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CHARACTER RINGS OF REPRESENTATIONS OF FINITELY GENERATED GROUPS

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ABSTRACT. Let $\Gamma = \langle g_1, g_2, \dots, g_m\rangle$ be a group with m generators. For an arbitrary yield K and a linear algebraic K-group G, the set of all representations $\mathrm{Hom}(\Gamma, G(K))$ can be identified in a natural way with the K-points of a certain algebraic variety. For any $g \in \Gamma$ we define a function τ_g on $\mathrm{Hom}(\Gamma, G(K))$ with values in K:

$$\tau_g(\rho) = \mathrm{tr}\,(\rho(g)), \qquad \rho \in \mathrm{Hom}(\Gamma, G(K)),$$

where $\mathrm{tr}\, X$ denotes the trace of a matrix X. Consider the ring $T(\Gamma, G(K))$ generated by the functions τ_g; it is called character ring of the representations of Γ in $G(K)$. Our main goal is to answer the question of whether the rings $T(\Gamma, \mathrm{GL}_n(K))$ and $T(\Gamma, \mathrm{SL}_n(K))$ and finitely generated. The answer is given in Theorems 1 and 2.

Bibliography: 9 titles.

§1. Introduction and formulation of the basic results

Let $\Gamma = \langle g_1, g_2, \dots, g_m\rangle$ be an arbitrary group with m generators. For a fixed K and a linear algebraic K-group G the collection $\mathrm{Hom}(\Gamma, G(K))$ of all representations can be identified in a natural way with the K-points of a certain algebraic variety. For each $g \in \Gamma$ we define the K-valued function τ_g on $\mathrm{Hom}(\Gamma, G(K))$ by

$$\tau_g(\rho) = \mathrm{tr}\,(\rho(g)), \qquad \rho \in \mathrm{Hom}(\Gamma, G(K)),$$

where $\mathrm{tr}\, X$ denotes the trace of a matrix X.

Consider the ring $T(\Gamma, G(K))$ generated by the functions τ_g (it would be more precise to write τ_g^G instead of τ_g, but each time it will be clear from the context which group the representations are into). It is called the ring of characters of representations of the group Γ in $G(K)$. This ring was first studied for the case $G(K) = \mathrm{SL}_2(\mathbb{C})$ by Vogt [9] and Fricke [4] almost a century ago. At the present time the ring $T(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is usually called the Fricke character ring for the group Γ. A survey of results on the ring $T(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ and its applications to various problems in group theory and linear differential equations is contained in Magnus' paper [6]. Interesting applications in three-dimensional topology are given in the recent important paper [3].

Here we consider a more general but also classical situation: the character rings $T(\Gamma, \mathrm{GL}_n(K))$ and $T(\Gamma, \mathrm{SL}_n(K))$ for arbitrary and special n-dimensional representations. One of the central points here is the question of whether the rings $T(\Gamma, \mathrm{GL}_n(K))$ and $T(\Gamma, \mathrm{SL}_n(K))$ are finitely generated. In 1972

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Horowitz [5] showed that the Fricke character ring $T(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is finitely generated. The question of whether the rings $T(\Gamma, \mathrm{GL}_n(K))$ and $T(\Gamma, \mathrm{SL}_n(K))$ are finitely generated was discussed by Bass and Lubotzky in [2].

The solution of the problem of whether $T(\Gamma, \mathrm{GL}_n(K))$ and $T(\Gamma, \mathrm{SL}_n(K))$ are finitely generated is contained in the following two theorems.

THEOREM 1. *Suppose that the group Γ has an infinite cyclic factor group. Then the following statements hold for a field K of zero characteristic:*

1) *for all $n \geq 2$ the ring $T(\Gamma, \mathrm{GL}_n(K))$ is not finitely generated;*
2) *for $n \geq 1$ the ring $T(\Gamma, \mathrm{SL}_n(K))$ is not finitely generated;*
3) *for $n = 3$ the ring $T(\Gamma, \mathrm{SL}_3(K))$ is finitely generated for any group Γ.*

The case of a field K of positive characteristic is considered in the second theorem. Note at once that for a finite field K the ring $T(\Gamma, G(K))$ is finitely generated for any group Γ. This follows easily from the fact that Γ has only finitely many subgroups of fixed index. Denote by $T_K(\Gamma, \mathrm{GL}_n(K))$ and $T_K(\Gamma, \mathrm{SL}_n(K))$ the K-algebra or characters for the corresponding representations.

THEOREM 2. *Let K be an infinite field of characteristic $p > 0$. Then the following statements hold:*

1) *for any group Γ the ring $T(\Gamma, \mathrm{GL}_n(K))$ is finitely generated for $n < p$, and the ring $T(\Gamma, \mathrm{SL}_n(K))$ is finitely generated for $n < 2p$;*

2) *if the group Γ has an infinite cyclic factor group, then the K-algebras $T_K(\Gamma, \mathrm{GL}_n(K))$ and $T_K(\Gamma, \mathrm{SL}_n(K))$ are not finitely generated for $n \geq p$ and $n \geq 2p$, respectively.*

We direct attention to the statement 2) in Theorem 2, which shows that the results of Procesi [8] on invariants of a finite collection of $n \times n$ matrices do not extend to the case of fields of arbitrary positive characteristic.

A brief exposition of the results in this paper was published in [1].

§2. Rings of characters of representations over fields of characteristic zero

We remark first that for any subgroup $H \subset \mathrm{GL}_n(K)$ the ring $T(\Gamma, H)$ is a homomorphic image of the ring $T(\Gamma, \mathrm{GL}_n(K))$, in particular, the ring $T(\Gamma, \mathrm{GL}_n(K))$ is infinitely generated if the ring $T(\Gamma, \mathrm{SL}_n(K))$ is infinitely generated. Similarly, if $\varphi: \Gamma_1 \to \Gamma_2$ is an epimorphism, then it induces an epimorphism $\varphi^*: T(\Gamma_1, G(K)) \to T(\Gamma_2, G(K))$, where $\varphi^*(\tau_g) = \tau_{\varphi(g)}$, $g \in \Gamma_1$. In particular, $T(\Gamma_1, G(K))$ is infinitely generated if $T(\Gamma_2, G(K))$ is infinitely generated. Obviously, among groups with m generators it is the free group that has the largest character ring.

To prove the theorems stated above we need the following lemmas.

LEMMA 1. *Suppose that the field K is infinite, $\Gamma = \langle g \rangle$ is an infinite cyclic group, and $K_1 \supset K$. Then the ring $T(\Gamma, \mathrm{SL}_n(K))$ (respectively, $T(\Gamma, \mathrm{GL}_n(K))$) is finitely generated if and only if the ring*

$$T(\Gamma, \mathrm{SL}_n(K_1)) \qquad (\textit{respectively}, \ T(\Gamma, \mathrm{GL}_n(K_1)))$$

is finitely generated.

PROOF. Suppose, for example, that τ_{g^i}, $-l \leq i \leq l$, is a finite system of generators of the ring $T(\Gamma, \mathrm{SL}_n(K))$. Then for any given $m \in \mathbb{Z}$ we have the equality

$$\tau_{g^m} = P(\tau_{g^{-l}}, \dots, \tau_{g^l}), \tag{1}$$

where $P \in \mathbb{Z}[y_{-l}, \dots, y_l]$. Since Γ is a cyclic group, (1) is equivalent to the equality

$$\operatorname{tr} X^m = P(\operatorname{tr} X^{-l}, \dots, \operatorname{tr} X^l), \tag{2}$$

where $X = (x_{ij})$ is an arbitrary matrix in $\mathrm{SL}_n(K)$. The equality (2) is equivalent in turn to the condition $Q(x_{ij}) = 0$, where Q is a regular function on $\mathrm{SL}_n(K)$. Since K is an infinite field,

$$Q(x_{ij}) = (\det X - 1)Q_1(x_{ij}),$$

from which it is clear that Q vanishes on $\mathrm{SL}_n(K_1)$. Consequently, (2) holds for any matrix X in $\mathrm{SL}_n(K_1)$, and this means that the functions τ^{g^i}, $-l \le i \le l$, generate the ring $T(\Gamma, \mathrm{SL}_n(K_1))$. The converse assertion in the lemma follows from our remarks above.

LEMMA 2. *If the ring $T(\Gamma, \mathrm{GL}_n(K))$ (respectively, $T(\Gamma, \mathrm{SL}_n(K))$) is not finitely generated for some n, then for any $m > n$ the ring $T(\Gamma, \mathrm{GL}_m(K))$ (respectively, $T(\Gamma, \mathrm{SL}_m(K))$) is infinitely generated.*

PROOF. Assume that $T(\Gamma, \mathrm{GL}_n(K))$ is not finitely generated, and for some $m > n$ the ring $T(\Gamma, \mathrm{GL}_m(K))$ is generated by the functions τ_{γ_i}, $1 \le i \le s$, $\gamma_i \in \Gamma$. We consider the standard imbedding of $\mathrm{GL}_n(K)$ in $\mathrm{GL}_m(K)$:

$$\varepsilon \colon X \mapsto \begin{pmatrix} X & 0 \\ 0 & E_{m-n} \end{pmatrix}.$$

The ring $T(\Gamma, \varepsilon(\mathrm{GL}_n(K)))$ is an epimorphic image of the ring $T(\Gamma, \mathrm{GL}_m(K))$, and hence is generated by the functions τ'_{γ_i}, $1 \le i \le s$ (a single prime will mark the functions in $T(\Gamma, \varepsilon(\mathrm{GL}_n(K)))$, and a double prime the functions in $T(\Gamma, \mathrm{GL}_n(K))$). Thus, for any $g \in \Gamma$ we must have the equality

$$\tau'_g = P(\tau'_{\gamma_1}, \dots, \tau'_{\gamma_s}), \tag{3}$$

where $P \in \mathbb{Z}[y_1, \dots, y_s]$. Since $\gamma'_\gamma = \tau''_\gamma + m - n$ for any $\gamma \in \Gamma$, the equality (3) can be written in the form

$$\tau''_\gamma = P_1(\tau''_{\gamma_1}, \dots, \tau''_{\gamma_s}) + l, \qquad l \in \mathbb{Z}, \tag{4}$$

where $P_1 \in \mathbb{Z}[y_1, \dots, y_s]$ is a polynomial without free term. We show that l is divisible by n. Taking the identity representation, we have from (4) that

$$n = P_1(n, \dots, n) + l,$$

from which it is clear that $l = rn$, $r \in \mathbb{Z}$. Since $\tau''_e = n$, where e is the identity of the group Γ, it follows that $l = r \cdot \tau''_e$. It now follows from (4) that the functions τ''_e, $\tau''_{\gamma_1}, \dots, \tau''_{\gamma_s}$ generate the ring $T(\Gamma, \mathrm{GL}_n(K))$, which contradicts our assumption that $T(\Gamma, \mathrm{GL}_n(K))$ is not finitely generated. The lemma is proved.

LEMMA 3. *Suppose that* Γ *is an infinite cyclic group. Then for a field* K *of characteristic zero the ring* $T(\Gamma, \mathrm{GL}_n(K))$ *is not finitely generated, for all* $n \geq 2$.

PROOF. Assume the contrary: some finite collection of functions τ_{g^i}, $-l \leq i \leq l$, generates the ring $T(\Gamma, \mathrm{GL}_n(K))$. Then for any $m > l$ we must have the equality

$$\tau_{g^m} = \sum_{(i_1, \dots, i_s)} a_{i_1 \cdots i_s} \tau_{g^{i_1}} \cdots \tau_{g^{i_s}}, \tag{5}$$

where $-l \leq i_j \leq l$, $j = 1, \dots, s$, $a_{i_1 \cdots i_s} \in \mathbb{Z}$. Since Γ is an infinite cyclic group, (5) is equivalent to the equality

$$\operatorname{tr} X^m = \sum_{(i_1, \dots, i_s)} a_{i_1} \cdots i_s \operatorname{tr} X^{i_1} \cdots \operatorname{tr} X^{i_s}, \tag{6}$$

where $X = (x_{ij})$ is an arbitrary matrix in $\mathrm{GL}_n(K)$. Lemma 1 enables us to take X to be a "general" matrix with independent elements x_{ij}. After multiplying both sides of (6) by a suitable power of $\det X$, we get two polynomials in x_{ij} that take the same values on $\mathrm{GL}_n(K)$, and are thus equal. Since the polynomial on the left-hand side is homogeneous, the polynomial on the right-hand side must also be homogeneous, and it follows from a comparison of powers that $i_1 + \cdots + i_s = m$ for each tuple $(i_1, \dots, i_s)$. Since $m > l$, it follows that $s \geq 2$. Setting $X = E_n$ in (6), we get that

$$n = \sum_{(i_1, \dots, i_s)} a_{i_1 \cdots i_s} n^s.$$

The last equality is impossible because the right-hand side is divisible by n^2 but the left-hand side is not. Lemma 3 is proved.

LEMMA 4. *Suppose that* Γ *is an infinite cyclic group, and* K *is a field of characteristic* 0. *Then for* $n \geq 4$ *the ring* $T(\Gamma, \mathrm{SL}_n(K))$ *is not finitely generated.*

PROOF. By Lemma 2, it suffices to consider the case $n = 4$. Assume that for some l the functions τ_{g^i}, $-l \leq i \leq l$, generate the ring $T(\Gamma, \mathrm{SL}_4(K(x)))$, where x is a transcendental element over K. Then for any $m > l$ we must have the equality

$$\tau_{g^m} = \sum_{i=-l}^{l} a_i \tau_{g^i} + \sum_{(i_1, \dots, i_s)} a_{i_1 \cdots i_s} \tau_{g^{i_1}} \cdots \tau_{g^{i_s}}, \tag{7}$$

where $-l \leq i_j \leq l$, $j = 1, \dots, s$, a_i, and $a_{i_1 \cdots i_s} \in \mathbb{Z}$. Consider the representation

$$\rho \colon g \mapsto X = \operatorname{diag}(x, x, x^{-1}, x^{-1}). \tag{8}$$

Obviously, $\operatorname{tr} X^d = 2(x^d + x^{-d})$ for any $d \in \mathbb{Z}$. For the representation (8) we get from (7) that

$$\begin{aligned} 2(x^m + x^{-m}) &= \sum_{i=-l}^{l} 2a_i (x^i + x^{-i}) \\ &\quad + \sum_{(i_1, \dots, i_s)} 2^s a_{i_1 \cdots i_s} (x^{i_1} + x^{-i_1}) \cdots (x^{i_s} + x^{-i_s}). \end{aligned} \tag{9}$$

Since $m > l$, the equality (9) can be written in the form

$$P(x) + P\left(\frac{1}{x}\right) = 0, \tag{9'}$$

where $P \in \mathbb{Z}[x]$, and $P \not\equiv 0$, since the coefficient of x^m in P is nonzero. The impossibility of (9′) is obvious enough. By Lemma 1, this contradiction completes the proof.

LEMMA 5. *For any* g, $h \in \Gamma$ *the following relation holds in* $T(\Gamma, \mathrm{SL}_3(K))$:

$$\tau_{g^2h} = \tau_g\tau_{gh} - \tau_{g^{-1}}\tau_h + \tau_{g^{-1}h}. \tag{10}$$

PROOF. We consider an arbitrary matrix $A \in \mathrm{SL}_3(K)$ and the roots α_1, α_2, and α_3 of its characteristic polynomial. By the Hamiltonian-Cayley theorem, A satisfies the equation

$$A^3 - A^2 \operatorname{tr} A + A(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - E = 0.$$

Note that

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} = \operatorname{tr} A^{-1},$$

because $\alpha_1\alpha_2\alpha_3 = 1$. Thus, we have the equality

$$A^3 - A^2 \operatorname{tr} A + A \operatorname{tr} A^{-1} - E = 0. \tag{11}$$

Let us multiply both sides of (11) by $A^{-1}B$ from the right, where $B \in \mathrm{SL}_3(K)$, and take the trace:

$$\operatorname{tr} A^2B = \operatorname{tr} A \operatorname{tr} AB - \operatorname{tr} A^{-1} \operatorname{tr} B + \operatorname{tr} A^{-1}B. \tag{12}$$

The equality (12) holds for any matrices A, $B \in \mathrm{SL}_3(K)$, and it immediately yields the equality (10).

The latter is often more convenient to use in the form

$$\tau_{tgh} = \tau_g\tau_{th} - \tau_{g^{-1}}\tau_{tg^{-1}h} + \tau_{ig^{-1}h}, \tag{13}$$

which is obtained from (10) by replacing h by $g^{-1}ht$. In the last two lemmas the group $\Gamma = \langle g_1, \dots, g_m\rangle$ is assumed to be free. For any arbitrary freely reduced word $W \in \Gamma$ we set $s_i(W) = (n_2, \dots, n_s)$, where $n_1, \dots, n_s$ are all the exponents with which the generators g_i appears in W, and $d_i(W) = |n_1| + \dots + |n_s|$. If g_i does not appear in W, then we set $s_i(W) = (0)$ and $d_i(W) = (0)$.

LEMMA 6. *Let* $g = g_r$ *be some generator of* Γ *and* $W = AB$ *an element of* Γ *such that* $d_{i_1}(A) = d_{i_2}(A) = \dots = d_{i_k}(A) = 0$ *for some* $i_1, \dots, i_k$ *(the tuple* $(i_1, \dots, i_k)$ *can also be empty). Then* $\tau_W = P(\tau_{W_1}, \dots, \tau_{W_s})$ *where* $P \in \mathbb{Z}[y_1, \dots, y_s]$, $W_i = A_iB_i$, $d_{i_1}(A_i) = \dots = d_{i_k}(A_i) = 0$, $B_i \in \{B, e\}$, $d_g(A_i) \le 2$, $1 \le i \le s$.

The proof of the lemma is by induction on $d_g(A)$. If $d_g(A) \le 2$, then there is nothing to prove. Suppose that the lemma is valid for all W such that $d_g(A) < n$. Consider $W = AB$, $d_g(A) = n$. Let $s_g(A) = (n_1, \dots, n_r)$.

Assume that for some i, say $i = 1$, we have that $|n_1| \ge 2$. For definiteness suppose that $n_1 \ge 2$. Then $W = U_1g^{n_1}U_2B$, and, using (13), we have that

$$\tau_W = \tau_{U_1g(g^{n_1-1}U_2B)} = \tau_g\tau_{U_1g^{n_1-1}U_2B} - \tau_{g^{-1}}\tau_{U_1g^{n_1-2}U_1B} + \tau_{U_1g^{n_1-3}U_2B}.$$

Since $d_g(U_1g^{n_1-1}U_2) < d_g(U_1g^{n_1}U_2) = d_g(A)$, $d_g(U_1g^{n_1-2}U_2) < d_g(A)$, and $d_g(U_1g^{n_1-3}U_2) < d_g(A)$, the required assertion follows by induction.

Suppose now that $|n_i| \le 1$, i.e., $n_i \in \{-1, -1\}$, $1 \le i \le r$, and $n_i = n_{i+1} = \varepsilon$ for some $i < r$. This means that $A = U_1g^\varepsilon U_2g^\varepsilon U_3$, and $d_g(U_2) = 0$. By (13),

$$\tau_W = \tau_{U_1(g^\varepsilon U_1)(g^\varepsilon U_3B)} = \tau_{g^\varepsilon U_2}\tau_{U_1g^\varepsilon U_3B} - \tau_{U_2^{-1}g^{-\varepsilon}}\tau_{U_1U_2^{-1}U_3B} + \tau_{U_1U_2^{-1}g^{-\varepsilon}U_2^{-1}U_3B}.$$

We set $A_1 = g^\varepsilon U_2$, $A_2 = U_1g^\varepsilon U_3$, $A_3 = g^{-\varepsilon}U_2^{-1}$, $A_4 = U_1U_2^{-1}U_3$, $A_5 = U_1U_2^{-1}g^{-\varepsilon}U_2^{-1}U_3$. Since $d_g(A_i) < d_g(A)$, $1 \le i \le 5$, the assertion of the lemma is valid for τ_W in view of the induction hypothesis.

It remains to consider the last possible case: $n_i \in \{-1, -1\}$, $1 \le i \le r$, $n_{i+1} = -n_i$ for all $i = 1, \dots, r-1$. Since $d_g(A) = n > 2$, it follows that $r > 2$ and A can be written in the form

$$A = U_1g^\varepsilon U_2g^{-\varepsilon}U_3g^\varepsilon U_4, \qquad \text{where } d_g(U_1) = d_g(U_2) = d_p(U_3) = 0.$$

Then, by (13),

$$\begin{aligned}\tau_W = \tau_{AB} &= \tau_{U_2g^\varepsilon(U_2g^{-\varepsilon}U_3g^\varepsilon U_4B)}\\ &= \tau_{g^\varepsilon}\tau_{U_1U_2g^{-\varepsilon}U_3g^\varepsilon U_4B} - \tau_{g^{-\varepsilon}}\tau_{U_1g^{-\varepsilon}U_2g^{-\varepsilon}U_3g^\varepsilon U_4B}\\ &\quad + \tau_{U_1g^{-2\varepsilon}U_2g^{-\varepsilon}U_3g^\varepsilon U_4B}.\end{aligned}$$

In turn,

$$\begin{aligned}\tau_{U_1g^{-2\varepsilon}U_2g^{-\varepsilon}U_3g^{-\varepsilon}U_4B} &= \tau_{(U_1g^{-\varepsilon})(g^{-\varepsilon}U_2)(g^{-\varepsilon}U_3g^\varepsilon U_4B)}\\ &= \tau_{g^{-\varepsilon}U_2}\tau_{U_1g^{-2\varepsilon}U_3g^\varepsilon U_4B} - \tau_{U_2^{-1}g^\varepsilon}\tau_{U_1g^{-\varepsilon}U_2^{-1}U_3g^\varepsilon U_4B}\\ &\quad + \tau_{U_1g^{-\varepsilon}U_2^{-1}g^\varepsilon U_2^{-1}U_3g^\varepsilon U_4B}.\end{aligned}$$

Thus,

$$\tau_w = \tau_{g^\varepsilon}\tau_{A_1B} - \tau_{g^{-\varepsilon}}\tau_{A_2B} + \tau_{g^{-\varepsilon}U_2}\tau_{A_3B} - \tau_{g^\varepsilon U_2^{-1}}\tau_{A_4B} + \tau_{A_5B},$$

where $A_1 = U_1U_2g^{-\varepsilon}U_3g^\varepsilon U_4$, $A_2 = U_1g^{-\varepsilon}U_2g^{-\varepsilon}U_4$, $A_3 = U_1g^{-2\varepsilon}U_3g^\varepsilon U_4$, $A_4 = U_1g^{-\varepsilon}U_2^{-1}U_3g^\varepsilon U_4$, $A_5 = U_1g^{-\varepsilon}U_2^{-1}g^\varepsilon U_2^{-1}U_3g^\varepsilon U_4$. Obviously, $d_g(A_1) < d_g(A)$ and $d_g(A_4) < d_g(A)$. Further, although $d_g(A_2) = d_g(A_3) = d_g(A_5) = d_g(A)$, the equalities $s_g(A_2) = (-\varepsilon, -\varepsilon, \cdots)$, $s_g(A_3) = (-2\varepsilon, \varepsilon, \cdots)$, and $s_g(A_5) = (-\varepsilon, \varepsilon, \varepsilon, \cdots)$ imply, as already shown above, that the statement of the lemma is valid for the functions τ_{A_2B}, τ_{A_3B}, τ_{A_5B}. The statement of the lemma is thereby obtained for τ_W by using the induction hypothesis for the functions τ_{A_1B} and τ_{A_4B}. The lemma is proved.

LEMMA 7. *Suppose that $W \in \Gamma$ is an arbitrary word. If $d_1(W) + \cdots + d_i(W) \le n$, then $\tau_W = P(\tau_{iW_1}, \dots, \tau_{W_s})$, $P \in \mathbb{Z}[y_1, \dots, y_s]$, and all the W_j, $1 \le j \le s$, have the property that*

$$d_1(W_j) + \cdots + d_i(W_j) + d_{i+1}(W_j) \le 3n.$$

PROOF. We employ induction on $d_{i+1}(W)$. If $d_{i+1}(W) = 0$, then the assertion of the lemma is obvious. Suppose that the assertion is valid for all $W \in \Gamma$ such that $d_1(W) + \cdots + d_i(W) \le n$, $d_{i+1}(W) < r$. Assume now that $d_{i+1}(W) =$

r. The word W can be written in the form $W = U_1 g_{i_1}^{n_1} \cdots U_k g_{i_k}^{n_k} U_{k+1}$, $1 \le i_j \le i$, $j = 1, \dots, k$, $d_s(U_m) = 0$, $1 \le s \le i$, $1 \le m \le k+1$. Since $\tau_W = \tau_{U_{k+1} u_1 g_{i_1}^{n_1} \cdots U_k, g_{i_k}^{n_k}}$, we can assume at once that $U_{k+1} = l$. Further, $d_1(W) + \cdots + d_i(W) = |n_1| + \cdots + |n_k| \le n$. Consequently, $k \le n$. If $d_{i+1}(U_j) \le 2$ for all $j = 1, \dots, k$, then $d_{i+1}(W) \le 2k \le 2n$ and $d_1(W) + \cdots + d_i(W) + d_{i+1}(W) \le n + 2n = 3n$. Therefore, we assume that for some j, say for $j = 1$, we have that $d_{i+1}(U_1) > 2$. Setting $A = U_1$, $B = g_{n_1}^{n_1} U_2 \cdots U_k g_{i_k}^{n_k}$, and $g = g_{i+1}$, we have from Lemma 6 that $\tau_W = P(\tau_{W_1}, \dots, \tau_{W_s})$, where $P \in \mathbb{Z}[y_1, \dots, y_s]$, $W_j = A_j B_j$, $d_1(A_j) = \cdots = d_i(A_j) = 0$, B_j is equal either to B or to e, and $d_{i+1}(A_j) \le 2$, $j = 1, 2, \dots, s$. If $B_j = e$, then $d_1(W_j) + \cdots + d_i(W_j) + d_{i+1}(W_j) \le 2 < 3n$. If $B_j = B$, then $d_{i+1}(W_j) = d_{i+1}(A_j B) \le d_{i+1}(W) - d_{i+1}(U_1) + 2 < d_{i+1}(W)$, and $d_1(W_j) + \cdots + d_i(W_j) \le n$. By the induction hypothesis, the assertion of the lemma is valid for all the functions τ_{W_j}, $1 \le j \le s$, and hence also for τ_W. Lemma 7 is proved.

Proof of Theorem 1. Assertions 1 and 2 of the theorem follow immediately from Lemmas 3 and 4. Assertion 3 will now be proved with the help of Lemmas 6 and 7. It suffices to consider the case of a free group Γ. If $W \in \Gamma$, then, by Lemma 6, $\tau_W = P(\tau_{W_1}, \dots, \tau_{W_s})$, $P \in \mathbb{Z}[y_1, \dots, y_s]$, $d_1(W_j) \le 2$, $j = 1, 2, \dots, s$. Thus, we can assume that $d_1(W) \le 2$. It follows from Lemma 7 that

$$\tau_W = F(\tau_{V_1}, \dots, \tau_{V_r}), \qquad F \in \mathbb{Z}[y_1, \dots, y_r],$$
$$d_1(V_i) + d_2(V_i) + \cdots + d_m(V_i) \le 2 \cdot 3^{m-1}, \qquad 1 \le i \le r.$$

It remains to see that $d_1(V_i) + \cdots + d_m(V_i)$ is the length of the word V_i, and there are finitely many words of bounded length in Γ. Accordingly, the functions $\tau_{W_1}, \dots, \tau_{W_s}$, where $W_1, \dots, W_s$ are all distinct words of length not exceeding $2 \cdot 3^{m-1}$, generate the ring $T(\Gamma, \mathrm{SL}_3(K))$. The proof of Theorem 1 is complete.

§3. Rings of characters of representations over fields of positive characteristic

In the case of fields of positive characteristic the situation becomes somewhat more complicated and requires a more detailed investigation. The contrast in comparison with fields of characteristic zero manifests itself, for example, in the fact that the K-algebra $T_K(\Gamma, \mathrm{GL}_n(K))$ for a field of characteristic zero is always finitely generated, while for a field of positive characteristic this algebra can fail to be finitely generated.

We introduce the following notation: $C_r = \{1, 2, \dots, r\}$, $D_r = \{\sigma : C_r \to C_p \mid \sigma \text{ is injective}\}$.

Lemma 8. *Suppose that k is a field of characteristic $p > 0$, and $(n_1, \dots, n_r)$ is an r-tuple of integers such that $r \le p$ and $n_1 + n_2 + \cdots + n_r \not\equiv 0 \pmod p$. Then the rational function*

$$\sum_{\sigma \in D_r} t_{\sigma(1)}^{n_1} t_{\sigma(2)}^{n_2} \cdots t_{\sigma(r)}^{n_r} \in K(t_1, \dots, t_p)$$

is not equal to zero.

PROOF. The rational function being considered does not change if the numbers in the r-tuple $(n_1, \dots, n_r)$ are permuted, therefore, it can be assumed without loss of generality that the numbers $n_1, \dots, n_r$ are arranged in nondecreasing order, i.e.,

$$n_1 = n_{i_0} = n_{i_0+1} = \cdots = n_{i_1} < n_{i_1+1} = \cdots = n_{i_2} < \cdots < n_{i_{s-1}+1} = \cdots = n_{i_s} = n_r.$$

It is easy to see that each of the s groups of equal exponents contain less than p exponents, because otherwise we would have that $r = p$ and $n_1 = \cdots = n_p$, i.e., $n_1 + \cdots + n_p \equiv 0 \pmod p$ despite the condition of the lemma. In other words, $i_k - i_{k-1} + 1 < p$ for $k = 1, \dots, s$. Suppose now that $\varphi, \sigma \in D_r$. We consider two monomials:

$$t_{\sigma(1)}^{n_1} \cdots t_{\sigma(r)}^{n_r} = t_{\sigma(i_0)}^{n_i} \cdots t_{\sigma(i_1)}^{n_i} \cdots t_{\sigma(i_{s-1}+1)}^{n_{i_s}} \cdots t_{\sigma(i_s)}^{n_{i_s}}$$

and

$$t_{\varphi(1)}^{n_1} \cdots t_{\sigma(r)}^{n_r} = t_{\varphi(i_0)}^{n_{i_1}} \cdots t_{\varphi(i_{s-1}+1)}^{n_{i_s}} \cdots t_{\varphi(i_s)}^{n_{i_s}}.$$

These monomials are equal if and only if the same variables appear in them with equal exponents, and this holds if and only if the numbers $\varphi(i_{j-1}+1), \dots, \varphi(i_j)$ are a permutation of the numbers $\sigma(i_{j-1}+1), \dots, \sigma(i_j)$ for all $j = 1, \dots, s$. Consequently, for any given monomial in the sum $\sum_{\sigma \in D_r} t_{\sigma(1)}^{n_i} \cdots t_{\sigma(r)}^{n_r}$ there are exactly

$$c(n_1, \dots, n_r) = (i_2 - i_0 + 1)!(i_2 - i_1 + 1)! \cdots (i_s - i_{s-1} + 1)! \qquad (14)$$

monomials equal to it in the sum (including the one being considered). The number $c_{(n_1, \dots, n_r)}$ does not depend on the choice of the monomial, but is completely determined by the r-tuple $(n_1, \dots, n_r)$. Obviously, $c_{(n_1, \dots, n_r)} \not\equiv \pmod p$, because $i_j - i_{j-1} + 1 < p$ for $1 \le j \le s$. Consequently, the rational function under consideration is not equal to zero. Lemma 8 is proved.

THEOREM 2. *Let K be an infinite field of characteristic $p > 0$. Then the following statements are valid:*

1) *for any group Γ the ring $T(\Gamma, \mathrm{GL}_n(K))$ is finitely generated for $n < p$, and the ring $T(\Gamma, \mathrm{SL}_n(K))$ is finitely generated for $n < 2p$;*

2) *if the group Γ has an infinite cyclic factor group, then the K-algebras $T_K(\Gamma, \mathrm{GL}_n(K))$ and $T_K(\Gamma, \mathrm{SL}_n(K))$ are not finitely generated for $n \ge p$ and $n \ge 2p$, respectively.*

PROOF. The first assertion of Theorem 2, namely that $T(\Gamma, \mathrm{GL}_n(K))$ is finitely generated, can be derived from the following result of Procesi [7]. Let $X_1, \dots, X_m$ be a collection of m general matrices in $M_n(\Omega)$, where $\Omega \supset K$. Procesi proved that the ring A generated by all the functions $\sigma_i(W)$, $i = 1, 2, \dots, n$, where σ_i denotes the ith coefficient of the characteristic polynomial of the matrix W, and W runs through all the monomials in $X_1, \dots, X_m$, is finitely generated. It follows from the condition $n < p$ that every $\sigma_i(W)$ can be expressed in the form of a polynomial in $\operatorname{tr} W$, $\operatorname{tr} W^2, \dots, \operatorname{tr} W^i$ with coefficients in the field $\mathbb{Z}/p\mathbb{Z}$. Hence, A can be generated by finitely many of the traces $\operatorname{tr} W_1, \dots, \operatorname{tr} W_d$. With respect to the system of generators $g_1, \dots, g_m$ of the group Γ we can assume without loss of generality that together with each g_i it also contains g_i^{-1}. Using the specialization $X_i \to g_i$, we get that the functions $\tau_{\overline{W}_1}, \dots, \tau_{\overline{W}_d}$ generate the ring $T(\Gamma, \mathrm{GL}_n(K))$, where $\overline{W}$ denotes the element of Γ obtained from the monomial W as a result of the specialization $X_i \to g_i$.

If in the case $n < p$ the ring $T(\Gamma, \mathrm{GL}_n(K))$ is finitely generated, then so is $T(\Gamma, \mathrm{SL}_n(K))$. But if $p \le n < 2p$, then we make the following observation. Let X be an arbitrary matrix in $\mathrm{SL}_n(K)$. In this case if $p \le i < n$, then $\sigma_i(X) = (1/\det X)\sigma_{n-i}(X^{-1}) = \sigma_{n-i}(X^{-1})$. Since $n - i < p$, it follows that $\sigma_{n-i}(X^{-1})$ can be expressed in terms of $\operatorname{tr} X^{-1}, \operatorname{tr} X^{-2}, \dots, \operatorname{tr} X^{i-n}$ which coefficients in the field $\mathbb{Z}/p\mathbb{Z}$. By the theory of Procesi, this implies that $T(\Gamma, \mathrm{SL}_n(K))$ is finitely generated for $p \le n < 2p$.

The main weight in the proof of Theorem 2 lies in the second assertion. According to Lemma 2, it suffices to prove this assertion for $n = p$. Moreover, it can be assumed that $\Gamma = \langle g \rangle$ is an infinite cyclic group. We assume the contrary, i.e., suppose that the algebra $T_K(\Gamma, \mathrm{GL}_p(K))$ has a finite system of generators τ_{g^i}, $-l \le i \le l$. Then for $m \in \mathbb{Z}$, $m > l$, $(m, p) = 1$, we have the equality

$$\tau_{g^m} = \sum_{(i_1, \dots, i_s)} a_{i_1 \cdots i_s}(m) \tau_{g^{i_1}} \cdots \tau_{g^{i_s}}, \tag{15}$$

where $-l \le i_j \le l$, $j = 1, 2, \dots, s$, and $a_{i_1 \cdots i_s}(m) \in K$. As in the proof of Lemma 3, we note that $i_1 + i_2 + \dots + i_s = m$; in particular, $s \ge 2$ in each tuple $(i_1, \dots, i_s)$. If for an arbitrary diagonal matrix $t = \operatorname{diag}(t_1, \dots, t_p)$ we consider the representation $g \mapsto t$, then (15) gives us the equality

$$t_1^m + \dots + t_p^m = \sum_{(i_1, \dots, i_s)} a_{i_1 \cdots i_s}(m) \prod_{j=1}^{s} (t_1^{i_j} + \dots + t_p^{i_j}), \tag{16}$$

which hold for all $t_1, \dots, t_p \in K^*$.

The product $\prod_{j=1}^{s}(t_1^{i_j} + \dots + t_p^{i_j})$ can be transformed into the following form:

$$\prod_{j=1}^{s} (t_1^{i_j} + \dots + t_p^{i_j}) = \sum_{(n_1, \dots, n_r)} b_{n_1 \cdots n_r}^{(i_1, \dots, i_s)} \left(\sum_{\sigma \in D_r} t_{\sigma(1)}^{n_1} \cdots t_{\sigma(r)}^{n_r} \right), \tag{17}$$

where $n_1 + n_2 + \dots + n_r = m$ and the summation is over the unordered collections $(n_1, \dots, n_r)$. The equality (17) is obtained as follows: if we multiply out the parentheses on the left-hand side of (17), then we get a sum of p^r monomials, where together with a monomial $t_{i_1}^{n_1} \cdots t_{i_r}^{n_r}$ the sum contains all monomials of the form $t_{\sigma(1)}^{n_1} \cdots t_{\sigma(r)}^{n_r}$, $\sigma \in D_r$; thus, we get the sum

$$\sum_{\sigma \in D_r} t_{\sigma(1)}^{n_1} \cdots t_{\sigma(r)}^{n_r},$$

and the integer $b_{n_1 \cdots n_r}^{(i_1, \dots, i_s)}$ shows how many sums like this accumulate after multiplying out the parentheses on the left-hand side of (17).

By Lemma 5, none of the sums $\sum_{\sigma \in D_r} t_{\sigma(1)}^{n_1} \cdots t_{\sigma(r)}^{n_r}$ is equal to zero, and the monomials in it differ from those of the analogous sum for another tuple $(m_1, \dots, m_q) \ne (n_1, \dots, n_r)$. The left-hand side of (17) is a sum of p^s monomials of the form $t_{i_1}^{m_1} \cdots t_{i_r}^{n_r}$, while the right-hand side is a sum of

$\sum_{(n_1,\dots,n_r)} b^{(i_1,\dots,i_s)}_{n_1\cdots n_r}|D_r|$ such monomials. Consequently,

$$\sum_{(n_1,\dots,n_r)} b^{(i_1,\dots,i_s)}_{n_1\cdots n_r}|D_r| = p^{\varepsilon}.$$

Since $|D_r| = p!/(p-r)!$ and $s \geq 2$, it follows that

$$\sum_{(n_1,\dots,n_r)} b^{(i_1,\dots,i_s)}_{n_1\cdots n_r} \frac{(p-r)!}{(p-1)!} \equiv 0 \pmod{p}. \tag{18}$$

We transform (16), using (17):

$$\begin{aligned} t_1^m + \cdots + t_p^m &= \sum_{(i_1,\dots,i_s)} a_{i_1,\dots,i_s}(m) \prod_{j=1}^{s} (t_1^{i_j} + \cdots + t_p^{i_j}) \\ &= \sum_{(i_1,\dots,i_s)} a_{i_1\cdots i_s}(m) \left(\sum_{(n_1,\dots,n_r)} b^{(i_1,\dots,i_s)}_{n_1\cdots n_r} \left(\sum_{\sigma\in D_r} t_{\sigma(1)}^{N_1} \cdots t_{\sigma(r)}^{n_r} \right)\right) \\ &= \sum_{(n_1,\dots,n_r)} c_{n_1}\cdots{}_{n_r} \left(\sum_{\sigma\in D_r} t_{\sigma(1)}^{n_1} \cdots t_{\sigma(r)}^{n_r} \right), \end{aligned} \tag{19}$$

where

$$c_{n_1\cdots n_r} = \sum_{(i_1,\dots,i_s)} a_{i_1\cdots i_s}(m) b^{(i_1,\dots,i_s)}_{n_1\cdots n_r} \in K.$$

Using (18), we get the following equality:

$$\begin{aligned} \sum_{(n_1,\dots,n_r)} c_{n_1\cdots n_r} \frac{(p-1)!}{(p-r)!} &= \sum_{(n_1,\dots,n_r)} \sum_{(i_1,\dots,i_s)} a_{i_1\cdots i_s}(m) b^{(i_1,\dots,i_s)}_{n_1,\dots,n_r} \frac{(p-1)!}{(p-r)!} \\ &= \sum_{(i_1,\dots,i_s)} a_{i_1\cdots i_s}(m) \left(\sum_{(n_1,\dots,n_s)} b^{(i_1,\dots,i_s)}_{n_1,\dots,n_r} \frac{(p-1)!}{(p-r)!} \right) \\ &= 0. \end{aligned}$$

On the other hand, it follows from (19) that $c_m = 1$, while all $c_{n_1,\dots,n_r} = 0$ for $r \geq 2$. Consequently,

$$\sum_{(n_1,\dots,n_r)} c_{n_1\cdots n_r} \frac{(p-1)!}{(p-r)!} = 1 \cdot \frac{(p-1)!}{(p-1)!} = 1.$$

This contradiction completes the proof that the algebra $T_K(\Gamma\,\mathrm{GL}_p(K))$ is infinitely generated.

To conclude the proof of Theorem 2 it must be shown that the algebra $T_K(\Gamma, \mathrm{SL}_{2p}(K))$ is not finitely generated. In view of Lemma 1 the field K can be assumed to be algebraically closed. We consider the imbedding

$$\varepsilon\colon \mathrm{GL}_p(K) \to \mathrm{SL}_{2p}(K), \qquad X \mapsto \begin{pmatrix} X & 0 \\ 0 & \alpha E_p \end{pmatrix},$$

where $\alpha = (\det X)^{-1/p}$. Since $\operatorname{tr} X = \operatorname{tr} \varepsilon(X)$, it follows that $T_K(\Gamma\,\mathrm{GL}_p(K)) = T_K(\Gamma, \varepsilon(\mathrm{GL}_p(K)))$. And since $T_K(\Gamma, \varepsilon(\mathrm{GL}_p(K)))$ is a homomorphic image of the K-algebra $T_K(\Gamma, \mathrm{SL}_{2p}(K))$ and (as we already showed) the K-algebra $T_K(\Gamma, \mathrm{GL}_p(K))$ is not finitely generated, the K-algebra $T_K(\Gamma, \mathrm{SL}_{2p}(K))$ is also not finitely generated. By Lemma 2, $T_K(\Gamma, \mathrm{SL}_n(K))$ is not finitely generated, for all $n \geq 2p$. Theorem 2 is proved.

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