KRAUSZ DIMENSION AND ITS GENERALIZATIONS IN SPECIAL GRAPH CLASSES

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A *krausz* (k, m)*-partition* of a graph G is the partition of G into cliques, such that any vertex belongs to at most k cliques and any two cliques have at most m vertices in common. The *m-krausz* dimension $kdim_m(G)$ of the graph G is the minimum number k such that G has a krausz (k, m)-partition. 1-krausz dimension is known and studied krausz dimension of graph $kdim(G)$. In this paper we prove, that the problem "$kdim(G) \leq 3$" is polynomially solvable for chordal graphs, thus partially solving the problem of P. Hlineny and J. Kratochvil. We show, that the problem of finding m-krausz dimension is NP-hard for every $m \geq 1$, even if restricted to $(1, 2)$-colorable graphs, but the problem "$kdim_m(G) \leq k$" is polynomially solvable for $(\infty, 1)$-polar graphs for every fixed $k, m \geq 1$.

Key words: Krausz dimension, intersection graph, linear k-uniform hypergraph, chordal graph, polar graph.

INTRODUCTION

In this paper we consider finite undirected graphs without loops and multiple edges. The vertex and the edge sets of a graph (hypergraph) G are denoted by $V(G)$ and $E(G)$ respectively. Given a graph G, let $G(X)$ and $\overline{G}$ denote, respectively, the subgraph of G induced by a set $X \subseteq V(G)$ and the complement of G.

A *krausz partition* of a graph G is the partition of G into cliques (called *clusters* of the partition), such that every edge of G belongs to exactly one cluster. If every vertex of G belongs to at most k clusters then the partition is called *krausz k-partition*. The *krausz dimension kdim*(G) of the graph G is the minimum k such that G has krausz k-partition.

Krausz k-partitions are closely connected with the representation of a graph as the intersection graph of a hypergraph. The *intersection graph* $L(H)$ of a hypergraph $H = (V(H), E(H))$ is defined as follows:

1. the vertices of $L(H)$ are in a bijective correspondence with the edges of H;
2. two vertices are adjacent in $L(H)$ if and only if the corresponding edges have a nonempty intersection.

Hypergraph H is called *linear*, if any two of its edges have at most one common vertex; *k-uniform*, if every edge contains k vertices; *Helly hypergraph*, if for every family of hyperedges $E_1,...,E_r \in E(H)$ such that $E_i \bigcap E_i \neq \varnothing$ for every $i, j = 1,\dots, r$ we have $\bigcap_{i=1}^{r} E_i \neq \varnothing$.

The *multiplicity* of the pair of vertices u, v of the hypergraph H is the number $m(u,v) = \left|\{E \in E(H) : u,v \in E\}\right|$; the *multiplicity* $m(H)$ of the hypergraph H is the maximum multiplicity of the pairs of its vertices. So, linear hypergraphs are the hypergraphs with the multiplicity 1.

Denote by H^* the dual hypergraph of H and by $H_{[2]}$ the 2-section of H (i. e. the simple graph obtained by transformation each edge into a clique). It follows immediately from the definition that

$$L(H) = (H^*)_{[2]} \tag{1}$$

(first this relation was implicitly formulated by C. Berge in [2]). This relation implies that a graph G has krausz k-partition if and only if it is intersection graph of linear k-uniform hypergraph.

A graph is called (p, q)-*colorable*, if its vertex set could be partitioned into p cliques and q stable sets. In this terms (1, 1)-colorable graphs are well-known split graphs.

Another generalization of split graphs is the class of polar graphs. A graph G is called *polar* if there exists a partition of its vertex set

$$V(G) = A \bigcup B, \quad A \bigcap B = \varnothing \tag{2}$$

(*bipartition* (A, B)) such that all connected components of the graphs $\overline{G}(A)$ and $G(B)$ are complete graphs. If, in addition, α and β are fixed integers, and the orders of connected components of the graphs $\overline{G}(A)$ and $G(B)$ are at most α and β respectively, then the polar graph G with bipartition (2) is called (α, β)-*polar*. Given a polar graph G with bipartition (2), if the order of connected components of the graph $\overline{G}(A)$ (the graph $G(B)$) is not restricted above, then the parameter α (respectively β) is supposed to be equal ∞. Thus an arbitrary polar graph is (∞, ∞)-polar, and a split graph is (1, 1)-polar.

Denote by *KDIM*(k) the problem of determining whether $kdim(G) \leq k$ and by *KDIM* the problem of finding the krausz dimension.

The class of line graphs (intersection graphs of linear 2-uniform hypergraphs, i. e. graphs with krausz dimension at most 2) have been studied for a long time. It is characterized by a finite list of forbidden induced subgraphs [1], efficient algorithms for recognizing it (i. e. solving the problem *KDIM*(2)) and constructing the corresponding krausz 2-partition are also known (see for example [4], [7], [13], [14]).

The situation changes radically if one takes $k = 3$ instead of $k = 2$: the problem *KDIM*(k) is NP-complete for every fixed $k \geq 3$ [5]. The case $k = 3$ was studied in the different papers (see [6], [10], [11], [12], [15]), and several graph classes, where the problem *KDIM*(3) is polynomially solvable or remains NP-complete, were found.

In [5] P. Hlineny and J. Kratochvil studied the computational complexity of the krausz dimension in detail. In particular, they proved, that for chordal graphs the problem *KDIM*(k) is NP-complete for every $k \geq 6$.

So, P. Hlineny and J. Kratochvil posed the problem of deciding the complexity of *KDIM*(k) restricted to chordal graphs for $k = 3, 4, 5$. In this paper we give a partial answer to this problem (namely, in the case $k = 3$).

Further we consider the natural generalization of the krausz dimension. The *krausz (k, m)-partition* of a graph G is the partition of G into cliques (called *clusters* of the partition), such that any vertex belongs to at most k clusters of the partition, and any two clusters have at most m vertices in common. As above, the relation (1) implies, that graphs with krausz (k, m)-partitions are exactly the intersection graphs of k-uniform hypergraphs with the multiplicity at most m. The *m-krausz dimension* $kdim_m(G)$ of the graph G is the minimum k such that G has a krausz (k, m)-partition. The krausz dimension in these terms is the 1-krausz dimension. In this paper we present some computational complexity results concerning the m-krausz dimension of graph.

FORMULATION OF THE RESULTS

Denote by $lc(H)$ and $\Delta(H)$ the length of a longest induced cycle and the maximum vertex degree of a graph H, respectively.

Lemma 1. There exists a polynomial-time algorithm, which takes a chordal graph G as an input and constructs the graph H with $\Delta(H) \leq 18$ and $lc(H) \leq 6$ such that $kdim(G) \leq 3$ if and only if $kdim(H) \leq 3$.

P. Hlineny and J. Kratochvil proved in [5], that the problem *KDIM* is polynomially solvable for graphs with bounded treewidth. The following relation was proved in [3]: if $lc(H) \leq s + 2$ and $\Delta(H) \leq \Delta$, then $treewidth(H) \leq \Delta\,(\Delta - 1)^{s-1}$. These two facts together with Lemma 1 imply the following statement.

Theorem 2. The problem *KDIM*(3) is polynomially solvable for chordal graph.

Denote by $KDIM_m$ the problem of determining the m-krausz dimension of graph, by $KDIM_m\,(k)$ the problem of determining whether $kdim_m(G) \leq k$ and by L_k^m the class of graphs with a krausz (k, m)-partition. It was proved in [8] that the class L_3^m could not be characterized by a finite set of forbidden induced subgraphs for every $m \geq 2$, but the complexity of the problem $KDIM_m$ for an arbitrary m was not established yet. We proved the following:

Theorem 3. The problem $KDIM_m$ is *NP*-hard for every $m \geq 1$, even if restricted to the class of (1, 2)-colorable graphs.

The class L_k^m is hereditary (i. e. closed with respect to deleting the vertices) and therefore can be characterized by the infinite list of forbidden induced subgraphs. We proved the following:

Theorem 4. There exists a finite set F of forbidden induced subgraphs such that an $(\infty, 1)$-polar graph G belongs to the class L_k^m if and only if no induced subgraph of G is isomorphic to an element of F.

Corollary 5. The problem $KDIM_m(k)$ is polynomially solvable in the class of $(\infty, 1)$-polar graphs for every fixed $k, m \geq 1$.

In particular, Corollary 5 generalizes the result of [5] and [9], that for every fixed k the problem *KDIM*(k) is polynomially solvable for split graphs.

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