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On Closability of Nonclosable Operators

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Abstract. Most of the linear operators acting in Banach spaces are assumed to be closed or closable. Yet in a series of problems some nonclosable operators arise in a natural way and consequently it is advisable to find a way of application of the general theory to such operators.

The present paper deals with a certain construction method which enables us to obtain a closed operator associated with a given nonclosable operator. Also some examples of such operators are given whose construction proves to be useful.

We would like to add that this construction has been applied to some particular cases but our purpose is to show a wide applicability of this approach.

1. Fundamental construction In this section the following notations will be used:

$(X, \|\,\|_1)$ and $(Y, \|\,\|_2)$ denote Banach spaces;

$A : X \supset D(A) \to Y$ denotes a linear operator with dense domain $D(A)$;

$M(A) := \{y \in Y : \forall n \in \mathbb{N} \quad \exists x_n \in D(A), \quad x_n \to 0, \quad Ax_n \to y\}$;

$G(A) := \{(x, Ax) \in X \times Y : x \in D(A)\}$ denotes the graph of A;

$P_X : X \times Y \to X$ and $P_Y : X \times Y \to Y$ denote the projections on X and Y respectively.

LEMMA 1. *The set $M(A)$ is a closed vector subspace of Y.*

Proof. It follows from the linearity of A that $M(A)$ is a vector subspace of Y.

Let (y_n) be a sequence of points $y_n \in M(A)$ such that $y_n \to y$. Then for each $n \in N$ there exists points $x_n^{(k)} \in D(A)$ such that $x_n^{(k)} \to 0$ and $Ax_n^{(k)} \to y_n$ when $k \to \infty$. For each $n \in N$ we choose an index $k(n)$ such

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that

$$\|x_n^{(k(n))}\|_1 \leq \frac{1}{n} \quad \text{and} \quad \|Ax_n^{(k(n))} - y_n\|_2 \leq \frac{1}{n}.$$

Then $z_n = x_n^{(k(n))} \to 0$ and $Az_n \to y$, i.e. $y \in M(A)$.

The linear subspace $M(A)$ is, in a certain sense, a measure of nonclosability of the operator A.

DEFINITION. A linar operator A is said to be *closable* [2] if the closure $\overline{G(A)}$ of the graph $G(A)$ is the graph of a certain operator $\bar{A}$, called the closure of A. Thus $D(A) \subset D(\bar{A})$ and $Ax = \bar{A}x$ for all $x \in D(A)$.

In the case of a nonclosable operator A, the set $\overline{G(A)} \subset X \times Y$ is only a relation which is not a mapping in the sense that

$$\text{if } (x, u_1), (x, u_2) \in \overline{G(A)}, \text{ then } u_1 = u_2.$$

LEMMA 2. *Let $A : X \to Y$ be an arbitrary linear operator and let $(x, y_1) \in \overline{G(A)}$. Then $(x, y_2) \in \overline{G(A)}$ if and only if $y_1 - y_2 \in M(A)$.*

Proof. Since $(x, y_1) \in \overline{G(A)}$ there exists a sequence (x_n) of points $x_n \in D(A)$ such that $x_n \to x$ and $Ax_n \to y_1$. Now, let $(x, y_2) \in \overline{G(A)}$. Then there exists a sequence (x'_n) such that $x'_n \in D(A)$, $x'_n \to x$ and $Ax'_n \to y_2$. Thus $x_n - x'_n \in D(A)$, $x_n - x'_n \to 0$ and $A(x_n - x'_n) \to y_1 - y_2$, i.e. $y_1 - y_2 \in M(A)$. Conversely, if $y_1 - y_2 \in M(A)$, then there exists a sequence (x''_n), $x''_n \in D(A)$ such that $x''_n \to 0$ and $Ax''_n \to y_1 - y_2$. Then $\bar{x}_n := x_n - x''_n \to x$ and $A\bar{x}_n = Ax_n - Ax''_n \to y_1 - (y_1 - y_2) = y_2$, i.e. $(x, y_2) \in \overline{G(A)}$.

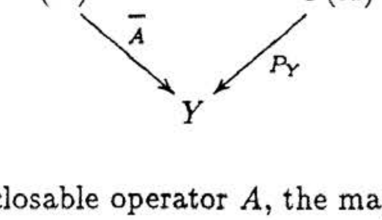
In other terminology, if $M(A) \neq \{0\}$, then $\overline{G(A)}$ determines a multivalued operator $\bar{A} : X \supset D(\bar{A}) \to Y$, where $D(\bar{A}) = P_X(\overline{G(A)})$. Thus $\bar{A}(x) = y + M(A)$, for $(x, y) \in \overline{G(A)}$.

If the operator A is closable, then the projection P_X induces a linear isomorphism

$$P_1 = P_X|_{\overline{G(A)}} : \overline{G(A)} \to D(\bar{A}),$$

so that the diagram

(1)



is commutative.

In the case of a nonclosable operator A, the mapping P_1 is not injective and the diagram (1) becomes

$$P_Y : \overline{G(A)} \to Y.$$

Let $\widetilde{Z}$ denote the set of all Cauchy sequences (x_n) in X such that $x_n \in D(A)$ and (Ax_n) is a Cauchy sequence in Y.

DEFINITION. Sequences $(x_n), (x'_n) \in \widetilde{\mathcal{Z}}$ are said to be equivalent if $x_n - x'_n \to 0$ and $Ax_n - Ax'_n \to 0$. The set of equivalence classes of sequences $(x_n) \in \widetilde{\mathcal{Z}}$ will be denoted by Z_A.

LEMMA 3. *For every linear operator $A : X \supset D(A) \to Y$, the mapping $\mathcal{J}_A : Z_A \to \overline{G(A)}$ defined by*

$$\mathcal{J}_A(z) = (\lim_{n\to\infty} x_n, \lim_{n\to\infty} Ax_n), \quad (x_n) \in z$$

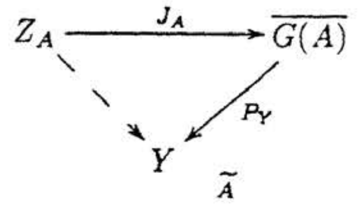
is an isometric isomorphism.

Proof. Since $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} Ax_n$ are independent of the choice of the representative $(x_n) \in \widetilde{\mathcal{Z}}$, the mapping $\mathcal{J}_A$ is well defined. Also the inclusion $\mathcal{J}_A(Z_A) \subset \overline{G(A)}$ holds. Now let $\mathcal{J}_A(z) = 0$, i.e. $\lim_{n\to\infty} x_n = 0$ and $\lim Ax_n = 0$. This implies that $z = 0$.

It remains to show that $\mathcal{J}_A$ is surjective. To this end let $(x, y) \in \overline{G(A)}$. This means that there exists a sequence (x_n) of points $x_n \in D(A)$ such that $x_n \to x$, $Ax_n \to y$ and so $(x_n) \in \widetilde{\mathcal{Z}}$. Let z be the equivalence class determined by (x_n). Then $J_A(z) = (x, y)$. This completes the proof.

If the operator A is closable, then Z_A is also algebraically isomorphic to $D(\overline{A})$. Therefore, in the case of an arbitrary operator A it is natural to consider the space Z_A instead of $D(\overline{A})$ and then to construct a new operator $\widetilde{A}$ by means of the diagram

(2)



which is analogous to diagram (1).

DEFINITION. The operator $\widetilde{A} : Z_A \to Y$ defined by the formula

$$\widetilde{A}(z) = \lim_{n\to\infty} Ax_n, \quad (x_n) \in z \tag{3}$$

is called the *extended closure* of A.

The total of our results can be summerized as follows. The equation of the form $Ax = y$ with a nonclosable operator A leads to equation $\overline{A}x = y$ with multivalued operator $\overline{A}$. Multivalence of the operator $\overline{A}$ means that the initial problem has not been posed correctly because the right-hand side of the equation is not uniquely defined by x.

For example, if in the initial problem an element x describes a certain process, then multivalence of the operator $\overline{A}$ means that there exist many different states of a system corresponding to the element x. Therefore each element x must be the whole class of elements of another type, each of which uniquely defines y, which means that each element x of new type defines a certain state of our system. The set of these new elements x, introduced above, forms the space Z_A whose elements are classes of equivalent Cauchy sequences of elements from $D(A)$.

Let us emphasize that this equivalence relation used in this construction is stronger than the one in the case of completion of space. Thus our approach leads to a decomposition into smaller equivalence classes than in the case of completion in the norm of the space X.

On the space Z_A, in a natural way, two topologies can be defined. The first one is defined by the seminorm

$$\varrho(z) = \lim_{n\to\infty} \|x_n\|_1, (x_n) \in z \tag{4}$$

and the second one by the norm

$$\|z\|_3 = \lim_{n\to\infty} (\|x_n\|_1 + \|Ax_n\|_2). \tag{5}$$

If the operator A is closable, then $\varrho(z)$ becomes $\|x\|_1$ as Z_A and $D(\overline{A})$ are isomorphic. If, however, A is nonclosable, then $\varrho(z)$ is not a norm. Therefore A is not a continuous operator from the nonseparated locally convex space Z_A to Y and its graph $G(A)$ is closed as it is isometrically isomorphic to $\overline{G(A)}$.

The space Z_A equipped with norm (5) is a Banach space as it is the completion of $D(A)$ with respect to its graph norm.

THEOREM 1. *The operator* $\widetilde{A} : (Z_A, \|\cdot\|_3) \to (Y, \|\cdot\|_2)$ *is a linear bounded operator and that*

$$\|\widetilde{A}\| = \begin{cases} < 1, & \text{if } A \text{ is bounded} \\ 1, & \text{if } A \text{ is unbounded.} \end{cases}$$

Proof. It follows from (3) and (5) that

$$\|\widetilde{A}z\| = \lim_{n\to\infty} \|Ax_n\|_2 \le \lim_{n\to\infty} (\|x_n\|_1 + \|Ax_n\|_2) = \|z\|_3, \tag{6}$$

i.e. $\|\widetilde{A}\| \le 1$.

Now let c be the least constant for which the inequality

$$\|Ax\|_2 \le c(\|x\|_1 + \|Ax\|_2), \quad x \in D(A) \tag{7}$$

holds. Then

$$c = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_1 + \|Ax\|_2} = \sup_{x\neq 0} \frac{1}{1 + \frac{\|x\|_1}{\|Ax\|_2}} = \frac{1}{1 + \inf_{x\neq 0} \frac{\|x\|_1}{\|Ax\|_2}}$$

$$= \frac{1}{1 + \left(\sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_1}\right)^{-1}} =$$

$$= \begin{cases} \dfrac{1}{1+\frac{1}{\|A\|}} < 1, & \text{if } A \text{ is bounded} \\ 1, & \text{if } A \text{ is unbounded.} \end{cases}$$

The domain $D(A)$ of A, in a natural way, (by means of stationary sequences), is imbedded in Z_A and that $\tilde{A}x = Ax$ for all $x \in D(A)$, i.e. $\tilde{A}$ is an extention of the operator A.

THEOREM 2. *Let $A_e : Z_A \supset D(A) \to Y$ be the operator defined by $A_e x = Ax$. Then A_e is closable and $\tilde{A}$ is its closure.*

Proof. The topology on $G(A_e)$ generated by the seminorm $\varrho(z) + \|Az\|_2$ coincides with the topology generated by the norm $\|x\|_1 + \|Ax\|_2$. Consequently the closure of $G(A_e)$ in both topologies coincides. This completes the proof.

In view of Lemma 2, $M(A)$ is isomorphic to $\ker P_X$, where $P_X : \overline{G(A)} \to X$ is the projection and so $M(A)$ is closed in $G(\bar{A})$.

As the vector space $\overline{G(A)}$ and consequently, in view of Lemma 2, the space Z_A is isomorphic to $D(\bar{A}) + M(A)$, where $D(\bar{A}) = P_X(\overline{G(A)})$.

In the case when $\ker P_X$ is topologically complemented in $\overline{G(A)}$ and N is its complement, then

$$Z_A \cong N \oplus M(A),$$

i.e. the norm on Z_A is equivalent to the norm $\|z\|_4 = \|z_1\| + \|z_2\|$, where $z = z_1 + z_2$, $z_1 \in N$ and $z_2 \in M(A)$. Therefore, e.g. in the case of Hilbert spaces, we have the following description of the space Z_A and the operator $\tilde{A}$.

THEOREM 3. *Let X and Y be Hilbert spaces and let $A : X \to Y$ be a linear operator with its domain $D(A)$. Let $A_0 : D(A) \to Y$ be the operator defined by $A_0 x = P_Y(P_N(x, Ax))$, where N is the orthogonal complement to $M(A)$ in $\overline{G(A)}$. Then the operator A_0 is closable, $D(\overline{A_0}) = D(\bar{A})$ and the operator $\tilde{A}$ can be expressed in the form*

$$\tilde{A}(x, y) = \overline{A_0}x + y.$$

The constructions given above become simpler in two extreme cases: (1) when $M(A) = \{0\}$, i.e. A is closable; (2) when $M(A) = \overline{A(X)}$.

In the second case the operator A will be called *maximally nonclosable*. In this case $A_0 x = 0$ and $\tilde{A}$ is defined on $X \oplus \overline{A(X)}$ by the formula $\tilde{A}(x, y) = y$.

THEOREM 4. *For any linear operator $A : X \to Y$ with dense domain $D(A)$, the following conditions are equivalent:*

(i) *A is maximally nonclosable;*

(ii) *for every $y \in \overline{A(X)}$ there exists a sequence (x_k) of points $x_k \in D(A)$ such that $x_k \to 0$ and $Ax_k \to y$;*

(iii) *for every* $x \in X$ *there exists a sequence* (u_k) *of points* $u_k \in D(A)$ *such that* $u_k \to x$, $Au_k \to 0$.

Proof. (i)⇔(ii) holds by the definition of maximally nonclosable operator.

(ii)⇒(iii). Let $x \in X$. As $\overline{D(A)} = X$, there exists a sequence (w_k) of points $w_k \in D(A)$ such that $w_k \to x$. Let us consider the sequence (Aw_k) which, generally speaking, is not convergent. For each $v_k = Aw_k$ by (ii), there exists a sequence (z_{k_l}) of points $z_{k_l} \in D(A)$ such that $z_{k_l} \to 0$ and $Az_{k_l} \to v_k$, when $l \to \infty$. Let us choose an integer $l(k)$ in such a way that

$$\|z_{k_{l(k)}}\|_1 \leq \frac{1}{k} \quad \text{and} \quad \|Az_{k_{l(k)}} - v_k\|_2 \leq \frac{1}{k}. \tag{8}$$

Put $u_k = w_k - z_{k_{l(k)}}$. Then we get

$$\|u_k - x\|_1 \leq \|w_k - x\|_1 + \|z_{k_{l(k)}}\|_1 \to 0, \quad \text{when } k \to \infty$$

and

$$\|Au_k\|_2 \leq \frac{1}{k} \to 0, \quad \text{when } k \to \infty.$$

(iii)⇒(ii). Let $y \in \overline{A(X)}$. Then elements $z_k \in D(A)(k = 1, 2, \ldots)$ can be found in such a way that $Az_k \to y$. In view of (iii), for each $z_k \in D(A)$ a sequence (u_{k_l}) of points $u_{k_l} \in D(A)$ can be chosen in such a way that $u_{k_l} \to z_k$ and $Au_{k_l} \to 0$, when $l \to \infty$. For each k we now choose $l(k)$ so as to have

$$\|u_{k_{l(k)}} - z_k\|_1 \leq \frac{1}{k} \quad \text{and} \quad \|Au_{k_{l(k)}}\|_2 \leq \frac{1}{k}.$$

On putting $x_k = z_k - u_{k_{l(k)}}$, we obtain $\|x_k\|_2 \leq \frac{1}{k} \to 0$ and $\|Ax_k - y\|_2 \leq \|Az_k - y\|_2 + \|Au_{k_{l(k)}}\|_2 \to 0$, when $k \to \infty$. This completes the proof.

COROLLARY 1. *If* $A : X \supset D(A) \to Y$ *is a maximally nonclosable operator with dense domain, then*

$$\overline{G(A)} = X \oplus M(A) = X \oplus \overline{A(X)}.$$

Proof. Since A is maximally nonclosable, then in view of Theorem 4, we have $(x, 0) \in \overline{G(A)}$ and $(0, y) \in \overline{G(A)}$ for every $x \in X$ and $y \in \overline{A(X)}$. Thus $X \subset \overline{G(A)}$ and $\overline{A(X)} \subset \overline{G(A)}$. Hence $X \oplus \overline{A(X)} \subset \overline{G(A)} \subset X \oplus \overline{A(X)}$.

COROLLARY 2. *If* $\overline{\ker A} = X$, *then* A *is maximally nonclosable.*

The proof follows immediately from (iii) of Theorem 4.

2. Examples. From the above considerations it follows that the natural space, on which the extended closure of A is uniqely defined, arises as a result of decomposition of a point $x \in X$ onto the whole family of element of different character. This family can be parametrised by points of the space $M(A)$. Thus the description of $M(A)$ is the detection of latent parameters intrinsic to the process described by the nonclosable operator A. From this point of view, in what follows, we consider several operators known in analysis and describe for them the measure of nonclosability.

2.1. *Completions in different norms.* Let a vector space X be equipped with two norms $\|.\|_1$, and $\|.\|_2$.

DEFINITION. The norm $\|\cdot\|_2$ is said to be *subordinate* to the norm $\|\cdot\|_1$ [3] if given a Cauchy sequence (x_n) in both these norms which converges to zero in norm $\|\cdot\|_1$, it follows that it converges to zero in norm $\|\cdot\|_2$.

DEFINITION. Two norms are said to be *compatible* [3] if one of them is subordinate to the other. We notice that the subordinativity of the norm $\|\cdot\|_2$ to the norm $\|\cdot\|_1$ means closability of the identity operator

$$I:(X,\|\cdot\|_1)\to(X,\|\cdot\|_2). \tag{9}$$

We recall [2] that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are said to be *comparable*, $\|\cdot\|_1$ being weaker and $\|\cdot\|_2$ being stronger, if there exists a constant $c>0$ such that, for all $x\in X$, the inequality

$$\|x\|_1\le c\|x\|_2 \tag{10}$$

holds.

It follows from inequality (10) that the norm $\|\cdot\|_1$ is subordinate to the norm $\|\cdot\|_2$, yet the norm subordinativity does not imply its comparability.

Every Cauchy sequence in a stronger norm is also a Cauchy sequence in a weaker norm. If the space X is complete in both norms, then by the Banach theorem on inverse operator, the comparability of norms implies their equivalence, i.e. there exist constants $c'>0$ and $c''>0$ such that

$$c'\|x\|_1\le\|x\|_2\le c''\|x\|_1 \quad \text{for all } x\in X.$$

However, if the space X is not complete (even in one of the norms), then the comparability of norms may not imply their equivalence.

In this case we can consider two complete spaces X_1 and X_2 with norm $\|\cdot\|_1$ and norm $\|\cdot\|_2$, respectively which are obtained by completion of the space X in norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. If $\|\cdot\|_2$ is stronger than $\|\cdot\|_1$, then we can establish the natural mapping $j:X_2\to X_1$. Every element $\bar{\bar{x}}\in X_2$ is defined by a Cauchy sequence (x_n) of elements in X_2. As it was said earlier, this sequence is a Cauchy sequence in X_1 and therefore it defines an element $\bar{x}\in X_1$. It is easy to check that the element $\bar{x}$ is uniquely determined by $\bar{\bar{x}}$, i.e. the mapping $j:X_2\to X_1$ is correctly defined. However this mapping j is not necessarily injective, i.e. different elements $\bar{\bar{x}},\bar{\bar{y}}\in X_2$ may be transformed onto some element $\bar{x}\in X_1$. To exclude this kind of possibility we introduce the notion of compatible norms. Indeed, if the norm $\|\cdot\|_2$ is subordinate to the norm $\|\cdot\|_1$, then the mapping $j:X_2\to X_1$ is injective.

Let us consider an example of a space whose two completions are related to each other in the case of noncompatible norms, e.g. the vector space $X = C^\infty[0,1]$ equipped with two norms

$$\|x\|_1 = \max_{0 \le t \le 1} |x(t)| \quad \text{and} \quad \|x\|_2 = \max_{0 \le t \le 1} |x(t)| + |x'(0)|.$$

The norm $\|\cdot\|_2$ is stronger than the norm $\|\cdot\|_1$. However the norm $\|\cdot\|_2$ is not subordinate to the norm $\|\cdot\|_1$. To this end let us consider the sequence (x_n), where $x_n(t) = \frac{1}{n}\sin nt$. It is easy to see that (x_n) is a Cauchy sequence in these two norms which converges to zero in norm $\|\cdot\|_1$ and since $\|x_n\|_2 = 1$, it does not converge to zero in norm $\|\cdot\|_2$.

In the given example let $X_1 = C[0,1]$, $X_2 = C[0,1] \oplus \mathbb{R}$. Let us look at this example from the point of view of the considered constructions in Section 1. In this case we may take for an unclosable operator A the operator $A : C[0,1] \to C[0,1] \oplus \mathbb{R}$ defined by

$$Ax = (x, x'(0)), \qquad D(A) = C^\infty[0,1],$$

whose measure of nonclosability is $M(A) = \mathbb{R}$ and is isomorphic to $\ker j$.

2.2. *The spaces $L_p(T, M, \mu)$ with different measures.* Let (T, M) be a measurable space and let μ_1, μ_2 be σ-additive measures on M. Let M_1 and M_2 denote the Lebesgue extensions of the σ-algebra M with respect to the given measures μ_1 and μ_2, respectively. The Lebesgue extensions of these measures will also be denoted by the same letters μ_1 and μ_2. We recall [4] that the measure μ_2 is absolutely continuous with respect to measure μ_1 if for every $E \in M$, $\mu_1(E) = 0$ implies that $\mu_2(E) = 0$.

THEOREM 5. *Let $L_p(T, M_1, \mu_1)$ and $L_p(T, M_2, \mu_2)$ $1 \le p < +\infty$ be Lebesgue spaces and let*

$$X \subset L_p(T, M_1, \mu_1) \cap L_p(T, M_2, \mu_2)$$

be a vector subspace dense in each of the spaces $L_p(T, M_k, \mu_k)$, $k = 1, 2$. Then the norm $\|\cdot\|_2 = \|\cdot\|_{L_p(T,M_2,\mu_2)}$ on X is subordinate to the norm $\|\cdot\|_{L_p(T,M_1,\mu_1)}$ if and only if the measure μ_2 is absolutely continuous with respect to measure μ_1.

Proof. Sufficiency. Let μ_2 be absolutely continuous with respect to μ_1. Let us choose a sequence of function $u_n \in X$ such that $u_n \to 0$ in $L_p(T, M_1, \mu_1)$ and which is a Cauchy sequence in $L_p(T, M_2, \mu_2)$. Then there exists $u \in L_p(T, M_2, \mu_2)$ such that $\|u_n - u\|_2 \to 0$. Consequently there exists a subsequence (u_{n_k}) of the sequence (u_n) such that $u_n(t) \to u(t)$ μ_2 - a.e. (μ_2 - almost everywhere) and $u_{n_k}(t) \to 0$ μ_1 - a.e. [4]. This means that there exists a subset $E \subset M_1$ such that $\mu_1(E) = 0$ and $\lim_{k\to\infty} u_{n_k}(t) = 0$ for all $t \in T \setminus E$.

As M_1 is the Lebesgue extention of M, there exists a subset $E_1 \in M$ such that $E \subset E_1$ and $\mu_1(E_1) = 0$. In view of absolute continuity of μ_2 with respect to μ_1 we have $\mu_2(E) = 0$ and $u_{n_k}(t) \to 0$ for all $t \in T \setminus E_1$. Consequently, $u(t) = 0$ for all $t \in T \setminus E_1$, i.e. $u(t) = 0$ μ_2 - a.e. and so $u = 0$ in $L_p(T, M_2, \mu_2)$. Thus the norm $\|\cdot\|_2$ is subordinate to the norm $\|\cdot\|_1$.

Necessity. Let $\|\cdot\|_2$ be subordinate to $\|\cdot\|_1$ and suppose that the measure μ_2 is not absolutely continuous with respect to measure μ_1, i.e. there exists a subset $E \in M$ such that $\mu_1(E) = 0$ but $\mu_2(E) > 0$. Let us consider the measure $\mu_3 = \mu_1 + \mu_2$ on M and let $L_p(T, M_3, \mu_3)$ be the Lebesgue space introduced in an analogous way as previously. Since $\|\cdot\|_3 = \|\cdot\|_1 + \|\cdot\|_2$, the norm $\|\cdot\|_1$ is subordinate to norm $\|\cdot\|_3$. Then, in accordance with our considerations in Section 2.1, the space $L_p(T, M_3, \mu_3)$ is imbedded in $L_p(T, M_k, \mu_k)$, $k = 1, 2$ and that X is dense in $L_p(T, M_3, \mu_3)$. Now let us choose a sequence (u_n), $u_n \in X$, such that $u_n \to \chi_E$ in $L_p(T, M_3, \mu_3)$. Moreover, without limiting the generality, we can also assume that $u_n \to \chi_E$ μ_3-a.e. and so does it μ_1-a.e. Since the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are weaker than the norm $\|\cdot\|_3$, so $u_n \to \chi_E$ in $L_p(T, M_k, \mu_k)$, $k = 1, 2$. But $\chi_E = 0$ a.e. in $L_p(T, M_1, \mu_1)$ and $\chi_E = 1$ a.e. in $L_p(T, M_2, \mu_2)$ which contradicts the subordinativity of $\|\cdot\|_2$ to $\|\cdot\|_1$. This completes the proof.

COROLLARY. *In order that the space $L_p(T, M_2, \mu_2)$ be naturally and continuously imbedded in the space $L_p(T, M_1, \mu_1)$ it is necessary and sufficient that the following conditions be satisfied:*

(i) *there exists a constant $c > 0$ such that $\mu_1(E) \leq c\mu_2(E)$ for every $E \in M$;*

(ii) *μ_2 is absolutely continuous with respect to μ_1.*

We notice that if inequality (i) in the Corollary to Theorem 5 is satisfied but the measure μ_2 is not absolutely continuous with respect to measure μ_1, then in this case the measure of nonclosability of the operator j defined by $j(x) = x$ for $x \in X$, can be obtained as follows. By the Lebesgue decomposition theorem, the measure μ_2 can be written as $\mu_2 = \mu_2^{ac} + \mu_2^{sing}$, μ_2^{ac} being the absolutely continuous measure with respect to μ_1 and μ_2^{sing} is the singular measure with respect to μ_1. Then

$$L_p(T, M_2, \mu_2) = L_p(T, M_1, \mu_2^{ac}) \oplus L_p(T, M_2, \mu_2^{sing})$$

and

$$M(j) = L_p(T, M_2, \mu_2^{sing}).$$

2.3. *Imbedings of the Sobolev spaces.* Let $S(\mathbb{R}^n)$ be the Schwartz space [8] of infinitely differentiable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that for arbitrary multi-indices $l = (l_1, \ldots, l_n)$, $k = (k_1, \ldots, k_n) \in N_o^n$, $N_o = \{0, 1, 2, \ldots\}$

$$\sup_{x \in \mathbb{R}^n} |x^l D^k u(x)| < +\infty. \tag{11}$$

For any $s \in \mathbb{R}$ let $\|\cdot\|_s$ be a norm defined on $S(\mathbb{R}^n)$ by the formula

(12) $$\|u\|_s = \left[\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi \right]^{\frac{1}{2}},$$

where

(13) $$\widehat{u}(\xi) = \int_{\mathbb{R}^n} u(x) e^{-i\langle x,\xi\rangle} \, dx$$

is the Fourier transform of the function u and

$$\langle x,\xi\rangle = x_1\xi_1 + \ldots + x_n\xi_n.$$

The completion of the space $S(\mathbb{R}^n)$ in the norm $\|\cdot\|_3$ is called the Sobolev space denoted by $H^s(\mathbb{R}^n)$ [5]. For $s > 0$ the space H^s is naturally understood as a subspace of $L_2(\mathbb{R}^n)$, $H^0(\mathbb{R}^n) = L_2(\mathbb{R}^n)$ and for $s < 0$, the elements of H^s are distributions.

It is obvious that for $s_1 < s_2$

(14) $$\|u\|_{s_1} \le \|u\|_{s_2}, \text{ for } u \in S(\mathbb{R}^n).$$

Let us turn our attention to the fact that the known continuous embedding of the spaces $H^{s_2}(\mathbb{R}^n) \subset H^{s_1}(\mathbb{R}^n)$ is not a consequence of inequality (14) (as sometimes it is maintained) but it is a consequence of subordinativity of the norms $\|\cdot\|_{s_1}$ and $\|\cdot\|_{s_2}$, since by Corollary to Theorem 5. the norm subordinativity is equivalent to absolute continuity of measure μ_{s_2} with respect to μ_{s_1}, where $\mu_k(\xi) = (1+|\xi|^2)^{s_k} d\xi$, $k = 1,2$.

Let

$$\varrho(x) = \frac{x^2}{1+x^2} \quad \text{for } x \in \mathbf{R}$$

and let the space $S(\mathbb{R})$ be equipped with the norm

$$\|u\|_\varrho = \left[\int_{\mathbb{R}} |u(x)|^2 \varrho(x) \, dx \right].$$

The completion of $S(\mathbb{R})$ in this norm is the space $L_2(\mathbf{R},\varrho)$ with weight ϱ. Under this norm, the embedding operatop $j_\varrho : S(\mathbb{R}) \to \mathbb{H}^{-1}(\mathbb{R})$ is nonclosable and $M(j_\varrho) = \{a\delta : a \in \mathbb{C}\}$, where δ is Dirac's function.

The explanation of nonclosability of the operator j_ϱ follows from the fact that the natural imbedding of the space $L_2(\mathbf{R},\varrho)$ into the space of distributions $\mathbb{H}^{-1}(\mathbb{R})$ does not exist. In particular, to the function $\frac{1}{x}$ there corresponds the whole class of distributions of the form $P(\frac{1}{x}) + a\delta$ [3].

2.4. *Traces in Sobolev's spaces.* In this section we shall deal with the trace operator [5]:

(15) $$T : H^s(\mathbb{R}^n) \to H^q(\mathbb{R}^{n-1}),$$

defined by the formula

$$(Tu)(x_1,\dots,x_n) := u(x_1,\dots,x_{n-1},0), \quad u \in S(\mathbb{R}^n),$$

whose value Tu is a function of $n-1$ variables.

In view of the trace theorem [5], the following is well known:

If $s > \frac{1}{2}$ and $q = s - \frac{1}{2}$, then the trace operator (15) is defined on the whole space $H^s(\mathbb{R}^n)$ which is continuous and surjective.

As a corollary of the trace theorem we have the assertion:

If $s > \frac{1}{2}$, then the operator (15) is bounded for $q \le s - \frac{1}{2}$ and is closable for $q > s - \frac{1}{2}$ and therefore unbounded.

Besides, it is quoted in [5] that for $s \le \frac{1}{2}$ the operator $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ is unbounded.

The following more exact assertion proves to be true.

THEOREM 6. *If* $s < \frac{1}{2}$, *then the trace operator* (15) *is maximally non-closable for any* q.

Proof. It is sufficient to show that there exists a dense vector subspace $V \subset H^q$ with the property that for $v \in V$ we can find a sequence (u_k) of elements $u_k \in H^s(\mathbb{R}^n)$ such that $u_k \to 0$ in H^s and $Tu_k \to v$ in H^q. For the space V we take the set

$$V = \{v : \widehat{v} \text{ are bounded functions with compact support}\}.$$

It is easy to check that V is contained in every H^q. In view of the Paley–Wiener theorem we observe that the functions in V are analytic on $\mathbb{R}^{n-1}$ and converge to 0 at infinity.

Let us consider the function φ such that

$$\widehat{\varphi}(\xi_n) = \pi\chi_{[-1,1]}(\xi_n) = \begin{cases} \pi, & |\xi_n| \le 1 \\ 0, & |\xi_n| > 1 \end{cases}, \tag{16}$$

i.e.

$$\varphi(x_n) = \begin{cases} \dfrac{\sin x_n}{x_n}, & x_n \ne 0 \\ 1, & x_n = 0. \end{cases} \tag{17}$$

Let us put

$$u_k(x', x_n) = \varphi(kx_n)v(x'), \quad \text{for } v \in V. \tag{18}$$

We notice that

$$\widehat{v}(\xi') \le C, \quad \xi' = (\xi_n,\dots,\xi_{n-1}) \in \mathbb{R}^{n-1} \quad \text{and} \quad v(\xi') = 0 for|\xi'| \ge R. \tag{19}$$

Then, for every q

$$Tu_k = v \in V \quad \text{and} \quad Tu_k \to v \text{ in } H^q(\mathbb{R}^{n-1}).$$

Now we shall estimate $\|u_k\|_s$:

$$\|u_k\|_s^2 = \frac{1}{k^2}\int\limits_{\mathbb{R}^n}\left|\widehat{\varphi}\left(\frac{\xi_n}{k}\right)\right|^2 |v(\xi')|^2(1+|\xi|^2)^s\,d\xi \le$$

$$\le \frac{C^2\pi^2}{k^2}\int\limits_{-k}^{k} d\xi_n \int\limits_{|\xi'|\le R}(1+|\xi|^2\,d\xi'.$$

Since $1+|\xi|^2 = 1+\xi_1^2+\ldots+\xi_n^2 = 1+|\xi'|^2+|\xi_n|^2 \le (1+|\xi'|)^2$ we obtain

$$\|u_k\|_s^2 \le \frac{C^2\pi^2}{k^2}\int\limits_{-k}^{k}\int\limits_{|\xi'|\le R}(1+|\xi'|+|\xi_n|)^{2s}\,d\xi_n\,d\xi' =$$

$$= \frac{2C^2\pi^2C_1}{k^2}\int\limits_0^R (1+r+k)^{2s+1}r^{n-2}\,dr - \frac{2\pi^2C^2C_1}{k^2(2s+1)} \le$$

$$\le \frac{C_3}{k^2}\int\limits_0^R (1+r+k)^{2s+1}\,dr \le \frac{C_3R(1+R+k)^{2s+1}}{k^2} \to 0,$$

as $2s+1<2$. This completes the proof.

COROLLARY. *The closure of the graph of the trace operator* (15) *for* $s<\frac{1}{2}$ *and each* q *coincides with* $H^s(\mathbb{R}^n)\times H^q(\mathbb{R}^{n-1})$.

This space was introduced by J. A. Rojtberg [7] for construction of the solvability theory for elliptic boundary value problems in the complete scale for Sobolev's spaces.

2.5. *Extention of the operator* $\frac{d}{dt}$. Let us consider the differential operator $A=\frac{d}{dt}: L_1[0,1]\to L_1[0,1]$ with the domain $D(A)=C^1[0,1]$, where $L_1[0,1]$ is the real normed space of Lebesgue integrable functions. It is known [6] that this operator is closable and the domain $D(\overline{A})$ of its closure $\overline{A}$ is the set of all absolutely continuous functions. Since continuous functions of bounded variation have the integrable derivative a.e. it follows thet the operator $\overline{A}$ can be further extended to the operator B, $Bx := \frac{dx}{dt}$ with the domain $D(B)$ to be the set of all continuous functions on $[0,1]$ of bounded variation. Stone noticed pathological properties of this operator [9]. Unlike the operator A, where $Ax=0$, it implies that $x=\text{const}$, so that the solution set of the equation $Bu=0$ is infinite dimensional and even dense in $L_1[0,1]$.

Following Stone we shall sketch the proof.

Since the set of linear combinations of characteristic functions of intervals is dense in $L_1[0,1]$, it suffices to prove that every characteristic function $\chi_{[a,b]}$, $[a,b]\subset[0,1]$ can be approximated by a continuous function of bounded variation whose derivative is equal to zero almost everywere. To this end we construct a sequence (φ_n) of functions as follows.

$$\varphi_n(t) = \begin{cases} 0, & t \in \left[0, a - \frac{1}{n}\right] \cup \left[b + \frac{1}{n}, 1\right]; \\ 1, & t \in [a, b]; \\ \varphi_n(t), & t \in \left[a - \frac{1}{n}, a\right] \text{ is continuous and increasing from 0 to 1} \\ & \qquad \text{with the zero derivative (e.g. the} \\ & \qquad \text{contractive Cantor function);} \\ \varphi_n(t), & t \in \left[b, b + \frac{1}{n}\right) \text{ is continuous and decreasing from 1 to 0} \\ & \qquad \text{with the zero derivative.} \end{cases}$$

Then $B\varphi_n = 0$ and $\varphi_n \to \chi_{[a,b]}$ in $L_1[0,1]$. Since $\ker B$ is everywhere dense in $L_1[0,1]$, it follows from Corollary 2 of Theorem 4, that the operator B is maximally nonclosable. We note that in original paper [9] the proof of nonclosability of the operator B is more complicated.

2.6. *δ-function and δ-sequences.* Dirac's definition of the δ-function is well known only as far as it is nonunderstandable and unnatural from the mathematical point of view as $\delta(x)$ was defined by means of the properties:

$$\int_{\mathbb{R}} \delta(x) = 1 \quad \text{and} \quad \delta(x) = 0 \quad \text{for all } x \neq 0.$$

We show that precisely this definition arises in investigation of a certain nonclosable operator and that the general approach, presented above, automatically leads to a correct definition of the δ-function.

Let us consider the equation

$$tu(t) = f(t), \qquad \text{where } f \in L_1[-1, 1; dt].$$

Then it would be natural to seek the function $u(t)$ in the space $L_1[-1, 1; |t|dt]$. Let us add a nonlocal condition

$$\int_{-1}^{1} u(t)dt = C, \qquad C \text{ being a constant.}$$

This problem arises, for example, as a limit case for a differentiable equation with a small parameter and nonlocal condition:

$$\begin{cases} \varepsilon u'(t) + tu(t) = f(t) \\ \int_{-1}^{1} u(t)\, dt = C. \end{cases}$$

In particular, when $f(t) = 0$ and $C = 1$, we obtain the problem:

$$(20) \qquad \begin{cases} tu(t) = 0, \\ \int_{-1}^{1} u(t)\, dt = 1, \end{cases}$$

from which we conclude that $u(t) \neq 0$ for $t \neq 0$ and $\int_{-1}^{1} u(t)\, dt = 1$, i.e. $u(t)$ is the δ-function in the sense of Dirac.

Let $D(A) := L_1[-1,1;dt] \subset L_1[-1,1;|t|dt] := X$ and let the operator A be defined by the formula:

$$Au = (su(s), \int_{-1}^{1} u(t)\, dt) \in Y := L_1[-1,1;dt] \oplus \mathbb{R}$$

This operator is nonclosable. The space $M(A)$ is one-dimensional and consists of elements of the form $(0,\xi)$, $\xi \in \mathbb{R}$.

Let us construct the space Z_A and the operator $\widetilde{A}$ according to the construction given above. We conclude that $\widetilde{Z}$ consists of sequences (u_n) of integrable functions satisfying the conditions:

$$\int_{-1}^{1} |t|\,|u_m(t) - u_n(t)|\, dt \to 0 \quad \text{and} \quad \int_{-1}^{1} u_n(t)\, dt \text{ converges for } n,m \to \infty.$$

Sequences (u_n) and (v_n) are said to be equivalent of

$$\int_{-1}^{1} |t|\,|u_n(t) - v_n(t)|\, dt \to 0 \quad \text{and} \quad \int_{-1}^{1} [u_n(t) - v_n(t)]\, dt \to 0$$

The set of equivalence classes forms the space Z_A. We note that the space Z_A is isomorphic to $X \oplus \mathbb{R}$ and that the operator A defines this isomorphism.

Consequently, the solution of problem (20) is the class consisting of sequences (u_n) of integrable functions such that

$$(21) \qquad \int_{-1}^{1} |t|\,|u_n(t)|\, dt \to 0, \qquad \int_{-1}^{1} u_n(t)\, dt \to 1.$$

This class is called a δ-function and its elements are called δ-sequences.

We note that in literature one can find different definitions of δ-sequences but the ones obtained here, in our opinion, are most relevent to the physical sense, because the first condition in (21) means that the masses or electic charges shrink to the point 0 and the second condition means that the whole mass or charge tends to the unit.

The space Z_A is imbedded in the Schwartz space of distributions in a natural manner.

The above examples show that nonclosable operators arise in analysis in a natural manner and the approach to these operators given above testifies that they can be sufficiently well investigated .

Let us remark that the operator of multiplication by a generalized function is a nonclosable operator and in the new theory of generalized functions analogous constructions arise [1].

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