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ОБ ОДНОМ КЛАССЕ ИНТЕРПОЛЯЦИОННЫХ МНОГОЧЛЕНОВ ДЛЯ НЕЛИНЕЙНЫХ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ ОПЕРАТОРОВ

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Статья посвящена построению интерполяционных формул Лагранжа и формул эрмитова типа с узлами второй кратности для обыкновенных дифференциальных операторов произвольного порядка, заданных в пространстве непрерывно дифференцируемых функций. Полученные формулы содержат интегралы Стилтьеса и дифференциалы Гато интерполируемого оператора. Эти формулы инвариантны относительно операторных многочленов специального класса. Построение операторных интерполяционных формул основано на интерполяционных полиномах для скалярных функций.

Ключевые слова: операторное интерполирование, операторный многочлен, интерполяция типа Лагранжа и Эрмита, погрешность интерполяции.

ON A CLASS OF INTERPOLATION POLYNOMIALS FOR NONLINEAR ORDINARY DIFFERENTIAL OPERATORS

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This article is devoted to the construction of Lagrange interpolation formulas and formulas of Hermite type with knots of second multiplicity for ordinary differential operators of arbitrary order given in the space of continuously differentiable functions. Obtained formulas contain Stieltjes integrals and differentials Gateaux of interpolated operator. These formulas are invariant for the operator polynomials of a special class. The construction of operator interpolation formulas is based on interpolation polynomials for scalar functions.

Key words: operator interpolation, operator polynomial, Lagrange and Hermite type interpolation, interpolation error.

Introduction

Operator interpolation, as one of the sections of applied functional analysis and the general theory of approximate methods, significantly generalizes the problem of function interpolation and is the foundation for construction of approximate methods and algorithms to solve applied problems. The main problems of operator interpolation are formulated similarly to classical problems of function interpolation.

We consider the ordinary differential operators $F: C^{(n)}(T) \to Y$ of the form

$$F(x) = f\left(t, x(t), x'(t), x''(t), ..., x^{(m)}(t)\right), \tag{1}$$

where $x^{(k)}(t) = \dfrac{d^k x(t)}{dt^k}$ $(k = 0,1,...,m)$, $C^{(m)}(T)$ is the space of continuously differentiable m times on $T \subseteq R$ functions $x(t)$, the function $y = f(t, u_0, u_1, ..., u_m)$ of variables t_0 u_0, $u_1, ..., u_m$ is defined on a rectangle $\Omega = T \times T_0 \times T_1 \times \cdots \times T_m$, T_i are sets of the number line $(i = 0,1,...,m)$, and Y is a function space.

Under Lagrange operator interpolation as in the case of scalar functions it is understood interpolation, at which interpolating polynomial coincides with interpolated operator at given knots. First we consider the formulas of this class.

1. Lagrange Interpolation Formulas

Let $X = X(T)$ be given space of smooth on $T \subseteq R$ functions. We consider operator polynomials of the form

$$P_n(x) = \sum_{k=0}^{n} a_{nk}(s,t)\phi_k(x(t)), \tag{2}$$

where $a_{nk}(s,t)$ are arbitrarily given functions of the variables s and t, and $\{\phi_k(t)\}_{k=0}^{n}$ is some Chebyshev system of functions defined on the number set T.

In the work [1] we have constructed an operator polynomial containing differentials Gateaux of interpolated operator of the form

$$L_n(F;x) = F(x_0) + \sum_{k=1}^{n} \int_0^1 \delta F[x_0 + \tau(x_k - x_0); \frac{l_{n,k}(x)}{\sigma_n(x)}(x_k - x_0)]d\tau, \tag{3}$$

where $l_{n,k}(x)$ are polynomials of the n-degree of the form (2), $l_{n,k}(x_j) = \delta_{kj}$ is the Kronecker symbol $(k, j = 0,1,...,n)$, and $\sigma_n(x) = \sum_{k=0}^{n} l_{n,k}(x)$ is a constant or a variable value. In particular, if $\left\{\phi_k(x) = x^k\right\}_{k=0}^{n}$ is algebraic system of functions, then the fundamental polynomials $l_{n,k}(x)$ are defined by the rule

$$l_{n,k}(x) = \frac{\omega_n(x)}{\omega'_n(x_j)(x - x_k)}, \qquad k = 0,1,...,n,$$

where $\omega_n(x)=\prod_{j=0}^{n}(x-x_j)$, the sum $\sigma_n(x)\equiv 1$, and from the interpolation nodes it is required that $x_i(t)-x_j(t)\neq 0$ for all $t\in T$.

Polynomial (3) is an interpolation operator for a given on the set *X* operator *F*(*x*) relative to the knots $x_0,x_1,...,x_n$. The formula (3) is invariant for the operator polynomials of the form

$$Q_n(x)=a_0(s)+\sum_{j=0}^{N}\sum_{i=0}^{n}\int_T a_{ij}(s,t)\frac{d^j}{dt^j}\left\{\frac{\phi_i(x(t))}{\sigma_n(x(t))}\right\}dt,$$

where $a_0(s),\ a_{ij}(s,t)$ are some given functions, N is fixed integer number. For interpolation error $r_n(x)=F(x)-L_n(F;x)$, where $L_n(F;x)$ is interpolation polynomial (3), the following representation holds:

$$r_n(x)=\sum_{k=1}^{n+1}\int_0^1\delta F[x_0+\tau(x_k-x_0);\left(\frac{l_{n+1,k}(x)}{\sigma_{n+1}(x)}-\frac{l_{n,k}(x)}{\sigma_n(x)}\right)(x_k-x_0)]d\tau\ , \tag{4}$$

where $x_{n+1}=x,\ l_{n,n+1}(x)\equiv 0.$

On base of the formula (3) we construct the interpolation polynomial for the operator (1). The differential operator (1) depends on one functional variable $x(t)$, for it the Gateaux differential $\delta F[x;h]$ at the point $x=x(t)$ in the direction $h=h(t)$ $(x,h\in C^{(m)}(T))$ is calculated according to the rule

$$\delta F[x;h]=\frac{\partial f}{\partial x}h(t)+\frac{\partial f}{\partial x'}h'(t)+\cdots+\frac{\partial f}{\partial x^{(m)}}h^{(m)}(t)=$$

$$=\sum_{k=0}^{m}\frac{\partial}{\partial x^{(k)}}f\left(t,x(t),x'(t),x''(t),...,x^{(m)}(t)\right)h^{(k)}(t). \tag{5}$$

Consequently, for the operators (1) the formula (3) is transformed to the form

$$L_n(F;x)=f\left(t,x_0(t),x'_0(t),x''_0(t),...,x_0^{(m)}(t)\right)+$$

$$+\sum_{k=1}^{n}\int_0^1\sum_{\nu=0}^{m}\frac{\partial}{\partial\upsilon_k^{(\nu)}}f\left(t,\upsilon_k(t,\tau),\frac{\partial}{\partial t}\upsilon_k(t,\tau),...,\frac{\partial^m}{\partial t^m}\upsilon_k(t,\tau)\right)\times \tag{6}$$

$$\times\frac{\partial^\nu}{\partial t^\nu}\left\{\frac{l_{n,k}(x(t))}{\sigma_n(x(t))}(x_k(t)-x_0(t))\right\}d\tau,$$

where the function $\upsilon_k=\upsilon_k(t,\tau)=x_0(t)+\tau\left(x_k(t)-x_0(t)\right),\ k=1,2,...,n.$ Error of interpolation of operator $F(x)$ by the polynomial (6) can be calculated on base of the formula (4) according to the rule

$$r_n(x) = \sum_{k=1}^{n+1} \int_0^1 \sum_{\nu=0}^{m} \frac{\partial}{\partial \upsilon_k^{(\nu)}} f\left(t, \upsilon_k(t,\tau), \frac{\partial}{\partial t}\upsilon_k(t,\tau), \ldots, \frac{\partial^m}{\partial t^m}\upsilon_k(t,\tau)\right) \times$$

$$\times \frac{\partial^\nu}{\partial t^\nu}\left\{\left(\frac{l_{n+1,k}(x(t))}{\sigma_{n+1}(x(t))} - \frac{l_{n,k}(x(t))}{\sigma_n(x(t))}\right)(x_k(t) - x_0(t))\right\} d\tau,$$

where $x_{n+1} = x,\ l_{n,n+1}(x) \equiv 0,$ as well as earlier.

Example. Let us consider the differential operator

$$F(x) = f\left(t, x(t), x'(t), x''(t)\right) = x''(t) + \alpha x'(t) + \beta x^p(t) + \gamma x(t), \tag{7}$$

where p is fixed non-negative integer number. We construct a linear interpolation polynomial $L_1(F;x)$ of the form (6) for the operator (7). As fundamental interpolation polynomials $l_{1,k}(x)$, $k = 0,1,$ we choose the algebraic polynomials $l_{1,0}(x) = \dfrac{x - x_1}{x_0 - x_1}$, $l_{1,1}(x) = \dfrac{x - x_0}{x_1 - x_0}$, and we take as interpolation nodes $x_k(t)$, $k = 0,1,$ the system of functions $x_0(t) \equiv 2,\ x_1(t) = \sin t$. In this case the function $\upsilon_1(t,\tau) = 2 + \tau(\sin t - 2)$, the sum $\sigma_1(x) \equiv 1$, and for the operators (7) the formula (6) takes the form

$$L_1(F;x) = f\left(t,2,0,0\right) + \int_0^1 \sum_{\nu=0}^{2} \frac{\partial}{\partial \upsilon_1^{(\nu)}} f\left(t, \upsilon_1(t,\tau), \frac{\partial}{\partial t}\upsilon_1(t,\tau), \frac{\partial^2}{\partial t^2}\upsilon_1(t,\tau)\right) \times$$

$$\times \frac{\partial^\nu}{\partial t^\nu}\left\{x(t) - 2\right\} d\tau = 2^p\beta + 2\gamma + \int_0^1 \left\{\left(\beta p \upsilon_1^{p-1}(t,\tau) + \gamma\right)\left(x(t) - 2\right) + \alpha x'(t) + x''(t)\right\} d\tau.$$

As $\upsilon_1^{p-1}(t,\tau) = \left(2 + \tau(\sin t - 2)\right)^{p-1} = 2^{p-1} \sum_{k=0}^{p-1} C_{p-1}^k \dfrac{\tau^k}{2^k}(\sin t - 2)^k$, then obtained polynomial is transformed to the form

$$L_1(F;x) = 2^p\beta + 2\gamma + \int_0^1 \left(2^{p-1}\beta p \sum_{k=0}^{p-1} C_{p-1}^k \frac{\tau^k}{2^k}(\sin t - 2)^k + \gamma\right) \times$$

$$\times\left(x(t) - 2\right) + \alpha x'(t) + x''(t) d\tau = 2^p\beta + 2^{p-1}\beta \times$$

$$\times \sum_{k=0}^{p-1} \frac{p!(\sin t - 2)^k}{(k+1)!(p-1-k)!2^k}\left(x(t) - 2\right) + \gamma x(t) + \alpha x'(t) + x''(t).$$

After some transformations for the operator (7) we obtain the formula of linear interpolation

$$F(x) \approx L_1(F;x) = x''(t) + \alpha x'(t) + \left(\beta \frac{\sin^p t - 2^p}{\sin t - 2} + \gamma\right) x(t) - 2\beta \frac{\sin^p t - 2^p}{\sin t - 2} + 2^p\beta,$$

for which the following equalities hold: $L_1(F;x_i)=F(x_i),\ \ i=0,1$. The error of interpolation is represented as

$$r_1(x)=F(x)-L_1(F;x)=\beta\left\{x^p(t)-\frac{\sin^p t-2^p}{\sin t-2}x(t)+2\frac{\sin^p t-2^p}{\sin t-2}-2^p\right\}.$$

2. Interpolation Formulas of Hermite Type

In the work [2] we have constructed an operator polynomial containing differentials Gateaux of interpolated operator of the form

$$H_{2n+1}(F;x)=F(x_0)+\sum_{k=0}^{n}\delta F[x_k;q_{n,k}(x)]+\sum_{k=1}^{n}\int_0^1\delta F[x_0+\tau(x_k-x_0);h_{n,k}(x)(x_k-x_0)]d\tau, \quad (8)$$

where $h_{n,k}(x)$ and $q_{n,k}(x)$ are fundamental Hermite polynomials in the case of second multiplicity knots $x_0,x_1,...,x_n$ with respect to Chebyshev system $\{\phi_k(t)\}_{k=0}^{2n+1}$ of functions, for which $h_{n,k}(x_j)=q'_{n,k}(x_j)=\delta_{kj}$, $h'_{n,k}(x_j)=q_{n,k}(x_j)=0 \quad (k,j=0,1,...,n)$. For example, if $\left\{\varphi_k(x)=x^k\right\}_{k=0}^{2n+1}$ is algebraic system of functions, then fundamental interpolation polynomials $h_{n,k}(x)$ and $q_{n,k}(x)$ are defined by the rules

$$h_{n,k}(x)=l_{n,k}^2(x)\left[1-\frac{\omega''_n(x_k)}{\omega'_n(x_k)}(x-x_k)\right], \qquad q_{n,k}(x)=l_{n,k}^2(x)(x-x_k),$$

where k=0,1,…,n, and $l_{n,k}(x)$ are the algebraic polynomials from example, the sum $\sigma_n(x)\equiv 1$. For the polynomial (8) the following interpolation conditions take true:

$$H_{2n+1}(F;x_k)=F(x_k), \qquad H'_{2n+1}(F;x_k)=F'(x_k) \quad (k=0,1,...,n).$$

If $\sigma_n(x)=\sum_{k=0}^{n}h_{n,k}(x)$ is constant value, then the formula (8) is invariant with respect to operator polynomials of the form

$$Q_{2n+1}(x)=a_0(s)+\sum_{j=0}^{N}\sum_{i=0}^{2n+1}\int_T a_{ij}(s,t)\frac{d^j}{dt^j}\varphi_i(x(t))dt,$$

where $a_0(s)$, $a_{ij}(s,t)$ are some given functions, N is fixed integer number. For interpolation error $r_{2n+1}(x)=F(x)-H_{2n+1}(F;x)$, where $H_{2n+1}(F;x)$ is interpolation polynomial (8), the following representation holds:

$$r_{2n+1}(x)=\sum_{k=1}^{n+1}\int_0^1\delta F[x_0+\tau(x_k-x_0);\left(h_{n+1,k}(x)-h_{n,k}(x)\right)(x_k-x_0)]d\tau+$$

$$+\sum_{k=0}^{n+1}\delta F[x_k;q_{n+1,k}(x)-q_{n,k}(x)], \quad (9)$$

where $x_{n+1} = x,\ \ h_{n,n+1}(x) = q_{n,n+1}(x) \equiv 0.$

We construct the interpolation polynomial for the operator (1) on base of the equality (8). Bearing in mind the rule (5), formula (8) for the operators (1) takes the form

$$H_{2n+1}(F;x) = f\left(t, x_0(t), x_0'(t), x_0''(t), \ldots, x_0^{(m)}(t)\right) +$$

$$+\sum_{k=0}^{n}\sum_{\nu=0}^{m}\frac{\partial}{\partial x_k^{(\nu)}} f\left(t, x_k(t), x_k'(t), x_k''(t), \ldots, x_k^{(m)}(t)\right)\frac{\partial^\nu}{\partial t^\nu} q_{n,k}(x(t)) + \tag{10}$$

$$+\sum_{k=1}^{n}\int_0^1\sum_{\nu=0}^{m}\frac{\partial}{\partial \upsilon_k^{(\nu)}} f\left(t, \upsilon_k(t,\tau), \frac{\partial}{\partial t}\upsilon_k(t,\tau), \ldots, \frac{\partial^m}{\partial t^m}\upsilon_k(t,\tau)\right)\frac{\partial^\nu}{\partial t^\nu}\left\{h_{n,k}(x(t))(x_k(t) - x_0(t))\right\} d\tau$$

where $\upsilon_k = \upsilon_k(t,\tau) = x_0(t) + \tau\left(x_k(t) - x_0(t)\right)$, $k = 1,2,\ldots,n,$ as well as earlier.

We may obtain submission of interpolation error of operator $F(x)$ by polynomial (10) on base of the equality (9). We have

$$r_{2n+1}(x) = \sum_{k=1}^{n+1}\int_0^1\sum_{\nu=0}^{m}\frac{\partial}{\partial \upsilon_k^{(\nu)}} f\left(t, \upsilon_k(t,\tau), \frac{\partial}{\partial t}\upsilon_k(t,\tau), \ldots, \frac{\partial^m}{\partial t^m}\upsilon_k(t,\tau)\right)\times$$

$$\times\frac{\partial^\nu}{\partial t^\nu}\left\{\left(h_{n+1,k}(x) - h_{n,k}(x)\right)(x_k(t) - x_0(t))\right\} d\tau +$$

$$+\sum_{k=0}^{n+1}\sum_{\nu=0}^{m}\frac{\partial}{\partial x_k^{(\nu)}} f\left(t, x_k(t), x_k'(t), x_k''(t), \ldots, x_k^{(m)}(t)\right)\frac{\partial^\nu}{\partial t^\nu}\left(q_{n+1,k}(x(t)) - q_{n,k}(x(t))\right),$$

where $x_{n+1} = x,\ \ h_{n,n+1}(x) = q_{n,n+1}(x) \equiv 0.$

Conclusion

A number of other interpolation formulas is given in the works [3-10]. The monography [3] deals with the problems of solvability of operator interpolation; construction of miscellaneous types of interpolation formulas (Lagrange, Newton, Hermite, Hermite-Birkhoff and others) in general linear, Hilbert and functional spaces, including functions of matrix variables and abstract functions of scalar argument; description of the general structure of interpolation polynomials; study of approximation errors in the formulas obtained and their possible applications to approximate solution of special classes of problems.

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