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THE PRODUCT OF GENERALIZED SUBNORMAL SUBGROUPS IN FINITE GROUPS

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Abstract: We study the structure of some classes of finite groups with a given system of commuting generalized subnormal subgroups. In particular, our results yield a characterization of superradical and Shemetkov formations.

Keywords: finite group, formation, saturated formation, superradical formation, Shemetkov formation

This article deals exclusively with finite groups.

Wielandt's theory of subnormal subgroups greatly influenced the development of the whole finite group theory. Shemetkov introduced the concept of an $\mathfrak{F}$-subnormal subgroup which is a natural generalization of subnormality in his book [1], where he also put forth the main problems of the theory of $\mathfrak{F}$-subnormal subgroups.

Recall some concepts. A *formation* is a class of groups closed under quotients and subdirect products. A formation is called *saturated* whenever it is closed with respect to Frattini extensions. We denote the intersection of all normal subgroups of G the quotients by which belong to $\mathfrak{F}$ by $G^{\mathfrak{F}}$, and call it the $\mathfrak{F}$-*coradical* of G.

Take some nonempty formation $\mathfrak{F}$. A subgroup K of G is called $\mathfrak{F}$-*subnormal* whenever either $K = G$ or there exists a maximal chain

$$G = K_0 \supset K_1 \supset \dots \supset K_n = K$$

such that $(K_{i-1})^{\mathfrak{F}} \subseteq K_i$ for all $i = 1, 2, \dots, n$.

If $\mathfrak{F} = \mathfrak{N}$ then in every soluble group G the set of all $\mathfrak{F}$-subnormal subgroups coincides with the set of subnormal subgroups of G. However, this fails for arbitrary groups.

Kegel introduced another generalization of the concept of $\mathfrak{F}$-subnormality in [2]. A subgroup H is called $\mathfrak{F}$-*subnormal in the sense of Kegel* or $\mathfrak{F}$-*accessible* whenever there exists a chain of subgroups

$$G = H_0 \supseteq H_1 \supseteq \dots \supseteq H_m = H$$

such that for every $i = 1, 2, \dots, m$ either H_i is normal in H_{i-1} or $(H_{i-1})^{\mathfrak{F}} \subseteq H_i$.

Observe that for every nonempty formation $\mathfrak{F}$ the set of all $\mathfrak{F}$-accessible subgroups of an arbitrary group G includes the set of all subnormal subgroups of G and the set of all $\mathfrak{F}$-subnormal subgroups of G. However, if $\mathfrak{F}$ is a nonempty nilpotent formation then the set of all $\mathfrak{F}$-accessible subgroups coincides precisely with the set of all subnormal subgroups for every group G.

As in the theory of subnormal subgroups, the development of the theory of $\mathfrak{F}$-subnormal and $\mathfrak{F}$-accessible subgroups started with the question: *In which cases the subgroup generated by two $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroups is also an $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroup?*

Shemetkov and Kegel posed this problem in 1978. The articles [3, 4] are devoted to its solution, and their results are presented in [5]. The subsequent development of these results is related to considering the products of commuting $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroups. The class of formations under study extends considerably if we replace the condition of the generation by $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroups by the weaker condition of being the product of commuting $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroups. In this case the problem of Shemetkov and Kegel extend as follows:

Problem 1. *Classify the saturated formations $\mathfrak{F}$ such that for every group G and all commuting $\mathfrak{F}$-accessible subgroups H and K of G the subgroup HK is $\mathfrak{F}$-accessible in G.*

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Problem 2. *Classify the saturated formations $\mathfrak{F}$ such that for every group G and all commuting $\mathfrak{F}$-subnormal subgroups H and K of G the subgroup HK is $\mathfrak{F}$-subnormal in G.*

In the Kourovka Notebook [6] Shemetkov posed the problem of classifying superradical formations. Recall that a formation $\mathfrak{F}$ is called *superradical* whenever it satisfies the following requirements:

(1) $\mathfrak{F}$ is a normally hereditary formation;

(2) every group $G = AB$, where A and B are $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups in G, belongs to $\mathfrak{F}$.

In this article we prove the equivalence of the above problems in the case that $\mathfrak{F}$ is a saturated hereditary formation.

The attempts to solve these and other classification problems revealed a special role of critical groups; these groups lie outside the class $\mathfrak{F}$, but all their proper subgroups belong to $\mathfrak{F}$. Refer to a group of this type as *minimal non-$\mathfrak{F}$-groups*, and denote there collection by $\mathscr{M}(\mathfrak{F})$.

In the Kourovka Notebook [6] Shemetkov posed the problem of classifying the saturated formations whose minimal non-$\mathfrak{F}$-groups are either Schmidt groups or groups of prime order. These formations are now called *Shemetkov formations*. In this article we obtain new characterizations of superradical formations and Shemetkov formations. We use standard definitions and notation [1, 7].

In the following lemmas we collect the available properties of $\mathfrak{F}$-subnormal and$\mathfrak{F}$-accessible subgroups needed for proving the main results of the article.

Lemma 1. *Take some nonempty hereditary formation $\mathfrak{F}$. Then*

(1) If K is a subgroup of G and $G^{\mathfrak{F}} \subseteq K$ then K is $\mathfrak{F}$-subnormal in G;

(2) If H is an $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroup of G then $H \cap K$ is an $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroup of K for every subgroup K of G;

(3) If H is an $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroup of K and K is an $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroup of G then H is an $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroup of G;

(4) If H_1 and H_2 are $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroups of G then $H_1 \cap H_2$ is an $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroup of G;

(5) If all composition factors of a group G belong to $\mathfrak{F}$ then every subnormal subgroup of G is $\mathfrak{F}$-subnormal;

(6) If H is an $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroup of G then H^x is $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) in G for every $x \in G$.

Lemma 2. *Take some nonempty formation $\mathfrak{F}$, some subgroup H of G, and some normal subgroup N in G. Then*

(1) If H is an $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroup of G then HN is an $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroup of G, while HN/N is an $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroup of G/N;

(2) If $N \subseteq H$ then H is $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) in G if and only if H/N is an $\mathfrak{F}$-subnormal ($\mathfrak{F}$-accessible) subgroup of G/N.

Lemma 3. *Take some nonempty hereditary formation $\mathfrak{F}$. If A is an $\mathfrak{F}$-accessible subgroup of G then $A^{\mathfrak{F}}$ is a subnormal subgroup of G.*

The proof of Lemmas 1–3 goes by simple verification.

Lemma 4. *Take some nonempty hereditary formation $\mathfrak{F}$. If $\mathfrak{H}$ is the formation of all groups whose composition factors belong to $\mathfrak{F}$ then $A^{\mathfrak{H}} = G^{\mathfrak{H}}$ for every $\mathfrak{F}$-subnormal subgroup A of a group G.*

Proof. By definition there exists a maximal chain $G = G_0 \supset G_1 \supset \dots \supset G_t = A$ such that $G_i \supseteq (G_{i-1})^{\mathfrak{F}}$ for all $i = 1, 2, \dots, t$. Since $G_{i-1}/(G_{i-1})^{\mathfrak{F}} \in \mathfrak{F}$, the heredity of $\mathfrak{F}$ implies that $G_i/(G_{i-1})^{\mathfrak{F}} \in \mathfrak{F}$. Hence, $(G_i)^{\mathfrak{F}} \subseteq (G_{i-1})^{\mathfrak{F}}$ for all $i = 1, 2, \dots, t$. Since $(G_i)^{\mathfrak{F}} \triangleleft G_i$ and $(G_i)^{\mathfrak{F}} \subseteq (G_{i-1})^{\mathfrak{F}} \subseteq G_i$, we have $(G_i)^{\mathfrak{F}} \triangleleft (G_{i-1})^{\mathfrak{F}}$. Since $\mathfrak{F}$ is a hereditary formation, so is $\mathfrak{H}$. Moreover, $\mathfrak{F} \subseteq \mathfrak{H}$. Therefore, $G_i/(G_i)^{\mathfrak{H}} \in \mathfrak{H}$ and $(G_i)^{\mathfrak{F}} \subseteq (G_{i-1})^{\mathfrak{F}}$ imply that $(G_{i-1})^{\mathfrak{F}}/(G_i)^{\mathfrak{F}} \in \mathfrak{H}$. Hence, the subgroup $A^{\mathfrak{H}}$ is subnormal in G, and in the segment from $A^{\mathfrak{H}}$ to G all composition factors belong to $\mathfrak{F}$. By Lemma 2 of [8] $A^{\mathfrak{H}}$ includes $G^{\mathfrak{H}}$. The inclusion $A^{\mathfrak{H}} \subseteq G^{\mathfrak{H}}$ follows from the heredity of $\mathfrak{H}$. The proof of the lemma is complete.

Lemma 5. *Take some nonempty hereditary formation $\mathfrak{F}$. If all composition factors of a group G belong to $\mathfrak{F}$ then the following are equivalent:*

(1) *H is an $\mathfrak{F}$-subnormal subgroup of G;*

(2) *H is an $\mathfrak{F}$-accessible subgroup of G.*

PROOF. Take some $\mathfrak{F}$-subnormal subgroup H of G. Then by definition H is an $\mathfrak{F}$-accessible subgroup of G. Take some $\mathfrak{F}$-accessible subgroup H of G. Then there exists a chain

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_t = H \qquad (*)$$

in which for every $i = 1, 2, \dots, t$ either G_i is normal in G_{i-1} or $(G_{i-1})^{\mathfrak{F}} \subseteq G_i$.

Suppose that $(G_{i-1})^{\mathfrak{F}} \subseteq G_i$. Refine the segment from G_i to G_{i-1} in the chain $(*)$ to a maximal $(G_{i-1} - G_i)$-chain.

By claim 1 of Lemma 1 all subgroups G_{i-1} that include G_i are $\mathfrak{F}$-subnormal in G_{i-1}. Suppose now that G_i is normal in G_{i-1}. We can assume that G_i is a maximal normal subgroup of G_{i-1}; otherwise, refine the segment from G_i to G_{i-1} to a composition $(G_{i-1} - G_i)$-chain. The hypotheses of the lemma yield $G_{i-1}/G_i \in \mathfrak{F}$, and so $(G_{i-1})^{\mathfrak{F}} \subseteq G_i$. We arrive at the case considered above. By claim 3 of Lemma 1 the subgroup H is $\mathfrak{F}$-subnormal in G. The proof of the lemma is complete.

A Shemetkov formation $\mathfrak{F}$ is called an *$\mathfrak{S}$-Shemetkov formation* whenever each soluble minimal non-$\mathfrak{F}$-group is either a Schmidt group or a group of prime order.

Lemma 6 [9]. *Take some saturated formation $\mathfrak{F}$ and some hereditary local formation $\mathfrak{H}$. If $G \in \mathscr{M}(\mathfrak{F})$ and $G/K \in \mathscr{M}(\mathfrak{H})$, where $K \subseteq \Phi(G)$, then $G \in \mathscr{M}(\mathfrak{H})$.*

Lemma 7. *Each nonempty saturated formation $\mathfrak{F}$ such that for every group G and all commuting $\mathfrak{F}$-subnormal subgroups H and K of G the subgroup HK is $\mathfrak{F}$-subnormal in G is an $\mathfrak{S}$-Shemetkov formation.*

PROOF. Take some saturated formation $\mathfrak{F}$ such that for each group G and all commuting $\mathfrak{F}$-subnormal subgroups H and K of G the subgroup HK is $\mathfrak{F}$-subnormal in G. Take an arbitrary soluble minimal non-$\mathfrak{F}$-group G. If $G \notin \mathfrak{S}_{\pi(\mathfrak{F})}$ then it is not difficult to observe that G is a group of prime order q with $q \notin \pi(\mathfrak{F})$.

Suppose that $G \in \mathfrak{S}_{\pi(\mathfrak{F})} \cap \mathscr{M}(\mathfrak{F})$. Consider the case that $\Phi(G) = 1$. Theorem 1.5 of [10] implies that $G = N \rtimes M$ for the unique minimal normal subgroup N in G, which is a p-group, and $M \in \mathscr{M}(f(p))$, where f is a maximal inner local screen of $\mathfrak{F}$. It is obvious that $C_G(N) = N$.

In order to show that M has a unique class of maximal conjugate subgroups, assume the contrary. Take two maximal nonconjugate subgroups M_1 and M_2 in M. Then $M = M_1M_2$ by Ore's theorem. Since $N = G^{\mathfrak{F}}$, it is obvious that NM_1 and NM_2 are $\mathfrak{F}$-maximal normal $\mathfrak{F}$-subgroups of G. Lemma 1 implies that M_1 and M_2 are $\mathfrak{F}$-subnormal subgroups of G. Then by hypothesis M is an $\mathfrak{F}$-subnormal subgroup of G. We have arrived at a contradiction. Thus, M has a unique class of maximal conjugate subgroups. Hence, M is a cyclic q-subgroup. Since $\mathfrak{F}$ is a saturated formation and $G \notin \mathfrak{F}$, we have $q \neq p$.

To show now that $|M| = q$, assume the contrary and take $|M| = q^n$ with $n > 1$. Take two cyclic groups E and L of order q^{n-1} and q respectively. Denote by T the regular wreath product $EwrL$. Denote by K the base of the wreath product: $T = K \rtimes L$. Since a subgroup of T is isomorphic to M, $T \notin f(p)$. It is obvious that K and L belong to the formation $f(p)$.

Put $R = PwrT$, where $|P| = p$. Denote by C the base of the wreath product R. Then $R = C \rtimes T = C \rtimes (K \rtimes L)$. Since $R/C \simeq K \rtimes L \in \mathfrak{F}$, we have $R^{\mathfrak{F}} \subseteq C$. Hence, the subgroups CK and CL are $\mathfrak{F}$-subnormal in R. It is easy to see that $CK \in \mathfrak{F}$ and $CL \in \mathfrak{F}$.

By Lemma 1 it is not difficult to show that K and L are $\mathfrak{F}$-subnormal in R. By hypothesis the subgroup KL is also $\mathfrak{F}$-subnormal in R, which is impossible.

The resulting contradiction shows that $n = 1$. Consequently, G is a Schmidt group. Thus, we have showed that $G/\Phi(G)$ is a Schmidt group. Lemma 6 implies that G is a Schmidt group and so $\mathfrak{F}$ is an $\mathfrak{S}$-Shemetkov formation. The proof of the lemma is complete.

Lemma 8. *Each nonempty saturated normally hereditary soluble formation $\mathfrak{F}$ such that for every group G and all commuting $\mathfrak{F}$-subnormal subgroups H and K of G the subgroup HK is $\mathfrak{F}$-subnormal in G is a hereditary formation.*

PROOF. Verify that $\mathfrak{F} = \mathfrak{F}^S$, where $\mathfrak{F}^S$ is the maximal hereditary subformation in $\mathfrak{F}$. Suppose that $\mathfrak{F} \setminus \mathfrak{F}^S$ is nonempty and choose in $\mathfrak{F} \setminus \mathfrak{F}^S$ a group G of the least order. Theorem 1 of [11] and Theorem 4.3 of [1] imply that $\mathfrak{F}^S$ is a saturated formation. Therefore, $\Phi(G) = 1$. It is obvious that G has a unique minimal normal subgroup N, and that $C_G(N) = F(G) = N$. Since $G \notin \mathfrak{F}^S$, G includes a minimal non-$\mathfrak{F}^S$-group H. The normal heredity of $\mathfrak{F}$ implies that $|\pi(H)| > 1$. It is clear that H is also a minimal non-$\mathfrak{F}$-group.

By hypothesis, H is a Schmidt group. In this case $H = RQ$, where Q is a normal Sylow q-subgroup, while R is a cyclic r-subgroup of H, with q and r being distinct primes.

If $H \cap N = 1$ then

$$HN/N \simeq H/H \cap N \simeq H \in \mathfrak{F}^S \subseteq \mathfrak{F}.$$

We have arrived at a contradiction with the choice of H. It remains to accept that $K = H \cap N \neq 1$. Together with $H/K \in \mathfrak{F}$ this implies that $K \subseteq Q$; hence, N is a q-group. Consider $H^* = H/\Phi(H)$. Then we can represent H^* as

$$H^* = Q\Phi(H)/\Phi(H) \leftthreetimes R\Phi(H)/\Phi(H),$$

where $Q\Phi(H)/\Phi(H)$ is an elementary abelian q-group, while $|R\Phi(H)/\Phi(H)| = r$. Since H^* lies outside $\mathfrak{F}$, Lemma 4.5 of [1] implies that

$$H^*/F_q(H^*) \simeq Z_r \notin f(q),$$

where f is a maximal inner local screen of $\mathfrak{F}$. Since $C_G(N) = N$ and $N = F_q(G)$; therefore, $F_q(G)$ is a q-group. Hence,

$$r \in \pi(G/N) \subseteq \pi(f(q)).$$

By Theorem 4.7 of [1] the normal heredity of $\mathfrak{F}$ implies that $f(q)$ is a normally hereditary formation. It is not difficult to show that $Z_r \in f(q)$. We have arrived at a contradiction. Therefore, $\mathfrak{F} = \mathfrak{F}^S$. The proof of the lemma is complete.

Given a set π of primes, denote by $\mathfrak{G}_\pi$ the class of all π-groups. Recall that a π-closed group is a group possessing a normal π-Hall subgroup; a π-special group is a group possessing a nilpotent normal π-Hall subgroup; a π-decomposable group is a group which is simultaneously π-special and π'-closed; a π-nilpotent group is a group which includes a normal p'-Hall subgroup for every $p \in \pi$.

In the following theorem I stands for some set of ordered pairs of nonnegative integers.

Theorem 1. *Every formation of the form $\mathfrak{F} = \bigcap_{(i,j)\in I} \mathfrak{G}_{\pi_i}\mathfrak{G}_{\pi_j}$ enjoys the property that for every group G and all commuting $\mathfrak{F}$-subnormal subgroups H and K of G the subgroup HK is $\mathfrak{F}$-subnormal in G.*

PROOF. By Theorems 1 and 2 of [12] every formation of the form $\mathfrak{F} = \bigcap_{(i,j)\in I} \mathfrak{G}_{\pi_i}\mathfrak{G}_{\pi_j}$ contains every group $G = AB$, where A and B are $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of G.

The proof continues by induction on the order of G. Take two commuting $\mathfrak{F}$-subnormal subgroups A and B of G. Put $T = AB$. Take a minimal normal subgroup N of G. Taking Lemma 2 into account, we find by induction that TN/N is an $\mathfrak{F}$-subnormal subgroup of GN. Lemma 2 implies that TN is an $\mathfrak{F}$-subnormal subgroup of G. If $TN \neq G$ then by induction T is an $\mathfrak{F}$-subnormal subgroup in TN. Hence, T is $\mathfrak{F}$-subnormal in G.

Suppose now that $TN = G$ for every normal subgroup of G. It is obvious that $T_G = 1$. If $A^{\mathfrak{F}} \neq 1$ then by Lemma 1 the subgroup $A^{\mathfrak{F}}$ is subnormal in G. By Wielandt's Theorem 7.10 of [1]

$$1 \neq (A^{\mathfrak{F}})^G = (A^{\mathfrak{F}})^{TN} \subseteq T.$$

Thereby, $T_G \neq 1$. We have arrived at a contradiction. Hence, $A^{\mathfrak{F}} = 1$. Similarly we can prove that $B^{\mathfrak{F}} = 1$.

In order to show that $AN \in \mathfrak{F}$ and $BN \in \mathfrak{F}$, consider the following cases.

1. Suppose that N is an abelian subgroup. Theorem 15.10 of [1] implies that $(AN)^{\mathfrak{F}} \subseteq \Phi(AN)$. Since $\mathfrak{F}$ is a saturated formation, $AN \in \mathfrak{F}$. Similarly we can prove that $BN \in \mathfrak{F}$.

2. Suppose that N is a nonabelian subgroup. Then $N = N_1 \times N_2 \times \dots \times N_t$ is the direct product of isomorphic nonabelian simple groups. Since $A \in \mathfrak{F}$, $(AN)^{\mathfrak{F}} \subseteq N$. If $(AN)^{\mathfrak{F}} = N$ then $AN = (AN)^{\mathfrak{F}}A$, which is impossible because A is $\mathfrak{F}$-subnormal in AN. Thus, $(AN)^{\mathfrak{F}} \subset N$. If $(AN)^{\mathfrak{F}} \neq 1$ then

$$(AN)^{\mathfrak{F}} = N_{i_1} \times N_{i_2} \times \dots \times N_{i_n}.$$

Since $\mathfrak{F}$ is a hereditary formation, $N/(AN)^{\mathfrak{F}} \in \mathfrak{F}$. However, then it is not difficult to observe that $N \in \mathfrak{F}$. Lemma 1 implies that N is an $\mathfrak{F}$-subnormal subgroup in AN. By hypothesis, $AN \in \mathfrak{F}$. We find similarly that $BN \in \mathfrak{F}$. Lemma 2 implies that AN and BN are $\mathfrak{F}$-subnormal subgroups of G. By hypothesis, $G \in \mathfrak{F}$. Since $\mathfrak{F}$ is a hereditary formation, T is an $\mathfrak{F}$-subnormal subgroup of G. The proof of the theorem is complete.

Corollary 1.1. *Take some hereditary formation $\mathfrak{F} = \mathfrak{F}\mathfrak{F}$. For every group G and all commuting $\mathfrak{F}$-accessible subgroups H and K of G the subgroup HK is $\mathfrak{F}$-accessible in G.*

Corollary 1.2. *Take the formation $\mathfrak{F}$ of all π-decomposable groups. Then for every group G and all commuting $\mathfrak{F}$-subnormal subgroups H and K of G the subgroup HK is $\mathfrak{F}$-subnormal in G.*

Corollary 1.3. *Take the formation $\mathfrak{F}$ of all π-nilpotent groups. Then for every group G and all commuting $\mathfrak{F}$-subnormal subgroups H and K of G the subgroup HK is $\mathfrak{F}$-subnormal in G.*

Corollary 1.4. *Take the formation $\mathfrak{F}$ of all π-closed groups. Then for every group G and all commuting $\mathfrak{F}$-subnormal subgroups H and K of G the subgroup HK is $\mathfrak{F}$-subnormal in G.*

Theorem 2. *Take some hereditary saturated formation $\mathfrak{F}$. The following are equivalent:*

(1) every group $G = AB$, where A and B are $\mathfrak{F}$-accessible $\mathfrak{F}$-subgroups in G, belongs to $\mathfrak{F}$;

(2) $\mathfrak{F}$ is a superradical formation;

(3) for every group G and all commuting $\mathfrak{F}$-subnormal subgroups H and K of G the subgroup HK is $\mathfrak{F}$-subnormal in G;

(4) for every group G and all commuting $\mathfrak{F}$-accessible subgroups H and K of G the subgroup HK is $\mathfrak{F}$-accessible in G.

Proof. The equivalence of claims 1 and 2 follows from Theorem 1 of [12].

Let us show that claim 2 implies claim 3 by induction on the order of G. Take two commuting $\mathfrak{F}$-subnormal subgroups A and B of G and put $T = AB$. Take a minimal normal subgroup N of G. Taking Lemma 2 into account, we find by induction that TN/N is an $\mathfrak{F}$-subnormal subgroup of GN. Lemma 2 implies that TN is an $\mathfrak{F}$-subnormal subgroup of G. If $TN \neq G$ then by induction T is an $\mathfrak{F}$-subnormal subgroup in TN; hence, T is $\mathfrak{F}$-subnormal in G.

Suppose now that $TN = G$ for every minimal normal subgroup N of G. It is obvious that $T_G = 1$. If $A^{\mathfrak{F}} \neq 1$ then by Lemma 1 the subgroup $A^{\mathfrak{F}}$ is subnormal in G. Then by Wielandt's Theorem 7.10 of [1],

$$1 \neq (A^{\mathfrak{F}})^G = (A^{\mathfrak{F}})^{TN} \subseteq T.$$

Consequently, $T_G \neq 1$. We have arrived at a contradiction. Thereby, $A^{\mathfrak{F}} = 1$. Similarly we can prove that $B^{\mathfrak{F}} = 1$.

In order to show that $AN \in \mathfrak{F}$ and $BN \in \mathfrak{F}$, consider the following cases.

1. Suppose that N is an abelian subgroup. Theorem 15.10 of [1] implies that $(AN)^{\mathfrak{F}} \subseteq \Phi(AN)$. Since $\mathfrak{F}$ is a saturated formation, $AN \in \mathfrak{F}$. Similarly we can prove that $BN \in \mathfrak{F}$.

2. Suppose that N is a nonabelian subgroup. Then $N = N_1 \times N_2 \times \dots \times N_t$ is the direct product of isomorphic nonabelian simple groups. Since $A \in \mathfrak{F}$, $(AN)^{\mathfrak{F}} \subseteq N$. If $(AN)^{\mathfrak{F}} = N$ then $AN = (AN)^{\mathfrak{F}}A$, which is impossible since A is $\mathfrak{F}$-subnormal in AN. Thus, $(AN)^{\mathfrak{F}} \subset N$. If $(AN)^{\mathfrak{F}} \neq 1$ then

$$(AN)^{\mathfrak{F}} = N_{i_1} \times N_{i_2} \times \dots \times N_{i_n}.$$

Since $\mathfrak{F}$ is a hereditary formation, $N/(AN)^{\mathfrak{F}} \in \mathfrak{F}$. It is not difficult to observe that $N \in \mathfrak{F}$. Lemma 1 implies that N is an $\mathfrak{F}$-subnormal subgroup in AN. By hypothesis, $AN \in \mathfrak{F}$. Similarly we find that $BN \in \mathfrak{F}$. Lemma 2 implies that AN and BN are $\mathfrak{F}$-subnormal subgroups of G. By hypothesis, $G \in \mathfrak{F}$. Since $\mathfrak{F}$ is a hereditary formation, T is an $\mathfrak{F}$-subnormal subgroup of G.

In order to show that claim 3 implies claim 2, take a counterexample G of the minimal order. Then $G = AB$, where A and B are $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups, but $G \notin \mathfrak{F}$. Take a minimal normal subgroup N of G. Since AN/N and BN/N are $\mathfrak{F}$-subnormal subgroups of G/N, by induction $G/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is a saturated formation, G includes a unique minimal normal subgroup $N = G^{\mathfrak{F}}$, and $\Phi(G) = 1$.

Consider the subgroups AN and BN. Since A is a proper $\mathfrak{F}$-subnormal subgroup of G and $N = G^{\mathfrak{F}}$, $AN \neq G$. Similarly, $BN \neq G$. Verify that $AN \in \mathfrak{F}$.

Suppose that N is an abelian group. Theorem 15.10 of [1] implies that $(AN)^{\mathfrak{F}} \subseteq \Phi(AN)$. Since $\mathfrak{F}$ is a saturated formation, $AN \in \mathfrak{F}$.

Suppose that N is a nonabelian group. In this case $N = N_1 \times N_2 \times \cdots \times N_t$ is the direct product of isomorphic nonabelian simple groups, and $C_G(N) = 1$.

Consider the subgroup $H = AN$. Lemma 2 implies that $H = AN$ is an $\mathfrak{F}$-subnormal subgroup of G. Consider the subgroup $A^H \subseteq H \neq G$. By the Dedekind identity,

$$A^H = A^H \cap G = A^H \cap AB = A(A^H \cap B).$$

Lemma 1 implies that $A^H \cap B$ is an $\mathfrak{F}$-subnormal subgroup of A^H. Since $\mathfrak{F}$ is a hereditary formation and $B \in \mathfrak{F}$, we have $A^H \cap B \in \mathfrak{F}$. By induction, $A^H \in \mathfrak{F}$. If $A^H \cap N = 1$ then $A^H \subseteq C_G(N) = 1$. We have arrived at a contradiction. Hence, $A^H \cap N \neq 1$. Since A^H is a normal subgroup in AN; therefore, $A^H \cap N$ is a normal subgroup in N. Then

$$A^H \cap N = N_{i_1} \times N_{i_2} \times \cdots \times N_{i_k},$$

where N_{i_j} are isomorphic nonabelian simple groups for $j = 1, 2, \dots, k$. Since $A^H \in \mathfrak{F}$ and $\mathfrak{F}$ is a hereditary formation, we have $A^H \cap N \in \mathfrak{F}$. This easily implies that $N \in \mathfrak{F}$. Since $N = G^{\mathfrak{F}}$, Lemma 1 implies that N is an $\mathfrak{F}$-subnormal subgroup of G. Since $AN \neq G$, it follows that by induction $AN \in \mathfrak{F}$. Similarly we can prove that $BN \in \mathfrak{F}$.

Take some complement K to N in G. Since $\Phi(G) = 1$, we have $K \neq G$. Since $\mathfrak{F}$ is a saturated formation, we deduce from $G/N = KN/N \simeq K/K \cap N \in \mathfrak{F}$ and $K \cap N \subseteq \Phi(K)$ that $K \in \mathfrak{F}$. By the Dedekind identity,

$$AN = AN \cap KN = N(AN \cap K).$$

If $AN \cap K = 1$ then $AN = N$. Thus, $G = AB = ANB = NB$, which is impossible because $N = G^{\mathfrak{F}}$ and B is a proper $\mathfrak{F}$-subnormal subgroup of G.

Thus, $AN \cap K \neq 1$. Since AN is an $\mathfrak{F}$-subnormal subgroup of G and $AN \in \mathfrak{F}$, the heredity of $\mathfrak{F}$ implies that $AN \cap K$ is an $\mathfrak{F}$-subnormal subgroup of G. Similarly we find that $BN = N(BN \cap K)$, where $BN \cap K$ is an $\mathfrak{F}$-subnormal subgroup of G. Then

$$G = AB = N(AN \cap K)(BN \cap K).$$

By hypothesis, $(AN \cap K)(BN \cap K)$ is an $\mathfrak{F}$-subnormal subgroup of G, which is impossible because $N = G^{\mathfrak{F}}$ and $(AN \cap K)(BN \cap K) \subseteq K$. Consequently, $G \in \mathfrak{F}$.

In order to show that claim 4 implies claim 3, take a group G, of the least order for which the implication fails. Take a minimal normal subgroup N of G. Given two commuting $\mathfrak{F}$-subnormal subgroups A and B of G, Lemma 2 implies that AN/N and BN/N are commuting $\mathfrak{F}$-subnormal subgroups in G/N. Since $|G/N| < |G|$, by induction $AN/N \cdot BN/N = ABN/N$ is an $\mathfrak{F}$-subnormal subgroup in G/N.

Take the formation $\mathfrak{H}$ of all groups all of whose composition factors belong to $\mathfrak{F}$. Lemma 4 yields $A^{\mathfrak{H}} = B^{\mathfrak{H}} = G^{\mathfrak{H}}$. If G lies outside $\mathfrak{H}$ then, taking N in $G^{\mathfrak{H}}$, we find that $AB = ABN$ is an $\mathfrak{F}$-subnormal subgroup of G. This is a contradiction; hence, $G \in \mathfrak{H}$. The claim of the theorem follows from Lemma 5.

We can prove the fact that claim 4 implies claim 1 in the same fashion as the fact that claim 3 implies claim 2. The proof of the theorem is complete.

Corollary 1.5. *Take some hereditary soluble saturated formation $\mathfrak{F}$. The following are equivalent:*

(1) *the formation $\mathfrak{F}$ is such that for every group G and all commuting $\mathfrak{F}$-subnormal subgroups H and K of G the subgroup HK is $\mathfrak{F}$-subnormal in G;*

(2) *the formation $\mathfrak{F}$ is such that for every group G and all commuting $\mathfrak{F}$-accessible subgroups H and K of G the subgroup HK is $\mathfrak{F}$-accessible in G;*

(3) *the formation $\mathfrak{F}$ is of the form $\mathfrak{F} = \bigcap_{(i,j)\in I} \mathfrak{S}_{\pi_i}\mathfrak{S}_{\pi_j}$.*

PROOF. The fact that claims 1 and 2 are equivalent follows from Theorem 2. Verify that claims 2 and 3 are equivalent. Indeed, $\mathfrak{F}$ is an $\mathfrak{S}$-Shemetkov formation by Lemma 6. Now claim 3 follows from Corollary 1 of [13]. Claim 2 follows from claim 3 of Theorem 2. The proof of the corollary is complete.

Lemma 9. *Take some hereditary saturated formation $\mathfrak{F}$ such that $\mathscr{M}(\mathfrak{F}) \subseteq \mathfrak{S}$. The following are equivalent:*

(1) *$\mathfrak{F}$ is a superradical formation;*

(2) *$\mathfrak{F}$ is of the form $\mathfrak{F} = \bigcap_{(i,j)\in I} \mathfrak{G}_{\pi_i}\mathfrak{G}_{\pi_j}$.*

PROOF. Suppose that $\mathfrak{F}$ is a superradical formation. Let us prove firstly that every minimal non-$\mathfrak{F}$-group is either a group of prime order or a Schmidt group.

Take an arbitrary minimal non-$\mathfrak{F}$-group G. By the hypothesis of the theorem, G is soluble. If $G \notin \mathfrak{G}_{\pi(\mathfrak{F})}$ then it is not difficult to observe that G is a group of prime order q with $q \notin \pi(\mathfrak{F})$.

Consider the case $\Phi(G) = 1$. Theorem 1.5 of [10] implies that $G = N \rtimes M$ for the unique minimal normal subgroup N of G, which is a p-group, and $M \in \mathscr{M}(f(p))$, where f is a maximal inner local screen of $\mathfrak{F}$. It is obvious that $C_G(N) = N$.

In order to show that M is a primary cyclic subgroup, assume the contrary. Since M is a soluble group, it includes maximal subgroups M_1 and M_2 such that $M = M_1M_2$. Since $N = G^{\mathfrak{F}}$, we see that NM_1 and NM_2 are $\mathfrak{F}$-maximal normal $\mathfrak{F}$-subgroups of G. However, then $G = NM_1 \cdot NM_2$. Since $\mathfrak{F}$ is a superradical formation, $G \in \mathfrak{F}$. This is a contradiction; thus, M has the unique class of maximal conjugate subgroups. Hence, M is a cyclic q-subgroup. Since $\mathfrak{F}$ is a saturated formation and $G \notin \mathfrak{F}$, we have $q \neq p$.

Show now that $|M| = q$. Assume the contrary. Take $|M| = q^n$, where $n > 1$. Take cyclic groups E and L of order q^{n-1} and q respectively. Denote by T the regular wreath product $EwrL$. Denote by K the base of the wreath product: $T = K \rtimes L$. Since a subgroup of T is isomorphic to M, we have $T \notin f(p)$. It is obvious that K and L belong to the formation $f(p)$.

Put $R = PwrT$, where $|P| = p$. Denote by C the base of the wreath product R. Then $R = C \rtimes T = C \rtimes (K \rtimes L)$. Since $R/C \simeq K \rtimes L \in \mathfrak{F}$; therefore, $R^{\mathfrak{F}} \subseteq C$. Hence, the subgroups CK and CL are $\mathfrak{F}$-subnormal in R. It is easy to see that $CK \in \mathfrak{F}$ and $CL \in \mathfrak{F}$. Since $\mathfrak{F}$ is a superradical formation, $R \in \mathfrak{F}$. However, $F_p(R) = C$, and therefore $T \simeq R/C \in f(p)$.

The resulting contradiction shows that $n = 1$. Consequently, G is a Schmidt group. Thus, we have showed that $G/\Phi(G)$ is a Schmidt group. Now Lemma 6 implies that G is a Schmidt group.

Take some maximal inner local screen f of the formation $\mathfrak{F}$. Verify that $\mathfrak{F}$ has a complete local screen h such that $h(p) = \mathfrak{G}_{\pi(f(p))}$ for every p in $\pi(\mathfrak{F})$. Indeed, take some formation $\mathfrak{F}^*$ with a local screen h and show that $\mathfrak{F}^* = \mathfrak{F}$. Since $f(p) \subseteq h(p)$ for every prime p in $\pi(\mathfrak{F})$, we see that $\mathfrak{F} \subseteq \mathfrak{F}^*$.

In order to verify the inverse inclusion, take a group G of the least order in $\mathfrak{F}^*\backslash\mathfrak{F}$. Since $h(p)$ is a hereditary formation, so is $\mathfrak{F}^*$. This means that $G \in \mathscr{M}(\mathfrak{F})$. Since $\mathfrak{F}$ is a saturated formation, it is not difficult to show that $\Phi(G) = 1$.

We have showed above that G is either a group of prime order or a Schmidt group. Suppose that G is a group of prime order and $|G| = q$. It is not difficult to show that $\pi(\mathfrak{F}^*) = \pi(\mathfrak{F})$. Since $G \in \mathfrak{F}^*$, we have $q \in \pi(\mathfrak{F})$. This implies that $G \in \mathfrak{F}$, which is a contradiction.

Suppose now that G is a Schmidt group. Since $\Phi(G) = 1$, the properties of Schmidt groups imply that $G = G_p \rtimes G_q$, where $G_p = F_p(G)$ and $|G_q| = q$. Since $G \in \mathfrak{F}^*$, we have $G/G_p \in h(p)$. It follows from $G/G_p \simeq G_q$ that $G_q \in h(p) = \mathfrak{G}_{\pi(f(p))}$. Since $q \in \pi(f(p))$ and $f(p)$ is a hereditary formation, $G_q \in f(p)$.

Now the facts that $G/G_p \in \mathfrak{F}$, where G_p is the unique minimal normal subgroup of G, and $G/G_p \in f(p)$ imply that $G \in \mathfrak{F}$. We have arrived at a contradiction. Thus, $\mathfrak{F}^* \subseteq \mathfrak{F}$. Hence, $\mathfrak{F}^* = \mathfrak{F}$.

Since h is a local screen of the formation $\mathfrak{F}$, we have

$$\mathfrak{F} = \bigcap_{p \in \pi(\mathfrak{F})} \mathfrak{S}_{p'} \mathfrak{S}_{\pi(f(p))} \cap \mathfrak{S}_{\pi(\mathfrak{F})}.$$

Thereby, $\mathfrak{F}$ is a formation from claim 2.

Put $\mathfrak{F} = \bigcap_{(i,j) \in I} \mathfrak{S}_{\pi_i} \mathfrak{S}_{\pi_j}$, and take a group G such that $G = AB$, where A and B are $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups in G. Let us show that $G \in \mathfrak{F}$ by induction on the order of G. Take a minimal normal subgroup N in G. It is obvious that AN/N and BN/N are $\mathfrak{F}$-subnormal subgroups of G. By induction, $G/N \in \mathfrak{F}$. This implies that G has a unique minimal normal subgroup N. It is obvious that $\Phi(G) = 1$ and $N = G^{\mathfrak{F}}$.

Show that $AN \in \mathfrak{F}$. If N is an abelian group then Theorem 15.10 of [1] implies that AN belongs to $\mathfrak{F}$. Suppose now that N is a nonabelian group. In this case $N = N_1 \times N_2 \times \dots \times N_t$ is the direct product of isomorphic nonabelian simple groups, and $C_G(N) = 1$. Consider the subgroup $H = AN$. It is clear that $H = AN$ is an $\mathfrak{F}$-subnormal subgroup of G. Consider the subgroup $A^H \subseteq H \neq G$. By the Dedekind identity,

$$A^H = A^H \cap G = A^H \cap AB = A(A^H \cap B).$$

It is obvious that $A^H \cap B$ is an $\mathfrak{F}$-subnormal subgroup of A^H. Since $\mathfrak{F}$ is a hereditary formation and $B \in \mathfrak{F}$, we have $A^H \cap B \in \mathfrak{F}$. By induction, $A^H \in \mathfrak{F}$. If $A^H \cap N = 1$ then $A^H \subseteq C_G(N) = 1$. This is a contradiction; hence, $A^H \cap N \neq 1$. Since A^H is a normal subgroup in AN, $A^H \cap N$ is a normal subgroup in N. However, then

$$A^H \cap N = N_{i_1} \times N_{i_2} \times \dots \times N_{i_k},$$

where N_{i_j} are isomorphic nonabelian simple groups. Since $A^H \in \mathfrak{F}$ and $\mathfrak{F}$ is a hereditary formation, $A^H \cap N \in \mathfrak{F}$. Thereby, $N \in \mathfrak{F}$. Since $N = G^{\mathfrak{F}}$ and A is a proper $\mathfrak{F}$-subnormal subgroup of G, we have $AN \neq G$. It is obvious that N and A are $\mathfrak{F}$-subnormal in AN. By induction, $AN \in \mathfrak{F}$. Hence, $AN \in \mathfrak{S}_{\pi_i} \mathfrak{S}_{\pi_j}$ for all (i,j) in I.

Similarly we can prove that $BN \in \mathfrak{S}_{\pi_i} \mathfrak{S}_{\pi_j}$ for all (i,j) in I. It follows from $AN \in \mathfrak{S}_{\pi_i} \mathfrak{S}_{\pi_j}$ that $AN/O_{\pi_i}(AN) \in \mathfrak{S}_{\pi_j}$.

Consider the cases $N \cap O_{\pi_i}(AN) \neq 1$ and $N \cap O_{\pi_i}(AN) = 1$.

Suppose that $N \cap O_{\pi_i}(AN) \neq 1$ and show that $N \in \mathfrak{S}_{\pi_i}$. If N is abelian then N is a p-group. Consequently, $N \in \mathfrak{S}_{\pi_i}$. If N is nonabelian then

$$N = N_1 \times N_2 \times \dots \times N_t$$

is the direct product of isomorphic nonabelian simple groups. Since $N \cap O_{\pi_i}(AN)$ is a normal subgroup in N,

$$N \cap O_{\pi_i}(AN) = N_{i_1} \times N_{i_2} \times \dots \times N_{i_n}.$$

Since $N \cap O_{\pi_i}(AN) \in \mathfrak{S}_{\pi_i}$, we have $N \in \mathfrak{S}_{\pi_i}$. Since $G/N \in \mathfrak{S}_{\pi_i} \mathfrak{S}_{\pi_j}$; therefore, $G \in \mathfrak{S}_{\pi_i} \mathfrak{S}_{\pi_j}$ for all (i,j) in I. Consequently, $G \in \mathfrak{F}$, which is a contradiction.

Suppose now that $N \cap O_{\pi_i}(AN) = 1$. If N is nonabelian then $C_G(N) = 1$. Then

$$O_{\pi_i}(AN) \subseteq C_G(N) = 1.$$

This implies that $O_{\pi_i}(AN) = 1$, and so $AN \in \mathfrak{S}_{\pi_j}$.

Consider the subgroup $N \cap O_{\pi_i}(BN)$. If $N \cap O_{\pi_i}(BN) \neq 1$ then $G \in \mathfrak{F}$ as above, which is a contradiction. If $N \cap O_{\pi_i}(BN) = 1$ then $BN \in \mathfrak{S}_{\pi_j}$ as above. Since $G = AN \cdot BN$, we have $G \in \mathfrak{S}_{\pi_j}$. Consequently, $G \in \mathfrak{F}$, which is a contradiction.

If N is an abelian group then $C_G(N) = N$. Thus,

$$O_{\pi_i}(AN) \subseteq C_G(N) = N.$$

We deduce from $G/N \in \mathfrak{S}_{\pi_i} \mathfrak{S}_{\pi_j}$ that $G \in \mathfrak{S}_{\pi_i} \mathfrak{S}_{\pi_j}$ for all (i,j) in I. This means that $G \in \mathfrak{F}$, which is a contradiction. Consequently, $\mathfrak{F}$ is a superradical formation. The proof of the lemma is complete.

Theorem 3. *Take some hereditary saturated formation $\mathfrak{F}$. The following are equivalent:*

(1) *$\mathfrak{F}$ is a Shemetkov formation;*

(2) *$\mathfrak{F}$ contains every group $G = AB$, where A and B are $\mathfrak{F}$-accessible $\mathfrak{F}$-subgroups in G, and $\mathscr{M}(\mathfrak{F}) \subseteq \mathfrak{S}$;*

(3) *$\mathfrak{F}$ is a superradical formation and $\mathscr{M}(\mathfrak{F}) \subseteq \mathfrak{S}$;*

(4) *$\mathfrak{F}$ is such that $\mathscr{M}(\mathfrak{F}) \subseteq \mathfrak{S}$ and for every group G and all commuting $\mathfrak{F}$-subnormal subgroups H and K of G the subgroup HK is $\mathfrak{F}$-subnormal in G;*

(5) *$\mathfrak{F}$ is such that $\mathscr{M}(\mathfrak{F}) \subseteq \mathfrak{S}$ and for every group G and all commuting $\mathfrak{F}$-accessible subgroups H and K of G the subgroup HK is $\mathfrak{F}$-accessible in G;*

(6) *$\mathfrak{F}$ is of the form $\mathfrak{F} = \bigcap_{(i,j)\in I} \mathfrak{G}_{\pi_i}\mathfrak{G}_{\pi_j}$, and $\mathscr{M}(\mathfrak{F}) \subseteq \mathfrak{S}$.*

The proof follows from Theorems 1, 2 and Lemmas 7, 9.

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