\partial t)u(t)-\widetilde{A}_0(t)\widetilde{w}(t),\quad x\in\Omega,\\ \Gamma_j(t)w(t)=0,\ j\in J_m,\ on\ S;\ \Gamma_{j,k}(t)w(t)=0,\ j\in J_m^-,\ on\ S_k^-,\ k=\overline{1,n},\end{gathered}$$

for almost all t; consequently,

$$w(t)=-A_0^{-1}(t)(\partial\widetilde{A}(t)/\partial t)u(t)-A_0^{-1}(t)\widetilde{A}_0(t)\widetilde{w}(t)$$

for almost all t, and the derivative $u'(t)$ can be expressed by formula (17).

For almost all t and for all $v(t)\in D(A(t))$, the squared distance $\|u'(t)-v(t)\|^2_{2m+1,\Omega}$ is equivalent to

$$\sum_{j=0}^{2m}\|\gamma_j(u'(t)-v(t))\|^2_{2m-j+1/2,S}=\sum_{j\in J_m}\|\gamma_j(u'(t)-w(t))+$$

$$+\sum_{i\in J_{-m}}^{i<j}a_{i,j}(t)\gamma_i(w(t)-v(t))\|^2_{2m-j+1/2,S}+\sum_{j\notin J_m\cup J_m^-}\|\gamma_j(u'(t)-v(t))\|^2_{2m-j+1/2,S}+$$

$$+\sum_{k=1}^{n}\sum_{j\in J_m^-}\|\gamma_j(u'(t)-w(t))+\sum_{i\in J_{-m}}^{i<j}a_{i,j}(t)\gamma_i(w(t)-v(t))\|^2_{2m-j+1/2,S_k^-}+$$

$$+\sum_{k=1}^{n}\sum_{j\in J_m^-}\|\gamma_j(u'(t)-v(t))\|^2_{2m-j+1/2,S_k^+},$$

whose minimum, by force of the boundary data (11) for $u=u(t)$, is obviously attained for $v(t)=w(t)$. It follows that $P(t)u'(t)=w(t)$ in $W_2^{2m+1}(\Omega)$ for almost all t. Therefore, the relation (18) is valid for almost all t. The proof of Lemma 3 is complete.

3. Thus, by passing to the conjugate in $L_2(\Omega)$ and by using formula (17), we have the relations

$$(u'(t),h)_{0,\Omega}=-(u(t),\,(\partial\widetilde{A}^*(t)/\partial t)A_0^{*-1}(t)h)_{0,\Omega}-(A_0^{-1}(t)\widetilde{A}_0(t)\widetilde{w}(t)-\widetilde{w}(t),\,h)_{0,\Omega}\qquad(22)$$

for almost all t and for all $h\in L_2(\Omega)$. For the first term on the right-hand side in this relation, the desired estimates (7) follow from the boundedness of the the operators

$(\partial\widetilde{A}^*(t)/\partial t)A_0^{*-1}(t) \in\ L_\infty(]0,T[,\ \mathfrak{L}(H))$ by virtue of inequalities (15). The use of Greens's formula for integration by parts to the inequality

$$\|A_0^{-1}(t)\widetilde{A}_0(t)v\|_{0,\Omega}^2\ \le\ c_8(\|v\|_{0,\Omega}^2+\sum_{j=0}^{2m}\|\gamma_j v\|_{0,S}^2)\quad\forall\ \ v{\in}W_2^{2m+1}(\Omega), \tag{23}$$

where $c_8{\ge}1$ is a constant independent of v and t. The conjugate of the isometry [4, pp 20–56]

$$\gamma:\mathcal{W}_2^{2m+1}(\Omega)\to\prod_{j=0}^{2m}W_2^{2m-j+1/2}(S)$$

is given by the isometry

$$\gamma^*:\prod_{j=0}^{2m}W_2^{-(2m-j+1/2)}(S)\to\mathcal{W}_2^{-2m-1}(\Omega)$$

of the antidual spaces $\prod_{j=0}^{2m}W_2^{-(2m-j+1/2)}(S)$ and $\mathcal{W}_2^{-2m-1}\Omega)$ onto the spaces $\prod_{j=0}^{2m}W_2^{2m-j+1/2}(S)$ and $\mathcal{W}_2^{2m+1}(\Omega)$, respectively. The restriction of the mapping γ^* to $\prod_{j=0}^{2m}W_2^{2m-j+1/2}(S)$ is the mapping γ^{-1}. By applying the main interpolation theorem [5, p. 41] to the operator

$$\gamma^*{\in}\mathfrak{L}\Big(\prod_{j=0}^{2m}W_2^{2m-j+1/2}(S),\mathcal{W}_2^{2m+1}(\Omega)\Big)\bigcap\mathfrak{L}\Big(\prod_{j=0}^{2m}W_2^{-(2m-j+1/2)}(S),\ \mathcal{W}_2^{-2m-1}(\Omega)\Big)$$

with respect to the parameter $\theta{\in}]0,1[$, we obtain the inclusion $\gamma^*\in\mathfrak{L}\Big(\prod_{j=0}^{2m}W_2^{-j+1/2}(S),\ L_2(\Omega)\Big)$, for $\theta=1/2$, i.e.,

$$\|v\|_{0,\Omega}^2\ \ \le\ c_9\sum_{j=0}^{2m}\|\gamma_j v\|_{-j+1/2,S}^2\quad\ \forall\ \ v{\in}W_2^{2m+1}(\Omega)^2, \tag{24}$$

where $c_9>0$ is a constant independent of v and t. Then the desired estimates (7) for the second term in (22) follow from inequalities (23), (24) and (16) by virtue of the boundary data (11) for $u=u(t)$. Property (iii) of the projections $P(t)$ is valid. The proof of the theorem is compete.

Remark 4. Theorem 5 generalizes Theorem 6.1 in [1, p. 142] for self-adjoint leading parts $A_1(t)$ of the operator coefficient $A(t)$ to the case of nonsymmetric operators $A_1(t)$ of accretive operators $A(t)$.

6. Applications. Let us apply the abstract obtained results to the investigation of well-posed solvability of new class of mixed problems for odd-order linear partial differential equations with time-dependent boundary conditions, which has not been researched by anybody yet.

In the cylinder $G=]0,T[\times\Omega$ of the variables t and $x=(x_1,...,x_n)$, where $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary $S{\in}C^\infty$, we investigate the nonclassical equations

$$\partial u(t,x)/\partial t\ +\sum_{|\alpha|\le 2m+1}a_\alpha(t,x)D_x^\alpha u(t,x)=f(t,x),\quad m=0,\ 1,..., \tag{25}$$

with the boundary conditions (8) on S for $t\in[0,T]$ and the initial condition

$$u(0,x) = u_0(x), \quad x \in \Omega. \tag{26}$$

Let the differential operators $\widetilde{A}(t) = \sum\limits_{|\alpha|\leq 2m+1} a_\alpha(t,x)D_x^\alpha$ with the boundary conditions (8) satisfy the following conditions.

I_3. The coefficients a_α of the equations (25) satisfy inclusion $D_x^\beta a_\alpha \in C(\Omega)$, $|\beta|\leq|\alpha|\leq 2m+1$, for each $t\in[0,T]$, the differential operators $\widetilde{A}(t)$ are lower semibounded in $L_2(\Omega)$ on $W_2^{2m+1}(\Omega;\ \{\Gamma_j(t)\}_{J_m\cup J_m^-}) = \{u(t)\in W_2^{2m+1}(\Omega):\ u(t)\in(8)\}$ and their conjugates in $L_2(\Omega)$ on $W_2^{2m+1}(\Omega;\ \{\Gamma_j(t)\}_{J_m\cup J_m^-})$ are given by some differential operators $\widetilde{A}^*(t) = \sum\limits_{|\alpha|\leq 2m+1} a_\alpha^*(t,x)D_x^\alpha$ with some adjoint boundary conditions $\{\Gamma_j^*(t)\}_{J_m^*}$.

The coefficients a_α^* of their conjugates satisfy inclusion $D_x^\beta a_\alpha^* \in C(\Omega)$, $|\beta|\leq|\alpha|\leq 2m+1$, for each $t\in[0,T]$, the differential operators $\widetilde{A}^*(t)$ are also lower semibounded in $L_2(\Omega)$ on $W_2^{2m+1}(\Omega;\ \{\Gamma_j(t)\}_{J_m\cup J_m^-}) = \{u(t)\in W_2^{2m+1}(\Omega):\ u(t)\in(\Gamma_j^*(t)\}_{J_m^*})\}$.

II_3. All coefficients a_α belong to $C(G)$, and their derivative $\partial a_\alpha/\partial t$ belongs to $L_\infty(G)$. The operators $\widetilde{A}_0(t) = \widetilde{A}(t) + c_0 I$ are coercive on $W_2^{2m+1}(\Omega;\ \{\Gamma_j(t)\}_{J_m\cup J_m^-})$ in $L_2(\Omega)$ for some $c_0 > 0$ and for any $t\in[0,T]$; i.e.,

$$\|u\|_{2m+1,\Omega} \ \leq\ c_{10}\|\widetilde{A}_0(t)u\|_{0,\Omega} \ \ \forall\ \ u\in W_2^{2m+1}(\Omega;\ \{\Gamma_j(t)\}_{J_m\cup J_m^-}),$$

where $c_{10} > 0$ is a constant independent of u and t. All coefficients a_α^* belong to $C(G)$, their derivative $\partial a_\alpha^*/\partial t$ belongs to $L_\infty(G)$, and

$$\|(\partial\widetilde{A}^*(t)/\partial t)v\|_{0,\Omega} \ \ \leq\ \ c_{11}\|\widetilde{A}_0^*(t)v\|_{0,\Omega} \quad \forall\ \ v\in W_2^{2m+1}(\Omega;\ \{\Gamma_j^*(t)\}_{J_m^*}), \tag{27}$$

where $c_{11}\geq 0$ is a constant independent of v and t. All coefficients $a_{i,j}(t)$ of boundary conditions belong to $C[0,T]$, their derivative $\partial a_{i,j}(t)/\partial t$ belongs to $L_\infty(0,T)$, they satisfy inequality (16), and functions admit the continuations (10) and (13) from S_k^- to the entire set $\overline{\Omega}$.

Since Conditions $I_3 - II_3$ provide the validity of properties $I_1 - II_1$ in $V = W_2^{2m+1}(\Omega)$ and, therefore, of Conditions $I - II$ in $H = L_2(\Omega)$, from Theorems 1, 3–5, we obtain the following assertion.

Theorem 6. *If Conditions $I_3 - II_3$ are satisfied, then for arbitrary functions $f \in L_2(]0,T[,\ W_2^{-m}(\Omega;\ \{\Gamma_j^*(t)\}_{J_m^*}))$ and $u_0\in L_2(\Omega)$, there exist unique weak solutions $u\in L_2(G)$ of the mixed problems (25), (8), (26).*

Here $W_2^{-m}(\Omega;\ \{\Gamma_j^*(t)\}_{J_m^*})$ are the antiduel spaces of the Hilbert spaces $W_2^m(\Omega;\ \{\Gamma_j^*(t)\}_{J_m^*})$, which are obtained by the closure of the sets $W_2^{2m+1}(\Omega;\ \{\Gamma_j^*(t)\}_{J_m^*})$ in Hermitian norms $\langle\cdot\rangle_{(t)}$, $t \in [0,T]$, corresponding to the formula (4).

For example, the Conditions $I_3 - II_3$ are satisfied for the following mixed problem for the linearized Korteweg-de Vries equation:

$$\frac{\partial u}{\partial t} \ -\ a(t)\frac{\partial^3 u}{\partial x^3} = f(t,x), \quad 0 < t < T, \quad 0 < x < l, \tag{28}$$

$$\frac{\partial^2 u(t,0)}{\partial x^2} = a_1(t)u(t,0), \quad \frac{\partial^2 u(t,l)}{\partial x^2} = -a_2(t)u(t,l), \quad \frac{\partial u(t,0)}{\partial x} = 0, \quad 0 \leq t \leq T, \tag{29}$$

$$u(0,x) = u_0(x), \quad 0 < x < l, \tag{30}$$

where $a\in C[0,T]$ is a strictly positive coefficients, $\partial a/\partial t\in L_\infty(0,T)$, $a_i(t)\in C[0,T]$ are nonnegative coefficients, $\partial a_i/\partial t\in L_\infty(0,T)$, $i=1,\ 2$, and

$$\textit{if}\ \exists\, t_0\in[0,T],\ \ \textit{such that}\ \ a_1(t_0)=0\ (\textit{respectively},\ a_2(t_0)=0),$$
$$\textit{then}\ \ a_1\equiv 0\ (\textit{respectively},\ a_2\equiv 0). \tag{31}$$

1). The differential operators $-\widetilde{A}(t)=a(t)\partial^3/\partial x^3$ with the boundary conditions (29) are dissipative in $L_2(0,l)$. Their conjugates in $L_2(0,l)$ are given by the differential expressions $-\widetilde{A}^*(t)=-a(t)\partial^3/\partial x^3$ with the boundary conditions

$$\frac{\partial^2 v(t,0)}{\partial x^2}=-a_1(t)v(t,0),\quad \frac{\partial^2 v(t,l)}{\partial x^2}=a_2(t)v(t,l),\quad \frac{\partial v(t,l)}{\partial x}=0,\quad 0\le t\le T, \tag{32}$$

which are also dissipative in $L_2(0,l)$.

2). The operators $\widetilde{A}_0(t)=\widetilde{A}(t)+I$ are obviously coercive on $W_2^3(\Omega;\ \{\Gamma_j(t)\}_{J_1\cup J_1^-})$ in $L_2(\Omega)$ for $c_0=1$ and for any $t\in[0,T]$. Here we note that the squared norm $\|\widetilde{A}_0(t)w\|^2_{0,\Omega}$ is equal to

$$\|\widetilde{A}_0(t)w\|^2_{0,\Omega}=2a(t)a_1(t)|w(0)|^2+2a(t)a_2(t)|w(l)|^2+$$
$$+a(t)|\partial w(l)/\partial x|^2+a^2(t)\|\partial^3 w/\partial x^3\|^2_{0,\Omega}+\|w\|^2_{0,\Omega},\ \Omega=]0,l[.$$

The inequality (27) for $m=1$ is also true. By virtue of condition (31), we have the inequality (16) in the Theorem 5, where the squared norm $[\cdot]^2_{(t)}$ is equal to

$$\sqrt{Re(\widetilde{A}_0(t)u,u)_{0,\Omega}}=(a(t)a_1(t)|u(0)|^2+\ +a(t)a_2(t)|u(l)|^2+(a(t)/2)|\partial u(l)/\partial x|^2+\|u\|^2_{0,\Omega})^{1/2}.$$

The Conditions (10) and (13) on the continuation of functions from the boundary on the entire domain are obviously valid for a segment [5]. The verification of conditions I_3-III_3 is finished. Therefore, Theorem 6 implies the following assertion.

Theorem 7. *If* $0<a_0\le a(t)$, $a_1(t)\ge 0$, $a_2(t)\ge 0$, $a(t)\in C[0,T]$, $\partial a(t)/\partial t\in L_\infty(0,T)$, $a_i\in C[0,T]$, $\partial a_i/\partial t\in L_\infty(0,T)$, $i=1,2$, *and condition (31) is true, then for each functions* $f\in L_2(]0,T[,\ \ W_2^{-1}(]0,l[;(32)_t))$ *and* $u_0\in L_2(0,l)$ *there exists a unique weak solution* $u\in L_2(]0,T[\times]0,l[)$ *of the mixed problem (28)–(30).*

Here $W_2^{-1}(]0,l[;\ (32)_t)$ – are the antidual spaces of the Hilbert spaces $W_2^1(]0,l[;\ (32)_t)$, which are obtained by the closure of the set of all functions $u(t)\in W_2^3(0,l)$ satisfying the boundary conditions (32) in the norms $\langle\cdot\rangle_{(t)}$ which is equal to

$$\sqrt{Re(\widetilde{A}_0^*(t)v,\ v)_{0,\Omega}}=(a(t)a_1(t)|v(0)|^2+a(t)a_2(t)|v(l)|^2+(a(t)/2)|\partial v(0)/\partial x|^2+\|v\|^2_{0,\Omega})^{1/2}.$$

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