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THE GEOMETRIC THEORY OF REPRESENTATIONS FOR THE FUNDAMENTAL GROUPS OF COMPACT ORIENTED SURFACES

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Let Γ be a finitely generated group. It is well known (see [1]–[3]) that for an arbitrary linear algebraic group G defined over a field K, the set of all representations $\rho\colon \Gamma \to G_K$ can be identified with the set $R(\Gamma, G)_K$ of K-points of a certain K-defined variety $R(\Gamma, C)$, called the *variety of representations*. Thus the variety $R(\Gamma, G)$, a basic object in geometric representation theory, furnishes a natural parametrization of the family of all representations of Γ in G; so that by obtaining a description of $R(\Gamma, G)$ we obtain extended information on the representations of Γ.

The present note examines the case that $\Gamma = \Gamma_g$ is the fundamental group of a compact oriented surface of genus g, i.e., the group given by the copresentation

$$\Gamma_g = \langle x_1, y_1, \dots, x_g, y_g | [x_y, y_1] \cdots [x_g, y_g] = 1 \rangle$$

where $[x, y] = xyx^{-1}y^{-1}$. Here the variety $R(\Gamma, \mathrm{SL}_2(\mathbb{R}))$ of real unimodular representations comes up in Riemann surface theory; specifically, the so-called Fricke space, closely connected with the variety of moduli, is a domain on $R(\Gamma, \mathrm{SL}_2(\mathbb{R}))$ (see [4]). We give a description of the variety $R(\Gamma, \mathrm{SL}_2(\mathbb{R}))$ of n-dimensional representations and of the corresponding variety $R_n(\Gamma) = R(\Gamma, \mathrm{GL}_n)$ of n-dimensional representations and of the corresponding variety $X_n(\Gamma)$ of n-dimensional characters for the case that the ground field has characteristic 0.

For a matrix $a \in M_n$ we denote by $f_a(\lambda)$ its characteristic polynomial $f_a(\lambda) = \det(\lambda E_n - a)$, and by $\sigma_1(a), \dots, \sigma_n(a)$ the coefficients of $f_a(\lambda)$; i.e.,

$$f_a(\lambda) = \lambda^n + \sigma_1(a)\lambda^{n-1} + \cdots + \sigma_n(a).$$

For $h \in \mathrm{SL}_n$, we denote by T_h the variety of M_n defined by the system

$$\sigma_1(a) = \sigma_1(ha), \dots, \sigma_{n-1}(a) = \sigma_{n-1}(ha). \tag{1}$$

Consider the following two conditions:

1) There exists a nonempty Zariski $\mathbb{Q}$-open subset $U \subset \mathrm{SL}_n$ such that for any $h \in U$ the variety T_h is irreducible.

2) For any $x, y \in \mathrm{GL}_n$, the set $xZ(y)$, where $Z(y)$ is the centralizer of y in GL_n, contains a regular element (i.e., an element Z such that $\dim Z(z) = n$).

Theorem 1. *Suppose conditions* 1) *and* 2) *are satisfied. Then the variety $R_n(\Gamma)$ is an (absolutely) irreducible $\mathbb{Q}$-rational variety, of dimension $(2g-1)n^2 + 1$ for $g > 1$ and $n^2 + n$ for $g = 1$.*

Theorem 2. *Under the hypothesis of Theorem* 1, *for $g > 1$ the variety $X_n(\Gamma)$ is an irreducible $\mathbb{Q}$-defined variety of dimension $(2g-2)n^2$. Moreover, the rational function*

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field $\mathbb{Q}(X_n(\Gamma))$ *is a purely transcendental extension of the field* $\mathbb{Q}(X_n(F_{2g-2}))$, *where* F_{2g-2} *is the free group of rank* $2g-2$.

Further, we show that condition 2) is "almost always" satisfied automatically; namely, it is certainly satisfied if the element y is semisimple (Proposition 4). Using this fact, we are able to verify the validity of 1) and 2) for $n \leq 4$. On the other hand, for $n \leq 4$ the variety $X_n(F_m)$ is rational for any m (see [7] and [8]). Thus, we obtain

Corollary. *For* $g > 1$ *and* $n \leq 4$, *the variety* $X_n(\Gamma)$ *is rational over* $\mathbb{Q}$.

We proceed to the proof of the theorems. The case $g = 1$ is easily worked out; everywhere below, therefore, $g > 1$ and $n > 1$. The first nontrivial fact here is the irreducibility of the variety $R_n(\Gamma)$.

We denote by F the subgroup of Γ generated by $x_1, y_1, \dots, x_{g-1}, y_{g-1}$, and by $\varphi\colon R_n(\Gamma) \to R_n(F)$ the corresponding morphism of the varieties of representations. Since, as we know, F is a free group of rank $2(g-1)$ (see [5]), the variety $R_n(F)$ coincides with the product $\mathrm{GL}_n \times \dots \times \mathrm{GL}_n$ ($2(g-1)$ times), and in particular is irreducible. On the other hand, the fact that every element of SL_n is a commutator in GL_n (see [6]) implies that φ is surjective.

Proposition 1. $\varphi(V) = R_n(F)$, *for any irreducible component* $V \subset R_n(\Gamma)$.

Proposition 2. *Suppose, for* $h \in \mathrm{SL}_n$, *that the variety* T_h *is irreducible. Then if condition* 2) *is satisfied, the variety*

$$W_h = \{(x, y) \in \mathrm{GL}_n \times \mathrm{GL}_n \,|\, [x, y] = h\}$$

is likewise irreducible.

Now suppose $R_n(\Gamma) = \bigcup_{i=1}^d V_i$ is a decomposition into irreducible components with $d > 1$. Put $U_i = V_i \backslash (\bigcup_{j \neq i} V_j)$, $i = 1, \dots, d$, and let $U_0 = \Psi^{-1}(U)$, where U is the open set of condition 1) and $\Psi\colon \mathrm{GL}_n \times \dots \times \mathrm{GL}_n \to \mathrm{SL}_n$ the morphism given by

$$\Psi(x_1, y_1, \dots, x_{g-1}, y_{g-1}) = [x_1, y_1] \cdots [x_{g-1}, y_{g-1}].$$

From Proposition 1 and the irreducibility of $R_n(F)$ we obtain that the intersection $\varphi(U_1) \cap \varphi(U_2) \cap U_0$ is nonempty; let a be a point in this intersection. Then the fiber $Z = \varphi^{-1}(a)$ is isomorphic to the variety $W_{\Psi(a)}$ and therefore, in view of our constructions and Proposition 2, irreducible. It follows that $Z \subset V_{i_0}$ for some $i_0 \in \{1, \dots, d\}$. But $a = \varphi(u_1) = \varphi(u_2)$ for some $u_i \in U_i$, $i = 1, 2$, such that u_1, $u_2 \in Z$. Each of u_1, u_2, however, lies in just one irreducible component, so that $V_1 = V_{i_0} = V_2$; and we have a contradiction.

Proof of Proposition 1. It suffices to show that any irreducible component $V \subset R_n(\Gamma)$ has a nonempty open subset V_0 such that for any point $v \in V_0$ the differential $d_v\varphi\colon T_v(V) \to T_{\varphi(v)}(R_n(F))$ is surjective. First one verifies:

Lemma 1. *Let* $v = (x_1, y_1, \dots, x_g, y_g) \in R_n(\Gamma)$ *be a point such that the elements* x_g *and* y_g *are regular and* $\dim(Z(x_g) \cap Z(y_g)) = 1$. *Then the mapping* $d_v\varphi\colon T_v(R_n(\Gamma)) \to T_{\varphi(v)}(R_n(F))$ *is surjective.*

Suppose now that for an irreducible component $V \subset R_n(\Gamma)$ we have $\overline{\varphi(V)} \neq R_n(F)$. Let V_1 be the open subvariety of V consisting of those points $(x_1, y_1, \dots, x_g, y_g)$ such that x_g and y_g are regular elements; from condition 2) it follows that $V_1 \neq \varnothing$. Then, by Lemma 1, $(V_1 \subset \mathrm{GL}_n^{2g-2} \times L$, where $L = \{(x, y) \in \mathrm{GL}_n \times \mathrm{GL}_n \,|\, x$ and y are regular and $\dim(Z(x) \cap Z(y)) > 1\}$.

Lemma 2. 1) $\dim L \leq 2n^2 - 2(n-1)$.

2) *For any* $h \in \mathrm{SL}_n$ *the dimension of any irreducible component* $T \subset W_h$ *lies in the interval* $n^2 + 1 \leq \dim T \leq n^2 + n$.

3) $\dim V \geq (2g-1)n^2 + 1$.

From parts 1) and 2) of Lemma 2 we obtain, using the theorem on the dimension of fibers of a morphism, that

$$\dim V \leq 2(g-2)n^2 + n^2 + n + 2n^2 - 2(n-1) = (2g-1)n^2 - n + 2.$$

Comparing this inequality with part 3) of Lemma 2 gives $n \leq 1$, a contradiction. This proves the irreducibility of $R_n(\Gamma)$. The dimension of $R_n(\Gamma)$ is easily computed by considering the morphism $\delta\colon \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{SL}_n$, $\delta(x, y) = [x, y]$. Since δ is surjective [6], there exists, by the theorem on dimension of fibers, an open set $W \subset \mathrm{SL}_n$ such that $\dim \delta^{-1}(w) = 2n^2 - (n^2 - 1) = n^2 + 1$ for any $w \in W$. Put $W_0 = \Psi^{-1}(W)$, where $\Psi\colon \mathrm{GL}_n^{2g-2} \to \mathrm{SL}_n$ is given by $\Psi(x_1, y_1, \ldots, g_{g-1}, y_{g-1}) = [x_1, y_1] \cdots [x_{g-1}, y_{g-1}]$. Then for any $v \in W_0$ we have $\dim \varphi^{-1}(v) = n^2 + 1$, so that $\dim R_n(\Gamma) = \dim R_n(F) + n^2 + 1 = (2g-1)n^2 + 1$.

The proof of the rationality of $R_n(\Gamma)$ is based on the following assertion.

Proposition 3. *There exists a nonempty* $\mathbb{Q}$*-open subset* $B \subset \mathrm{SL}_n$ *such that for any extension* $K/\mathbb{Q}$ *and any point* $h \in B_k$ *the variety* W_h *is an irreducible* K*-rational variety of dimension* $n^2 + 1$.

Proof. Denote by B_1 a $\mathbb{Q}$-open subset of SL_n with the following properties.

1) B_1 consists of regular semisimple elements.

2) For $h \in B_1$ the variety T_h is irreducible, and W_h has dimension $n^2 + 1$.

Suppose $h \in B_1$. Consider the projection $\pi\colon W_k \to \mathrm{GL}_n$, $\pi(x, y) = y$, and put $T = \overline{\mathrm{Im}\,\pi}$. Let T^0 (resp. T_h^0) be the open subset of T (resp. of $T_h \cap \mathrm{GL}_n$) formed by the regular semisimple elements. It is easily seen that $T \subset T_h$ and $T^0 \subset T_h^0 \subset \mathrm{Im}\,\pi$, so that in fact $T^0 = T_h^0$. Since obviously $T^0 \neq \varnothing$, it follows from the irreducibility of T_h that $T = T_k \cap \mathrm{GL}_n$; in particular, T is open in T_h.

We examine now the system (1) defining T_h. From the definition of the characteristic polynomial $f_a(\lambda)$ of a matrix $a = (a_{ij})$ it follows that the coefficient $\sigma_r(a)$ of λ^{n-r} is, up to sign, the sum of all the principal minors of order r. Expanding by elements of the first column those principal minors that contain a_{11}, we obtain for $\sigma_r(a)$ a representation of the form

$$\sigma_r(a) = \sum_{l=1}^{n} P_{lr} a_{l1} + Q_r,$$

where

$$P_{lr}, Q_r \in \mathcal{O} = K[a_{ij}]_{\substack{i=1,\ldots,n \\ j=2,\ldots,n}}.$$

Let a'_{ij} be the element in position (i, j) in the matrix ha. Then

$$\sigma_r(ha) = \sum_{l=1}^{n} P'_{lr} a'_{l1} + Q'_r,$$

where the P'_{lr} and Q'_r are the polynomials obtained by substituting the a'_{ij} for the a_{ij}. Using the expressions for the a'_{ij} in terms of the a_{ij}, we now easily establish the existence of polynomials $\overline{P}_{lr}$, $\overline{Q}_r \in \mathcal{O}$ such that

$$\sigma_r(ha) = \sum_{l=1}^{n} \overline{P}_{lr} a_{l1} + \overline{Q}_r.$$

We see, therefore, that (1) reduces to a system of $n-1$ linear equations in the elements of the first column of the matrix a:

$$\sum_{l=1}^{n} p_{lr} a_{l1} = q_r, \qquad r = 1, \ldots, n-1, \tag{2}$$

where $p_{lr}, q_r \in \mathscr{O}$.

Let B_2 be the subset of SL_n consisting of those h for which the corresponding system (2) is of rank $n-1$. It is easily seen that B_2 is $\mathbb{Q}$-open and nonempty. We show now that the $\mathbb{Q}$-open subset $B = B_1 \cap B_2 \neq \varnothing$ in SL_n has the desired property. Suppose $h \in B$. Then in the matrix of the system (2) defining the variety T_h there exists a minor of order $n-1$ identically not equal to zero. From Cramer's rule it follows that this minor is different from zero on T_h and that the coordinates $a_{i_1 1}, \ldots, a_{i_{n-1} 1}$ connected with it can be expressed in a rational fashion in terms of the others. This proves the rationality of T_h, and therefore of T, since T is open in T_h. The rationality of W_h now follows automatically, since finding the first coordinate of the point $(x, y) \in W_h$ when the second is fixed reduces to solving the matrix equation $xy = hyx$, which is equivalent to a linear system in the elements of the matrix x. This completes the proof of Proposition 3.

There is now no difficulty in completing the proof of Theorem 1. Considering "generic" $n \times n$ matrices $x_1, y_1, \ldots, x_{g-1}, y_{g-1}$, let K be the field generated over $\mathbb{Q}$ by the elements of these matrices, and construct the matrix $h = [x_1, y_1] \cdots [x_{g-1}, y_{g-1}] \in \mathrm{SL}_n(K)$. Then the $\mathbb{Q}$-rational function field $\mathbb{Q}(R_n(\Gamma))$ is isomorphic to the K-rational function field $K(W_{h^{-1}})$. Since h is a "generic" point of the group SL_n over $\mathbb{Q}$, it lies in the $\mathbb{Q}$-open subset B constructed in Proposition 3; therefore, by that proposition, $W_{h^{-1}}$ is K-rational, i.e., the extension $K(W_{h^{-1}})/K$ is purely transcendental. But K is a purely transcendental extension of Q, and therefore the same is true of $\mathbb{Q}(R_n(\Gamma)) \simeq K(W_{h^{-1}})$.

Proof of Theorem 2. There exists a commutative diagram

$$\begin{array}{ccc} R_n(\Gamma) & \xrightarrow{\sigma} & X_n(\Gamma) \\ \varphi\downarrow & & \downarrow\delta \\ R_n(F) & \xrightarrow[\tau]{} & X_n(F), \end{array}$$

in which σ and τ are the natural projections of the varieties of representations onto the corresponding varieties of characters (see [2]), and φ and δ are induced by the restriction. It is easily seen that the subset $W_0 \subset R_n(F)$ of irreducible representations is a nonempty open $\mathbb{Q}$-defined subvariety. Let $W \subset X_n(F)$ be an open subset contained in $\tau(W_0)$ (clearly $\tau^{-1}(W) \subset W_0$), and suppose $w \in W$ and $\tilde{w} \in \delta^{-1}(w)$. It is easily seen that σ indices a bijective $\mathbb{Q}$-defined morphism

$$\tilde{\sigma}: F = \varphi^{-1}(\tilde{w}) \to \delta^{-1}(w) = D.$$

Thus, the fibers F and D are birationally isomorphic over the field over which they are both defined.

Now let ρ be a generic point over $\mathbb{Q}$ of the variety $R_n(\Gamma)$; $\omega = \sigma(\rho)$ and $\mu = \delta(w)$ generic points of $X_n(\Gamma)$ and $X_n(F)$, respectively; and $G = \delta^{-1}(\mu)$ a generic fiber of δ. Since condition 1) and Proposition 2 imply that G is irreducible, the field $L = \mathbb{Q}(X_n(\Gamma))$ is isomorphic to the rational function field $K(G)$, where $K = \mathbb{Q}(\mu) = \mathbb{Q}(X_n(F))$. Clearly, there exists a preimage $\tilde{\omega} \in \tau^{-1}(\omega)$,

$\tilde{\omega} = (x_1, y_1, \dots, x_{g-1}, y_{g-1})$, such that $h = [x_1, y_1]\cdots[x_{g-1}, y_{g-1}] \in \mathrm{SL}_n(K)$. It follows from the above that G is isomorphic over K to the variety $W_{h^{-1}}$, which is K-rational. This gives the desired result.

Proposition 4. *Suppose $x, y \in \mathrm{GL}_n$, with y semisimple. Then the set $xZ(y)$ contains a regular semisimple element.*

Explicit computations in each of the remaining cases yield a complete verification of 2) for $n \le 4$.

Remark. All the above results remain valid for a group Γ with $n \ge 4$ generators $x_1, \dots, x_n$ and one defining relation of the form $r = r_1[x_{n-3}, x_{n-2}][x_{n-1}, x_n]$, where r_1 lies in the commutator subgroup of the free group $F(x_1, \dots, x_{n-4})$.

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