THE GEOMETRIC THEORY OF REPRESENTATIONS FOR THE FUNDAMENTAL GROUPS OF COMPACT ORIENTED SURFACES
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Let $\Gamma$ be a finitely generated group. It is well known (see [1]–[3]) that for an arbitrary linear algebraic group $G$ defined over a field $K$, the set of all representations $\rho: \Gamma \to G_K$ can be identified with the set $R(\Gamma, G)_K$ of $K$-points of a certain $K$-defined variety $R(\Gamma, C)$, called the variety of representations. Thus the variety $R(\Gamma, G)$, a basic object in geometric representation theory, furnishes a natural parameterization of the family of all representations of $\Gamma$ in $G$; so that by obtaining a description of $R(\Gamma, G)$ we obtain extended information on the representations of $\Gamma$.

The present note examines the case that $\Gamma = \Gamma_g$ is the fundamental group of a compact oriented surface of genus $g$, i.e., the group given by the copresentation $\Gamma_g = \langle x_1, y_1, \ldots, x_g, y_g \mid [x_i, y_i] \cdots [x_g, y_g] = 1 \rangle$

where $[x, y] = xyx^{-1}yx^{-1}$. Here the variety $R(\Gamma, SL_2(\mathbb{R}))$ of real unimodular representations comes up in Riemann surface theory; specifically, the so-called Fricke space, closely connected with the variety of moduli, is a domain on $R(\Gamma, SL_2(\mathbb{R}))$ (see [4]). We give a description of the variety $R(\Gamma, SL_2(\mathbb{R}))$ of $n$-dimensional representations and of the corresponding variety $X_n(\Gamma)$ of $n$-dimensional characters for the case that the ground field has characteristic 0.

For a matrix $a \in M_n$ we denote by $f_a(\lambda)$ its characteristic polynomial $f_a(\lambda) = \det(\lambda E_n - a)$, and by $\sigma_1(a), \ldots, \sigma_n(a)$ the coefficients of $f_a(\lambda)$; i.e.,

$$f_a(\lambda) = \lambda^n + \sigma_1(a)\lambda^{n-1} + \cdots + \sigma_n(a).$$

For $h \in SL_n$, we denote by $T_h$ the variety of $M_n$ defined by the system

$$\sigma_1(a) = \sigma_1(ha), \ldots, \sigma_{n-1}(a) = \sigma_{n-1}(ha).$$

Consider the following two conditions:

1) There exists a nonempty Zariski $Q$-open subset $U \subset SL_n$ such that for any $h \in U$ the variety $T_h$ is irreducible.

2) For any $x, y \in GL_n$, the set $xZ(y)$, where $Z(y)$ is the centralizer of $y$ in $GL_n$, contains a regular element (i.e., an element $Z$ such that $\dim Z(z) = n$).

Theorem 1. Suppose conditions 1) and 2) are satisfied. Then the variety $R_n(\Gamma)$ is an (absolutely) irreducible $Q$-rational variety of dimension $(2g - 1)n^2 + 1$ for $g > 1$ and $n^2 + n$ for $g = 1$.

Theorem 2. Under the hypothesis of Theorem 1, for $g > 1$ the variety $X_n(\Gamma)$ is an irreducible $Q$-defined variety of dimension $(2g - 2)n^2$. Moreover, the rational function
field \( \mathbb{Q}(X_n(\Gamma)) \) is a purely transcendental extension of the field \( \mathbb{Q}(X_n(F_{2g-2})) \), where \( F_{2g-2} \) is the free group of rank \( 2g-2 \).

Further, we show that condition 2) is "almost always" satisfied automatically; namely, it is certainly satisfied if the element \( y \) is semisimple (Proposition 4). Using this fact, we are able to verify the validity of 1) and 2) for \( n \leq 4 \). On the other hand, for \( n \leq 4 \) the variety \( X_n(F_m) \) is rational for any \( m \) (see [7] and [8]). Thus, we obtain

**Corollary.** For \( g > 1 \) and \( n \leq 4 \), the variety \( X_n(\Gamma) \) is rational over \( \mathbb{Q} \).

We proceed to the proof of the theorems. The case \( g = 1 \) is easily worked out; everywhere below, therefore, \( g > 1 \) and \( n > 1 \). The first nontrivial fact here is the irreducibility of the variety \( R_n(\Gamma) \).

We denote by \( F \) the subgroup of \( \Gamma \) generated by \( x_1, y_1, \ldots, x_{g-1}, y_{g-1} \), and by \( \varphi: R_n(\Gamma) \to R_n(F) \) the corresponding morphism of the varieties of representations. Since, as we know, \( F \) is a free group of rank \( 2(g-1) \) (see [5]), the variety \( R_n(F) \) coincides with the product \( GL_n \times \cdots \times GL_n \) \( (2g-1) \) times, and in particular is irreducible. On the other hand, the fact that every element of \( SL_n \) is a commutator (see [6]) implies that \( \varphi \) is surjective.

**Proposition 1.** \( \varphi(V) = R_n(F) \), for any irreducible component \( V \subset R_n(\Gamma) \).

**Proposition 2.** Suppose, for \( h \in SL_n \), that the variety \( T_h \) is irreducible. Then if condition 2) is satisfied, the variety

\[ W_h = \{(x, y) \in GL_n \times GL_n | [x, y] = h\} \]

is likewise irreducible.

Now suppose \( R_n(\Gamma) = \bigsqcup_{i=1}^{d} V_i \) is a decomposition into irreducible components with \( d > 1 \). Put \( U_i = V_i \setminus (\bigsqcup_{j \neq i} V_j) \), \( i = 1, \ldots, d \), and let \( U_0 = \Psi^{-1}(U) \), where \( U \) is the open set of condition 1) and \( \Psi: GL_n \times \cdots \times GL_n \to SL_n \) the morphism given by

\[ \Psi(x_1, y_1, \ldots, x_{g-1}, y_{g-1}) = [x_1, y_1] \cdots [x_{g-1}, y_{g-1}] \].

From Proposition 1 and the irreducibility of \( R_n(F) \) we obtain that the intersection \( \varphi(U_i) \cap \varphi(U_0) \) is nonempty; let \( a \) be a point in this intersection. Then the fiber \( Z = \varphi^{-1}(a) \) is isomorphic to the variety \( W_{\Psi(a)} \), and therefore, in view of our constructions and Proposition 2, irreducible.

It follows that \( Z \subset V_0 \), for some \( i_0 \in \{1, \ldots, d\} \). But \( a = \varphi(u_1) = \varphi(u_2) \) for some \( u_1 \in U_i \), \( i = 1, 2 \), such that \( u_1 \), \( u_2 \in Z \). Each of \( u_1 \), \( u_2 \), however, lies in just one irreducible component, so that \( V_i = V_{i_0} = V_2 \); and we have a contradiction.

**Proof of Proposition 1.** It suffices to show that any irreducible component \( V \subset R_n(\Gamma) \) has a nonempty open subset \( V_0 \) such that for any point \( v \in V_0 \) the differential \( d_v \varphi: T_v(V) \to T_{\varphi(v)}(R_n(F)) \) is surjective. First one verifies:

**Lemma 1.** Let \( v = (x_1, y_1, \ldots, x_g, y_g) \in R_n(\Gamma) \) be a point such that the elements \( x_g \) and \( y_g \) are regular and \( \dim(Z(x_g) \cap Z(y_g)) = 1 \). Then the mapping

\[ d_v \varphi: T_v(R_n(\Gamma)) \to T_{\varphi(v)}(R_n(F)) \]

is surjective.

Suppose now that for an irreducible component \( V \subset R_n(\Gamma) \) we have \( \varphi(V) \neq R_n(F) \). Let \( V_1 \) be the open subvariety of \( V \) consisting of those points \( (x_1, y_1, \ldots, x_g, y_g) \) such that \( x_g \) and \( y_g \) are regular elements; from condition 2) it follows that \( V_1 \neq \emptyset \). Then, by Lemma 1, \( V_1 \subset GL_{2g-2}^n \times L \), where \( L = \{(x, y) \in GL_n \times GL_n | x \) and \( y \) are regular \}.
Lemma 2. 1) \( \dim L \leq 2n^2 - 2(n - 1) \).

2) For any \( h \in \text{SL}_n \) the dimension of any irreducible component \( T \subset W_h \) lies in the interval \( n^2 + 1 \leq \dim T \leq n^2 + n \).

3) \( \dim V \geq (2g - 1)n^2 + 1 \).

From parts 1) and 2) of Lemma 2 we obtain, using the theorem on the dimension of fibers of a morphism, that

\[
\dim V \leq 2(g - 2)n^2 + n^2 + n + 2n^2 - 2(n - 1) = (2g - 1)n^2 - n + 2.
\]

Comparing this inequality with part 3) of Lemma 2 gives \( n \leq 1 \), a contradiction.

This proves the irreducibility of \( R_n(\Gamma) \). The dimension of \( R_n(\Gamma) \) is easily computed by considering the morphism \( \delta: \text{GL}_n \times \text{GL}_n \to \text{SL}_n \), \( \delta(x, y) = [x, y] \). Since \( \delta \) is surjective [6], there exists, by the theorem on dimension of fibers, an open set \( W \subset \text{SL}_n \) such that \( \dim \delta^{-1}(w) = 2n^2 - (n^2 - 1) = n^2 + 1 \) for any \( w \in W \). Put \( W_0 = \Psi^{-1}(W) \), where \( \Psi: \text{GL}_{2g}^+ \to \text{SL}_n \) is given by \( \Psi(x_1, y_1, \ldots, x_g-1, y_g-1) = [x_1, y_1] \cdots [x_{g-1}, y_{g-1}] \). Then for any \( v \in W_0 \) we have \( \dim \varphi^{-1}(v) = n^2 + 1 \), so that \( \dim R_n(\Gamma) = \dim R_n(F) + n^2 + 1 = (2g - 1)n^2 + 1 \).

The proof of the rationality of \( R_n(\Gamma) \) is based on the following assertion.

Proposition 3. There exists a nonempty \( \mathbb{Q} \)-open subset \( B \subset \text{SL}_n \) such that for any extension \( K/\mathbb{Q} \) and any point \( h \in B_k \) the variety \( W_h \) is an irreducible \( K \)-rational variety of dimension \( n^2 + 1 \).

Proof. Denote by \( B_1 \) a \( \mathbb{Q} \)-open subset of \( \text{SL}_n \) with the following properties.

1) \( B_1 \) consists of regular semisimple elements.

2) For \( h \in B_1 \) the variety \( T_h \) is irreducible, and \( W_h \) has dimension \( n^2 + 1 \).

Suppose \( h \in B_1 \). Consider the projection \( \pi: W_h \to \text{GL}_n \), \( \pi(x, y) = y \), and put \( T = \text{Im} \pi \). Let \( T^0 \) (resp. \( T^0_h \)) be the open subset of \( T \) (resp. of \( T_h \cap \text{GL}_n \)) formed by the regular semisimple elements. It is easily seen that \( T \subset T^0 \) and \( T^0 \subset T^0_h \subset \text{Im} \pi \), so that in fact \( T^0 = T^0_h \). Since obviously \( T^0 \neq \varnothing \), it follows from the irreducibility of \( T_h \) that \( T = T_h \cap \text{GL}_n \); in particular, \( T \) is open in \( T_h \).

We examine now the system (1) defining \( T_h \). From the definition of the characteristic polynomial \( f_a(\lambda) \) of a matrix \( a = (a_{ij}) \) it follows that the coefficient \( \sigma_r(a) \) of \( \lambda^{n-r} \) is, up to sign, the sum of all the principal minors of order \( r \). Expanding by elements of the first column those principal minors that contain \( a_{11} \), we obtain for \( \sigma_r(a) \) a representation of the form

\[
\sigma_r(a) = \sum_{i=1}^{n} P_{ir}a_{11} + Q_r,
\]

where

\[
P_{ir}, Q_r \in \mathcal{O} = \mathbb{K}[a_{ij}]_{i=1, \ldots, n}.
\]

Let \( a'_{ij} \) be the element in position \( (i, j) \) in the matrix \( ha \). Then

\[
\sigma_r(ha) = \sum_{i=1}^{n} P'_{ir}a'_{11} + Q'_r,
\]

where the \( P'_{ir} \) and \( Q'_r \) are the polynomials obtained by substituting the \( a'_{ij} \) for the \( a_{ij} \). Using the expressions for the \( a'_{ij} \) in terms of the \( a_{ij} \), we now easily establish the existence of polynomials \( \overline{P}_{ir}, \overline{Q}_r \in \mathcal{O} \) such that

\[
\sigma_r(ha) = \sum_{i=1}^{n} \overline{P}_{ir}a_{11} + \overline{Q}_r.
\]
We see, therefore, that (1) reduces to a system of \( n - 1 \) linear equations in the elements of the first column of the matrix \( a \):

\[
\sum_{l=1}^{n} p_{lr} a_{1l} = q_r, \quad r = 1, \ldots, n - 1,
\]

where \( p_{lr}, q_r \in \mathbb{C} \).

Let \( B_2 \) be the subset of \( SL_n \) consisting of those \( h \) for which the corresponding system (2) is of rank \( n - 1 \). It is easily seen that \( B_2 \) is \( \mathbb{Q} \)-open and nonempty. We show now that the \( \mathbb{Q} \)-open subset \( B = B_1 \cap B_2 \neq \emptyset \) in \( SL_n \) has the desired property. Suppose \( h \in B \). Then in the matrix of the system (2) defining the variety \( T_h \) there exists a minor of order \( n - 1 \) identically not equal to zero. From Cramer's rule it follows that this minor is different from zero on \( T_h \) and that the coordinates \( a_{1l}, \ldots, a_{n-1} \) connected with it can be expressed in a rational fashion in terms of the others. This proves the rationality of \( T_h \), and therefore of \( T \), since \( T \) is open in \( T_h \). The rationality of \( W_h \) now follows automatically, since finding the first coordinate of the point \((x, y) \in W_h \) when the second is fixed reduces to solving the matrix equation \( xy = hyx \), which is equivalent to a linear system in the elements of the matrix \( x \). This completes the proof of Proposition 3.

There is no difficulty in completing the proof of Theorem 1. Considering "generic" \( n \times n \) matrices \( x_1, y_1, \ldots, x_{g-1}, y_{g-1} \), let \( K \) be the field generated over \( \mathbb{Q} \) by the elements of these matrices, and construct the matrix \( h = [x_1, y_1] \cdots [x_{g-1}, y_{g-1}] \in SL_n(K) \). Then the \( \mathbb{Q} \)-rational function field \( \mathbb{Q}(R_n(\Gamma)) \) is isomorphic to the \( K \)-rational function field \( K(W_{h-1}) \). Since \( h \) is a "generic" point of the group \( SL_n \) over \( \mathbb{Q} \), it lies in the \( \mathbb{Q} \)-open subset \( B \) constructed in Proposition 3; therefore, by that proposition, \( W_{h-1} \) is \( K \)-rational, i.e., the extension \( K(W_{h-1})/K \) is purely transcendental. But \( K \) is a purely transcendental extension of \( \mathbb{Q} \), and therefore the same is true of \( \mathbb{Q}(R_n(\Gamma)) \simeq K(W_{h-1}) \).

**Proof of Theorem 2.** There exists a commutative diagram

\[
\begin{array}{ccc}
R_n(\Gamma) & \overset{\sigma}{\longrightarrow} & X_n(\Gamma) \\
\sigma \downarrow & & \downarrow \delta \\
R_n(F) & \overset{\tau}{\longrightarrow} & X_n(F),
\end{array}
\]

in which \( \sigma \) and \( \tau \) are the natural projections of the varieties of representations onto the corresponding varieties of characters (see [2]), and \( \varphi \) and \( \delta \) are induced by the restriction. It is easily seen that the subset \( W_0 \subset R_n(F) \) of irreducible representations is a nonempty open \( \mathbb{Q} \)-defined subvariety. Let \( W \subset X_n(F) \) be an open subset contained in \( \tau(W_0) \) (clearly \( \tau^{-1}(W) \subset W_0 \)), and suppose \( w \in W \) and \( \bar{w} \in \delta^{-1}(w) \). It is easily seen that \( \sigma \) induces a bijective \( \mathbb{Q} \)-defined morphism

\[
\bar{\delta}: F = \varphi^{-1}(\bar{w}) \rightarrow \delta^{-1}(w) = D.
\]

Thus, the fibers \( F \) and \( D \) are birationally isomorphic over the field over which they are both defined.

Now let \( \rho \) be a generic point over \( \mathbb{Q} \) of the variety \( R_n(\Gamma) \); \( \omega = \sigma(\rho) \) and \( \mu = \delta(\omega) \) generic points of \( X_n(\Gamma) \) and \( X_n(F) \), respectively; and \( G = \delta^{-1}(\mu) \) a generic fiber of \( \delta \). Since condition 1) and Proposition 2 imply that \( G \) is irreducible, the field \( L = \mathbb{Q}(X_n(\Gamma)) \) is isomorphic to the rational function field \( K(G) \), where \( K = \mathbb{Q}(\mu) = \mathbb{Q}(X_n(F)) \). Clearly, there exists a preimage \( \tilde{\omega} \in \tau^{-1}(\omega) \),
\(\omega = (x_1, y_1, \ldots, x_{g-1}, y_{g-1})\), such that \(h = [x_1, y_1] \cdots [x_{g-1}, y_{g-1}] \in SL_n(K)\).

It follows from the above that \(G\) is isomorphic over \(K\) to the variety \(W_{h-1}\), which is \(K\)-rational. This gives the desired result.

**Proposition 4.** Suppose \(x, y \in GL_n\), with \(y\) semisimple. Then the set \(xZ(y)\) contains a regular semisimple element.

Explicit computations in each of the remaining cases yield a complete verification of 2) for \(n \leq 4\).

**Remark.** All the above results remain valid for a group \(\Gamma\) with \(n \geq 4\) generators \(x_1, \ldots, x_n\) and one defining relation of the form \(r = r_1[x_{n-3}, x_{n-2}][x_{n-1}, x_n]\), where \(r_1\) lies in the commutator subgroup of the free group \(F(x_1, \ldots, x_{n-4})\).

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**Bibliography**


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