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Distributions and Mnemofunctions on Adeles. Fourier Transform

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Abstract—Some classical results are recalled, and a finite part distribution is interpreted as the zero-order term in the expansion of a homogeneous distribution. An adelic finite part distribution and a generalization of the Tate distribution are defined, and their Fourier transforms are calculated. The machinery of mnemofunctions (nonlinear generalized functions) is adapted to p-adic and adelic cases, and product formulas for some specific distributions are given.

INTRODUCTION

In this paper, we study some specific local (p-adic, real) and adelic distributions and also deal with the problem of multiplying distributions in p-adic and adelic cases.

In Section 1, we recall some well-known distributions and interpret local finite parts as zero-order terms in the expansions of homogeneous distributions. In Section 2, we consider an adelic finite part and a generalization of the Tate distribution and obtain analogues of a functional equation. Fourier transforms are given both in local and adelic cases. For some other distributions on adeles and their application in quantum mechanics, see [9].

Sections 3 and 4 concern the basics of mnemofunction theory in p-adic and adelic cases. The first example of mnemofunctions was presented by Colombeau (the so-called new generalized functions [8]). Later, the essential part of the construction was derived, and a number of analogous algebras with real domains were constructed and studied [1, 5, 7, 10]. Here, we construct an algebra of mnemofunctions with p-adic and adelic domains with well-defined Fourier transform and convolution (for the p-adic case, see also [6]).

These algebras are of great interest in connection with the development of p-adic mathematical physics [3] and as an example for the general theory of mnemofunctions.

1. LOCAL DISTRIBUTIONS

Let $\mathbb{Q}_\nu$ be the completion of rational numbers $\mathbb{Q}$ with respect to the valuation $|\cdot|_\nu$, where $\nu = \infty$ corresponds to the ordinary absolute value and $\nu = p$, to a p-adic valuation. We often omit the index ν when a formula is valid for all local cases.

Let I_X denote the characteristic function of a subset X in an appropriate space.

Let dx_ν be an additive Haar measure on $\mathbb{Q}_\nu$ (which coincides with the standard Lebesgue measure on $\mathbb{Q}_\infty = \mathbb{R}$ and takes the value 1 in the set $\mathbb{Z}_p$ of p-adic integers on $\mathbb{Q}_p$). We choose normalizing constants λ_ν for the multiplicative Haar measures $d^*x_\nu = \lambda_\nu dx_\nu/|x|_\nu$ on $\mathbb{Q}_\nu^\times$ to be $\lambda_\infty = 1$ and $\lambda_p = (1 - p^{-1})^{-1}$, respectively.

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Let θ_ν be a character on a multiplicative subgroup $U_\nu = \{x \in \mathbb{Q}_\nu : |x| = 1\}$. It is continued to the whole $\mathbb{Q}_\nu^\times$ by

$$\theta_\infty(\alpha x) = \theta_\infty(x), \quad \alpha > 0, \qquad \theta_p(p^k x) = \theta_p(x), \quad k \in \mathbb{Z}.$$

There is only one nontrivial character for $\mathbb{R}$ and infinitely many for $\mathbb{Q}_p$. Any *quasicharacter* on $\mathbb{Q}_\nu$ (homomorphism $\mathbb{Q}_\nu^\times \to \mathbb{C}^\times$) is uniquely represented as $\theta(x)|x|^s$, $s \in \mathbb{C}$ [3]. We often omit θ in the notations when it is trivial.

Consider the Schwartz–Bruhat space $\mathcal{S}(\mathbb{Q}_\nu)$ (consisting of smooth rapidly decreasing functions on $\mathbb{R}$ and locally constant compactly supported functions on $\mathbb{Q}_p$) and a distribution $\Delta_{s,\theta}^\nu$ on it,

$$\Delta_{s,\theta}^\nu(\varphi) = \int\limits_{\mathbb{Q}_\nu^\times} \varphi(x)\theta(x)|x|^s d^*x, \qquad \varphi \in \mathcal{S}(\mathbb{Q}_\nu), \quad \operatorname{Re} s > 0.$$

Now, we continue $\Delta_{s,\theta}^\nu$ to $\operatorname{Re} s > -1$ in the real case and to all s in the p-adic case. Choose a specific function $\omega_\nu \in \mathcal{S}(\mathbb{Q}_\nu)$ such that $\omega_\nu(0) = 1$. It is convenient to use $\exp(-\pi x_\infty^2)$ in the real case and $\omega(x_p) = I_{\mathbb{Z}_p}(x_p)$, the characteristic function of $\mathbb{Z}_p$, in the p-adic case. Then,

$$\begin{aligned}\Delta_{s,\theta}^\nu(\varphi) &= \int\limits_{\mathbb{Q}_\nu} (\varphi(x) - \varphi(0)\omega(x))\theta(x)|x|^s\, d^*x + \varphi(0) \int\limits_{\mathbb{Q}_\nu} \omega(x)\theta(x)|x|^s\, d^*x \\ &= \Lambda_{s,\theta}^{\nu,\omega}(\varphi) + \delta(\varphi)\Delta_{s,\theta}^\nu(\omega) \qquad \text{(here, } \delta \text{ is the Dirac delta).}\end{aligned}$$

The distribution $\Lambda_{s,\theta}^{\nu,\omega}$, which depends on ω, is entire on the domain of interest, and $\Delta_{s,\theta}^\nu(\omega)$ is easily computed. It is zero for nontrivial θ and a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 0$ for trivial θ:

$$\begin{aligned}\Delta_s^p(\omega) &= (1 - p^{-s})^{-1} = \frac{1}{\ln p} s^{-1} + \frac{1}{2} + \frac{\ln p}{2} s + \dots, \\ \Delta_s^\infty(\omega) &= \frac{1}{2}\pi^{-s/2}\Gamma(s/2) = s^{-1} - \frac{1}{2}(\gamma_e + \ln \pi) + \dots\end{aligned}$$

($\gamma_e = -\Gamma'(1)$ is the Euler–Mascheroni constant). Introducing constants $R = R^\nu(\omega)$ and $T = T^\nu(\omega)$ such that $\Delta_s(\omega) = Rs^{-1} + T + \dots$, we obtain a Laurent series for Δ_s around $s = 0$: $\Delta_s = (R\delta)s^{-1} + (\Lambda_0^\omega + T\delta) + \dots$. The uniqueness of the expansion implies that the coefficient $R^\nu(\omega)$ and the distribution $\Lambda_0^\omega + T\delta$ do not depend on ω.

Definition. We define a *local finite part distribution* $\mathcal{P}_\theta^\nu$ to be the zero term in the expansion of $\Delta_{s,\theta}^\nu$ at $s = 0$: $\mathcal{P}_\theta^\nu = \Delta_{0,\theta}^\nu$ for $\theta \not\equiv 1$ and $\mathcal{P}^\nu = \Lambda_{0,1}^{\nu,\omega} + T^\nu(\omega)\delta$ for $\theta \equiv 1$.

Example 1.1. The p-adic finite part $\mathcal{P}^p$ equals $(1 - p^{-1})^{-1}\mathcal{P}\frac{1}{|x|_p} + \frac{\delta(x)}{2}$, where $\mathcal{P}\frac{1}{|x|_p}$ is the finite part introduced by Vladimirov [3]:

$$\langle \mathcal{P}\tfrac{1}{|x|_p}, \varphi \rangle = \int\limits_{\mathbb{Z}_p} (\varphi(x) - \varphi(0)) \frac{dx_p}{|x|_p} + \int\limits_{\mathbb{Q}_p \setminus \mathbb{Z}_p} (\varphi(x)) \frac{dx_p}{|x|_p}.$$

Example 1.2. The real finite part $\mathcal{P}_{\mathrm{sgn}}^\infty$ is the classical Cauchy finite part $\mathcal{P}\frac{1}{x}$.

The Fourier transform $\mathcal{F}\psi$ of $\psi \in \mathcal{S}(\mathbb{Q}_\nu)$ is defined by

$$(\mathcal{F}\psi)(\xi) = \int\limits_{\mathbb{R}} \psi(x) \exp(-2\pi i x\xi)\, dx_\infty, \qquad (\mathcal{F}\psi)(\xi) = \int\limits_{\mathbb{Q}_p} \psi(x) \exp(2\pi i \{x\xi\}_p)\, dx_p$$

for the real and p-adic cases, respectively, where $\{y\}_p$ is the p-adic fractional part [3]. The Fourier transform is a unitary operator on $\mathcal{S}(\mathbb{Q}_\nu)$ and, for any distribution $u \in \mathcal{S}'(\mathbb{Q}_\nu)$, is defined by duality: $\langle \mathcal{F}u, \varphi\rangle = \langle u, \mathcal{F}\varphi\rangle$, $\varphi \in \mathcal{S}(\mathbb{Q}_\nu)$.

The distribution $\Delta^\nu_{s,\theta}$ is known to satisfy the functional equation [3]

$$\mathcal{F}(\Delta^\nu_{s,\theta}) = \Gamma_\nu(s,\theta)\Delta^\nu_{1-s,\overline{\theta}}$$

with Γ_ν called a *local Gamma function.* Since $\mathcal{F}(\omega_\nu) = \omega_\nu$, we have

$$\Gamma_p(s) = \frac{\Delta_s(\omega_p)}{\Delta_{1-s}(\omega_p)} = \frac{1-p^{s-1}}{1-p^{-s}}, \qquad \Gamma_\infty(s) = \frac{\Delta_s(\omega_\infty)}{\Delta_{1-s}(\omega_\infty)} = 2(2\pi)^{-s}\cos\frac{\pi s}{2}\Gamma(s)$$

for $\theta \equiv 1$. Introduce γ_ν and τ_ν such that $\Gamma_\nu(1-\varepsilon) = \gamma\varepsilon + \tau\varepsilon^2 + \dots$ as $\varepsilon \to 0$, namely,

$$\Gamma_\infty(1-\varepsilon) = \frac{1}{2}\,\varepsilon + \frac{1}{2}(\gamma_e + \ln 2\pi)\varepsilon^2 + \dots, \qquad \Gamma_p(1-\varepsilon) = \frac{\ln p}{1-p^{-1}}\,\varepsilon + \frac{(\ln p)^2 p(p-3)}{2(p-1)^2}\,\varepsilon^2 + \dots.$$

On the one hand, $\Gamma_\nu(\varepsilon)\Delta^\nu_\varepsilon = (\gamma R\delta) + (\gamma\mathcal{P} + \tau R\delta)\varepsilon + \dots$. On the other hand, $|x|^{1-\varepsilon}\,d^*x_\nu = \lambda\exp(-\varepsilon\ln|x|)\,dx_\nu$ and $\mathcal{F}(\Delta^\nu_{1-\varepsilon}) = \lambda_\nu\delta + \lambda_\nu\mathcal{F}(-\ln|y|)\varepsilon + \dots$. Thus, from the functional equation, we obtain $\lambda_\nu\mathcal{F}(-\ln|y|) = \gamma\mathcal{P} + \tau R\delta$, i.e.,

$$\mathcal{F}(-\ln|y|_\infty) = \mathcal{P}^\infty + (\gamma_e + \ln(2\pi))\delta, \qquad \mathcal{F}(-\ln|y|_p) = \ln p\cdot\mathcal{P}^p + \frac{(\ln p)^2(p-3)}{2(p-1)}\delta.$$

2. ADELIC DISTRIBUTIONS

Consider a multiplicative *quasicharacter* on the group of idele classes $\mathbb{A}^\times/\mathbb{Q}^\times$, i.e., a quasicharacter on ideles such that $\theta(q) = 1$ for $q \in \mathbb{Q}^\times$. It can be uniquely represented as $\theta(x)|x|^s$, where $|x| = \prod_\nu |x_\nu|_\nu$ and $\theta(x) = \prod_\nu \theta_\nu(x_\nu)$; here, θ_ν are as above and $\theta_\nu \equiv 1$ for all but finitely many ν.

The space $\mathcal{S}(\mathbb{A})$ of Schwartz–Bruhat functions consists of finite linear combinations of cylindrical functions $\varphi(\lambda) = \prod_\nu \varphi_\nu(\lambda_\nu)$, where $\varphi_\nu \in \mathcal{S}(\mathbb{Q}_\nu)$ is the characteristic function of $\mathbb{Z}_p$ for large p; hence, $\mathcal{S}(\mathbb{A})$ is a restricted tensor product of $\mathcal{S}(\mathbb{Q}_\nu)$. The Fourier transform $\mathcal{F}\colon \mathcal{S}(\mathbb{A}) \to \mathcal{S}(\mathbb{A})$ acts "componentwise" on cylindrical functions, $\mathcal{F}\colon \bigotimes_p \varphi_p \mapsto \bigotimes_p \widetilde{\varphi}_p$.

The Fourier transform on $\mathcal{S}'(\mathbb{A}_0)$ is defined via duality; it is a linear isomorphism both on $\mathcal{S}$ and $\mathcal{S}'$. There is also a convolution on $\mathcal{S}(\mathbb{A})$, which is related to the Fourier transform in a usual way.

Tate distribution and finite parts. The adelic analogue of $\Delta^\nu_{s,\theta}$ is the Tate distribution $\Delta_{s,\theta} \in \mathcal{S}'(\mathbb{A})$:

$$\Delta_{s,\theta}(\varphi) = \int\limits_{\mathbb{A}^\times} \varphi(x)\theta(x)|x|^s\,d^*x, \qquad d^*x = \prod_\nu d^*x_\nu, \quad \varphi \in \mathcal{S}(\mathbb{A}).$$

The integral converges for $\operatorname{Re} s > 1$. Consider an entire analytic distribution

$$\Delta^+_{s,\theta}(\varphi) = \int\limits_{|x|\ge 1} \varphi(x)\theta(x)|x|^s\,d^*x.$$

The following equality gives an analytic continuation of $\Delta_{s,\theta}$ to the whole $\mathbb{C}$ except $s = 0$ and $s = 1$ for trivial θ where the Tate distribution has simple poles:

$$\Delta_{s,\theta}(\varphi) = \Delta^+_{s,\theta}(\varphi) + \Delta^+_{1-s,\overline{\theta}}(\widetilde{\varphi}) + \varepsilon_\theta\left(\frac{\widetilde{\varphi}(0)}{s-1} - \frac{\varphi(0)}{s}\right).$$

Here, ε_θ equals 1 if θ is trivial and 0 otherwise. (The above formula follows from the more general considerations given below; see also [2, 11].)

Definition. Two poles are possible in the adelic case. So, we define *adelic finite part distributions* $\mathcal{P}_{0,\theta}$ and $\mathcal{P}_{1,\theta}$ as the zero-order coefficients of the Laurent series for $\Delta_{s,\theta}$ at $s=0$ and $s=1$, respectively:

$$\mathcal{P}_{0,\theta}(\varphi) = -\delta(\widetilde{\varphi})\varepsilon_\theta + \Delta^+_{0,\theta}(\varphi) + \Delta^+_{1,\overline{\theta}}(\widetilde{\varphi}), \qquad \mathcal{P}_{1,\theta}(\varphi) = -\delta(\varphi)\varepsilon_\theta + \Delta^+_{1,\theta}(\varphi) + \Delta^+_{0,\overline{\theta}}(\widetilde{\varphi}).$$

This definition immediately implies the following theorem.

Theorem 2.1. *Two adelic finite part distributions are Fourier dual:*

$$\mathcal{F}\mathcal{P}_{0,\theta} = \mathcal{P}_{1,\overline{\theta}}, \qquad \mathcal{F}\mathcal{P}_{1,\theta} = \mathcal{P}_{0,\overline{\theta}}.$$

Adelic distribution $\Delta_{g,\theta}$. Consider a function $g\colon \mathbb{R}_+ \to \mathbb{C}$, $\mathbb{R}_+ = (0,+\infty)$, and a character θ on idele classes as above. For now, we impose only one condition on g, namely, that it is locally integrable on $\mathbb{R}_+$ and defines a tempered distribution there:

$$g \in L_1^{\mathrm{loc}}(\mathbb{R}_+) \cap \mathcal{S}'(\mathbb{R}_+). \tag{A}$$

For example, $g(t) = \sqrt{t}\ln t$ will do. We define

$$\Delta_{g,\theta}(\varphi) = \int\limits_{\mathbb{A}^\times} \varphi(\lambda)\theta(\lambda)g(|\lambda|)\,d^*\lambda, \qquad \varphi \in \mathcal{S}(\mathbb{A}).$$

Let us formally calculate $\Delta_{g,\theta}(\varphi)$ for *cylindrical* $\varphi(\lambda) = \prod_\nu \varphi_\nu(\lambda_\nu) \in \mathcal{S}(\mathbb{A})$. To this end, we partition the domain of integration into cosets qT, $q \in \mathbb{Q}^\times$, $T = \mathbb{R}_+ \times \prod_p U_p$, where U_p is the p-adic unit sphere. (Any idele λ can be uniquely represented as $q\lambda_0$, where $q \in \mathbb{Q}^\times$ and $\lambda_0 \in T$; so, T is a *fundamental domain* under the action of $\mathbb{Q}^\times$ [2].) Thus,

$$\int\limits_{\mathbb{A}^\times} \varphi(\lambda)\theta(\lambda)g(|\lambda|)\,d^*\lambda = \sum_{q\in\mathbb{Q}^\times} \int\limits_{(q\mathbb{R}_+)\times\prod_p (qU_p)} \varphi(\lambda)\theta(\lambda)g(|\lambda|)\,d^*\lambda.$$

The function $|\lambda|$ on qT depends only on λ_∞ and equals $\frac{|\lambda_\infty|_\infty}{|q|_\infty}$; so, we proceed to

$$\begin{aligned}
&\sum_{q\in\mathbb{Q}^\times} \int\limits_{(q\mathbb{R}_+)\times\prod_p (qU_p)} \varphi_\infty(\lambda_\infty)\theta_\infty(\lambda_\infty)g\left(\frac{|\lambda_\infty|_\infty}{|q|_\infty}\right)d^*\lambda_\infty \prod_p \varphi_p(\lambda_p)\theta_p(\lambda_p)\,d^*\lambda_p \\
&= \sum_{q\in\mathbb{Q}^\times} \left(\theta_\infty(q) \int\limits_{q\mathbb{R}_+} g\left(\frac{|\lambda_\infty|_\infty}{|q|_\infty}\right)\varphi_\infty(\lambda_\infty)\,d^*\lambda_\infty\right)\left(\prod_p \int\limits_{qU_p} \theta_p(\lambda_p)\varphi_p(\lambda_p)\,d^*\lambda_p\right).
\end{aligned} \tag{$*$}$$

Remark. For large p, the local φ_p is the characteristic function of the unit ball. If q has such p in the prime decomposition of its denominator, then $\mathbb{Z}_p \cap qU_p = \varnothing$, and the integration yields zero; i.e., the sum above does not contain a term for such q.

The application of the above formula to $\varphi(\lambda) = \prod_\nu \omega_\nu(\lambda_\nu)$ and $g(t) = I_{(0,1)}(t)$ yields a divergent series

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \int\limits_0^{|k|} \exp(-\pi t^2)\,d^*t,$$

and we can see that problem arises from the integration near zero in the real term.

Theorem 2.2. *If* $\operatorname{supp} g \subset [T, +\infty)$ *for some* $T > 0$, *then* $\Delta_{g,\theta}(\varphi)$ *is well defined by* $(*)$.

Indeed, the inner product over p is estimated as

$$\left| \prod_p \int_{qU_p} \theta_p(\lambda_p)\varphi_p(\lambda_p)\, d^*\lambda_p \right| \le \prod_p \max_{\lambda_p \in \mathbb{Q}_p} |\varphi_p(\lambda_p)| < +\infty.$$

We have $\operatorname{supp}\varphi_p \subset p^{a_p}\mathbb{Z}_p$, $a_p \in \mathbb{Z}$, and $a_p = 0$ for almost all p. Define $Q = \prod_p p^{a_p}$. If $q \notin \frac{1}{Q}\mathbb{Z} \setminus \{0\}$, then the corresponding term in $(*)$ vanishes, and we obtain

$$\sum_{q \in \frac{1}{Q}\mathbb{Z}\setminus\{0\}} \theta_\infty(q) \int_{q\mathbb{R}_+} g\left(\frac{|\lambda_\infty|_\infty}{|q|_\infty}\right) \varphi_\infty(\lambda_\infty)\, d^*\lambda_\infty = \int_{\mathbb{R}_+} \sum_{k=1}^{+\infty} g\left(\frac{Qt}{k}\right) \big(\varphi_\infty(t) + \theta_\infty(-1)\varphi_\infty(-t)\big)\, d^*t.$$

The distribution $\sum_{k=1}^{+\infty} g(Qt/k)$ has support in $[T/Q, +\infty)$ and is tempered. Thus, the theorem is proved.

By this theorem, $\Delta^+_{g,\theta}$ is defined as $\Delta_{f,\theta}$, where $f(t) = I_{[1,+\infty)}g(t)$ is a regular distribution. To regularize the distribution $\Delta_{g,\theta}$, we will use the following *Poisson–Tate summation formula* [2] (the general Poisson summation formula applies to any locally compact group with a discrete subgroup):

$$\sum_{q\in\mathbb{Q}} \varphi(q\lambda) = \frac{1}{|\lambda|} \sum_{q\in\mathbb{Q}} \widetilde{\varphi}\left(\frac{q}{\lambda}\right), \qquad \varphi \in \mathcal{S}(\mathbb{A}), \quad \lambda \in \mathbb{A}^\times.$$

Denote $\Phi(g,\theta) = \Delta_{g,\theta}(\varphi)$ and divide the integral for $\Phi(g,\theta)$ into two parts:

$$\Phi(g,\theta) = \Phi^+(g,\theta) + \Phi^-(g,\theta) = \left(\int_{|\lambda|\ge 1} + \int_{|\lambda|\le 1} \right) \varphi(\lambda)\theta(\lambda)g(|\lambda|)\, d^*\lambda.$$

The strictness of the inequality signs does not matter here because $\{\lambda \in \mathbb{A}^\times : |\lambda| = 1\}$ has zero measure.

A rational number $q \in \mathbb{Q}^\times$ acts on $\{\lambda \in \mathbb{A}^\times : |\lambda| \le 1\}$ via $q\colon \lambda \mapsto q\lambda$. The corresponding fundamental domain is $E = (0,1] \times \prod_p U_p$. Using the Poisson–Tate summation formula, one obtains

$$\begin{aligned}\Phi^-(g,\theta) &= \int_E \sum_{q\in\mathbb{Q}^\times} \varphi(q\lambda)\theta(q\lambda)g(|q\lambda|)\, d^*\lambda \\ &= \int_E \left(\sum_{q\in\mathbb{Q}^\times} \widetilde{\varphi}\left(\frac{q}{\lambda}\right)\right) \theta(\lambda)\frac{1}{|\lambda|} g(|\lambda|)\, d^*\lambda + \widetilde{\varphi}(0) \int_E \theta(\lambda)\frac{g(|\lambda|)}{|\lambda|}\, d^*\lambda - \varphi(0)\int_E \theta(\lambda)g(|\lambda|)\, d^*\lambda.\end{aligned}$$

Changing the variable $\lambda \mapsto 1/\lambda$ in the first integral, one has

$$\int_E \left(\sum_{q\in\mathbb{Q}^\times} \widetilde{\varphi}\left(\frac{q}{\lambda}\right)\right) \theta(\lambda)\frac{1}{|\lambda|} g(|\lambda|)\, d^*\lambda = \int_{E^{-1}} \left(\sum_{q\in\mathbb{Q}^\times} \widetilde{\varphi}(q\lambda)\right) \overline{\theta}(\lambda)|\lambda| g\left(\frac{1}{|\lambda|}\right) d^*\lambda.$$

Observe that $g(|\lambda|)$ does not depend on λ_p for all p. For nontrivial θ, the second and the third integrals are zero. So, the whole thing reduces to

$$\Phi^-(g,\theta) = \widetilde{\Phi}^+(g^\tau, \overline{\theta}) + \varepsilon_\theta \widetilde{\varphi}(0) \int_{(0,1]} \frac{g(t)}{t}\, d^*t - \varepsilon_\theta\varphi(0) \int_{(0,1]} g(t)\, d^*t,$$

where $g^\tau(t) = tg(t^{-1})$ and $\varepsilon_\theta = 1$ if $\theta \equiv 1$ and 0 otherwise.

Thus, $\Delta_{s,\theta}$ is well defined for nontrivial θ. To define it for $\theta \equiv 1$, we should assign values to the second and the third integrals. Assume that

$$\int\limits_{(0,1]} \frac{g(t)}{t}\, d^*t, \quad \int\limits_{(0,1]} g(t)\, d^*t \quad \textit{have values.} \tag{B$'$}$$

Remark. When $g(t) = t^s$, the integrals in (B$'$) represent analytic functions $(s-1)^{-1}$ and s^{-1}, and we come to the formula for analytic continuation used in the previous subsection.

Fourier transform and functional equation. The distribution $\Delta_{g,\theta}$ belongs to $\mathcal{S}'(\mathbb{A})$ and has a Fourier transform. To write it down in a suitable form, we assume that

$$\int\limits_{(0,1]} g^\tau(t)\, d^*t, \quad \int\limits_{(0,1]} \frac{g^\tau(t)}{t}\, d^*t \quad \textit{have values.} \tag{B$''$}$$

Subtracting two regularization formulas for $\widetilde{\Phi}^-(g,\theta)$ and $\Phi^-(g^\tau,\overline{\theta})$, we obtain the following formula for the Fourier transform (functional equation):

$$\begin{aligned}\widetilde{\Phi}(g,\theta) &= \Phi(g^\tau,\overline{\theta}) + \varepsilon_\theta\varphi(0)\left(\int\limits_{(0,1]}\frac{g(t)}{t}\, d^*t + \int\limits_{(0,1]} g^\tau(t)\, d^*t\right) - \varepsilon_\theta\widetilde{\varphi}(0)\left(\int\limits_{(0,1]} g(t)\, d^*t + \int\limits_{(0,1]}\frac{g^\tau(t)}{t}\, d^*t\right)\\ &= \Phi(g^\tau,\overline{\theta}) + \varepsilon_\theta\varphi(0)\int\limits_{(0,+\infty)}\frac{g(t)}{t}\, d^*t - \varepsilon_\theta\widetilde{\varphi}(0)\int\limits_{(0,+\infty)} g(t)\, d^*t.\end{aligned}$$

The integrals in (B$'$) and (B$''$) exist when g is compactly supported on $(0,+\infty)$ or $g \in \mathcal{S}(\mathbb{R}_+)$. As a consequence, we have the following statement.

Theorem 2.3. *Any Schwartz–Bruhat function on the group of idele classes $\varpi \in \mathcal{S}(\mathbb{A}^\times/\mathbb{Q}^\times)$ can be regarded as a distribution on adeles*; *i.e., there is an embedding $\mathcal{S}(\mathbb{A}^\times/\mathbb{Q}^\times) \to \mathcal{S}'(\mathbb{A})$.*

Indeed, considering the Pontryagin dual of $\mathbb{A}^\times/\mathbb{Q}^\times = T$, we see that any such ϖ is a finite linear combination of $g(\lambda_\infty) \in \mathcal{S}(\mathbb{R}_+)$ twisted with a multiplicative character θ as above.

Fourier transform of specific distributions. Let

$$g(t) = \begin{cases} t^\alpha, & t \in (0,1],\\ t^\beta, & t \in [1,+\infty).\end{cases}$$

Then,

$$g^\tau(t) = \begin{cases} t^{1-\beta}, & t \in (0,1],\\ t^{1-\alpha}, & t \in [1,+\infty),\end{cases}$$

and

$$\widetilde{\Phi}(g) = \widetilde{\Phi}(g^\tau) + \varphi(0)\left(\frac{1}{\alpha-1} - \frac{1}{\beta-1}\right) - \widetilde{\varphi}(0)\left(\frac{1}{\alpha} - \frac{1}{\beta}\right).$$

Here, we applied the analytic continuation in α and β; they should not be 0 or 1. When $\alpha = \beta$, we obtain the Tate formula $\widetilde{\Phi}(t^\alpha) = \Phi(t^{1-\alpha})$.

For the characteristic function of a segment $g(t) = I_{[a,b]}(t)$, $a, b > 0$, one has

$$\widetilde{\Phi}\big(I_{[a,b]}\big) = \Phi\big(t\, I_{[1/b,1/a]}\big) + \varphi(0)(\ln b - \ln a) - \widetilde{\varphi}(0)(b-a).$$

For $g(t) = t^s h(t)$ depending on $s \in \mathbb{C}$, our functional equation simplifies to

$$\widetilde{\Phi}(t^s h(t)) = \Phi(t^{1-s} h(t^{-1})) + \varphi(0)(Mh)(s-1) - \widetilde{\varphi}(0)(Mh)(s),$$

where $(Mh)(s)$ stands for the Mellin transform of h at s. In this case, we can apply the analytic continuation in s. For example,

$$\widetilde{\Phi}(t^s e^{-t}) = \Phi(t^{1-s} e^{-1/t}) + \varphi(0)\Gamma(s-1) - \widetilde{\varphi}(0)\Gamma(s), \qquad s \neq 1, 0, -1, -2, \dots,$$

$$\widetilde{\Phi}\left(\frac{t^s}{a+t^r}\right) = \Phi\left(\frac{t^{1-s}}{a+t^{-r}}\right) + \varphi(0)\frac{\pi a^{\frac{s-1}{r}-1}}{r\sin\frac{\pi(s-1)}{r}} - \widetilde{\varphi}(0)\frac{\pi a^{\frac{s}{r}-1}}{r\sin\frac{\pi s}{r}}, \qquad a > 0, \quad s \notin r\mathbb{Z}.$$

3. MNEMOFUNCTIONS ON $\mathbb{Q}_p$

Denote by $B_n = p^{-n}\mathbb{Z}_p \subset \mathbb{Q}_p$, $n \in \mathbb{Z}$, a closed ball of radius p^n centered at 0.

The space of Schwartz–Bruhat functions $\mathcal{S} = \mathcal{S}(\mathbb{Q}_p)$ consists of $\varphi\colon \mathbb{Q}_p \to \mathbb{C}$ satisfying the conditions

(1) $\exists l \in \mathbb{Z}\colon (x-y) \in B_{-l} \Rightarrow \varphi(x) = \varphi(y)$ (local constancy);

(2) $\exists k \in \mathbb{Z}\colon \operatorname{supp}\varphi \subset B_k$ (support is compact).

Define the spaces

$$\mathcal{S}_n = \{\varphi\colon (x-y) \in B_{-n} \Rightarrow \varphi(x) = \varphi(y),\ \operatorname{supp} f \subset B_n\}, \qquad n = 0, 1, 2, \dots.$$

Then,

(1) $\mathcal{S}_n$ is finite-dimensional over $\mathbb{C}$, $\dim \mathcal{S}_n = p^{2n}$;

(2) $n \le m \Rightarrow \mathcal{S}_n \subset \mathcal{S}_m$, $\bigcup_{n=0}^{\infty} \mathcal{S}_n = \mathcal{S}$;

(3) the space $\mathcal{S}$ is the inductive limit of $\mathcal{S}_n$: $\mathcal{S} = \lim\operatorname{ind}_{n\to\infty} \mathcal{S}_n$.

Equipped with the topology of inductive limit, $\mathcal{S}$ is a locally convex algebra with pointwise operations.

Let us fix a norm $\|\cdot\|$ on $\mathcal{S}$, say, $\|\varphi\| = \sup_{x\in\mathbb{Q}_p} |\varphi(x)|$ (all norms are equivalent on each $\mathcal{S}_n$ because of its finite dimensionality). Let $\Lambda\colon 0 < \lambda_0 < \lambda_1 < \lambda_2 < \dots$ be an increasing sequence of real numbers. We define a seminorm $\pi_\Lambda(f)$ of $f \in \mathcal{S}$ as follows. We choose a minimal n such that $f \in \mathcal{S}_n$. Then, $\pi_\Lambda(f) := \lambda_n \|f\|$.

Theorem 3.1. *The topology of $\mathcal{S}$ is given by a family of seminorms π_Λ indexed by all increasing sequences. This topology is not metrizable.*

Algebra of Egorov-type mnemofunctions. Consider an algebra $\mathcal{G}_M$ formed by all sequences from $\mathcal{S}(\mathbb{Q}_p)$ and an ideal $\mathcal{N} \subset \mathcal{G}_M$ consisting of finite sequences:

$$\mathcal{G}_M = \{g = (g_0, g_1, g_2, \dots)\colon g_n \in \mathcal{S}(\mathbb{Q}_p),\ n = 0, 1, 2, \dots\},$$
$$\mathcal{N} = \{g \in \mathcal{G}_M\colon \exists m\, \forall n \ge m \Rightarrow g_n \equiv 0\}.$$

Definition. We define an *algebra of Egorov-type mnemofunctions on* $\mathbb{Q}_p$ as the quotient algebra $\mathcal{G} = \mathcal{G}_M/\mathcal{N}$. (A similar quotient algebra consisting of smooth functions on $\mathbb{R}$ with an ideal of finite sequences was considered by Egorov [5].)

Definition. We define a *canonical δ-sequence* as

$$\delta_n = p^n I_{p^n\mathbb{Z}_p} \in \mathcal{S}, \qquad n = 0, 1, 2, \dots.$$

The sequence δ_n converges to the p-adic Dirac δ weakly in the space $\mathcal{S}' = \mathcal{S}'(\mathbb{Q}_p)$ of Schwartz–Bruhat distributions. The Dirac δ is given by $\langle\delta,\varphi\rangle = \varphi(0)$, $\varphi \in \mathcal{S}$. The Fourier images $\widetilde{\delta}_n = \mathcal{F}\delta_n = I_{B_n}$ converge weakly to a constant function equal to 1.

Definition. We define a *regularization operator* $R\colon \mathcal{S}' \to \mathcal{G}_M$ and an *embedding* $\tau\colon \mathcal{S}' \to \mathcal{G}$ of the linear space of distributions into the algebra of mnemofunctions by

$$R\colon u \mapsto R(u) = \big((u*\delta_0)\cdot\widetilde{\delta}_0, (u*\delta_1)\cdot\widetilde{\delta}_1, (u*\delta_2)\cdot\widetilde{\delta}_2, \dots\big),$$

$$\tau\colon u \mapsto [R(u)]; \qquad \text{here, } [R(u)] = R(u) + \mathcal{N} \text{ is a coset in } \mathcal{G}_M.$$

Note that $R(u)_n = (u*\delta_n)\cdot\widetilde{\delta}_n = (u\cdot\widetilde{\delta}_n)*\delta_n \in \mathcal{S}_n$, $n = 0,1,2,\dots$, and $R(u)$ converges to u in $\mathcal{S}'$. Thus, the regularization operator maps any distribution to *specific* sequence of functions in $\mathcal{S}$ that converges to this distribution. The separability of $\mathcal{S}'$ yields the injectivity of the embedding τ.

Remark. As in the case of mnemofunctions containing $\mathcal{S}'(\mathbb{R})$ [7] and $\mathcal{D}'(\mathbb{R})$ [8], other regularization operators are possible, for example, $R(u)_n = (u*\varphi_n)\cdot\widetilde{\varphi}_n$, where φ_n is a δ-sequence different from δ_n. Our definition of the regularization operator allows us to reach agreement between the multiplication of distributions by locally constant functions and the multiplication of mnemofunctions (Theorem 3.4), as well as between the Fourier transform of distributions and the Fourier transform of mnemofunctions (Theorem 3.6). These results do not hold in general for another regularization operator and have no immediate analogue for mnemofunctions with real domain.

Example 3.1. The distribution δ is represented by the mnemofunction

$$R(\delta) = (\delta_0, \delta_1, \delta_2, \dots).$$

Example 3.2. The homogeneous distribution Δ_s, $s \notin -1 + \frac{2\pi i}{\ln p}\mathbb{Z}$, is represented by the mnemofunction

$$R(\Delta_s)_n = (1-p^{-1})^{-1}|x|_p^\alpha I_{B_n\setminus B_{-n}} + \frac{p^{-\alpha n}}{1-p^{-1-\alpha}} I_{B_{-n}}.$$

Example 3.3. The Vladimirov finite part $\mathcal{P}\frac{1}{|x|_p}$ is represented by the mnemofunction

$$R\big(\mathcal{P}\tfrac{1}{|x|_p}\big)_n = |x|_p^{-1} I_{B_n\setminus B_{-n}}(x) - (1-p^{-1})np^n I_{B_{-n}}(x).$$

Example 3.4. The locally integrable function $\ln|x|_p \in L_1^{\mathrm{loc}}(\mathbb{Q}_p)$ yields the mnemofunction

$$R(\ln|x|_p)_n = \ln|x|_p \cdot I_{B_n\setminus B_{-n}}(x) - (n + (p-1)^{-1})\ln p \cdot I_{B_{-n}}.$$

Example 3.5. A multiplicative character θ is a locally integrable function on $\mathbb{Q}_p$ and defines the mnemofunction

$$R(\theta(x))_n = \theta(x)\,\Delta_{B_n\setminus B_{-n}}(x).$$

Remark. In the cases of $\mathcal{S}'(\mathbb{R})$ [7] and $\mathcal{D}'(\mathbb{R})$ [8], we can embed distributions into rather small algebras of sequences of elements from $\mathcal{S}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$, respectively. These algebras are formed of sequences g_n such that, for any seminorm π (on $\mathcal{S}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$, respectively), a sequence $\pi(g_n)$ grows at most polynomially; i.e., $\pi(g_n) < f(n)$ for some polynomial f depending only on π and (g_n).

In the p-adic case, we should not bound the growth of seminorms in the algebra of mnemofunctions; otherwise, we cannot embed distributions into it.

Theorem 3.2. *There exists $u \in \mathcal{S}'(\mathbb{Q}_p)$ (even $u \in L_1^{\mathrm{loc}}(\mathbb{Q}_p)$) and a seminorm π on $\mathcal{S}(\mathbb{Q}_p)$ such that $\pi(R(u)_n) \ge \lambda_n$, $n = 0,1,2,\dots$, for any given sequence λ_n.*

Indeed, let $u = \theta$ be the character from Example 3.5. Without loss of generality, assume that $\Lambda = (\lambda_n)$ is a positive and increasing sequence. For a seminorm π_Λ on $\mathcal{S}$, one has $\pi_\Lambda(R(u)_n) = \lambda_n$.

Multiplication of distributions and mnemofunctions. There is a p-adic analogue of the Schwartz example that shows the impossibility of defining a product of distributions that reasonably generalizes the multiplication of distributions by smooth functions.

Example 3.6 (contradictory equality).

$$0 = 0 \cdot |x|_p^{-2} = (\delta(x) \cdot |x|_p^2) \cdot |x|_p^{-2} = \delta(x) \cdot (|x|_p^2 \cdot |x|_p^{-2}) = \delta(x) \cdot 1 = \delta(x).$$

Note that, for general $u, v \in \mathcal{S}'$, there is no $w \in \mathcal{S}'$ such that $\tau(u)\tau(v) = \tau(w)$. Multiplying mnemofunctions, we go beyond our linear space. For example, it is impossible to imagine the square of the δ function as a usual functional.

There is a *canonical embedding* $\mathcal{S} \to \mathcal{G}$: $a \mapsto [\langle a \rangle]$, $\langle a \rangle = (a, a, a, \dots) \in \mathcal{G}_M$, in addition to the restriction of the embedding $\tau: \mathcal{S} \to \mathcal{G}_M$.

Theorem 3.3. *The embedding τ is in agreement with the canonical embedding and the multiplication in the algebra $\mathcal{S} \subset \mathcal{S}'$; i.e.,*

$$\tau(a) = [\langle a \rangle], \qquad \tau(ab) = \tau(a)\tau(b) \qquad \forall a, b \in \mathcal{S}.$$

Indeed, the equality $\tau(a) = [\langle a \rangle]$ is equivalent to $R(a) - \langle a \rangle \in \mathcal{N}$. Take m such that $a \in \mathcal{S}_m$. For all $n \geq m$, one has $R(a)_n = a$. Thus, $R(a) - \langle a \rangle$ is finite and belongs to the ideal $\mathcal{N}$.

The equality $\tau(ab) = \tau(a)\tau(b)$ is equivalent to $R(ab) - R(a)R(b) \in \mathcal{N}$. One has $R(ab) - R(a)R(b) = R(ab) - \langle ab \rangle - \langle a \rangle(R(b) - \langle b \rangle) - R(b)(R(a) - \langle a \rangle) \in \mathcal{N}$ because each term belongs to $\mathcal{N}$.

Consider an algebra $\mathcal{E}$ of locally constant functions $\mathbb{Q}_p \to \mathbb{C}$, $\mathcal{S} \subset \mathcal{E}$. In distribution theory, one defines the product of $u \in \mathcal{S}'$ and $a \in \mathcal{E}$ by the formula $\langle au, \varphi \rangle = \langle u, a\varphi \rangle$ for any test functions $\varphi \in \mathcal{S}$.

Theorem 3.4. *The embedding τ is in agreement with the multiplication of distributions by locally constant functions; i.e.,*

$$\tau(au) = \tau(a)\tau(u) \qquad \forall a \in \mathcal{E}, \ u \in \mathcal{S}'.$$

Definition. Two mnemofunctions g and h are called *associated* if $(g - h)_n \to 0$ weakly.

Let $u, v \in C(\mathbb{Q}_p)$ be continuous functions and uv be their pointwise product. In general, $\tau(uv) \neq \tau(u)\tau(v)$, i.e., $R(uv) - R(u)R(v) \notin \mathcal{N}$. However, the following theorem holds.

Theorem 3.5. *For all $u, v \in C(\mathbb{Q}_p)$, $R(uv)$ and $R(u)R(v)$ are associated.*

The following examples demonstrate the multiplication of distributions (or of the corresponding mnemofunctions, to be more precise).

Example 3.7 (the square of the δ function).

$$R(\delta)^2 = (\delta_0, p\delta_1, p^2\delta_2, \dots) = C \cdot R(\delta), \qquad \text{where} \quad C = (1, p, p^2, \dots).$$

The "infinite" coefficient C here is a *mnemonumber*, i.e., an element of the quotient algebra of all complex sequences with respect to the ideal of finite sequences.

Example 3.8 (product $\delta \cdot \Delta_\alpha$, $\alpha \neq -1$).

$$R(\delta) \cdot R(\Delta_\alpha) = C \cdot R(\delta), \qquad \text{where} \quad C_n = \frac{p^{-\alpha n}}{1 - p^{-1-\alpha}}, \quad n = 0, 1, 2, \dots.$$

Example 3.9 (product $\Delta_\alpha \cdot \Delta_\beta$). Let $\alpha, \beta \in \mathbb{C}$ and $\alpha, \beta, \alpha+\beta \neq -1$.

$$R(\Delta_\alpha) \cdot R(\Delta_\beta) = R(\Delta_{\alpha+\beta}) + C \cdot R(\delta),$$

where

$$C_n = \frac{p^{-(\alpha+1)n}}{1-p^{-(\alpha+1)}} \frac{p^{-(\beta+1)n}}{1-p^{-(\beta+1)}} p^n - \frac{p^{-(\alpha+\beta+1)n}}{1-p^{-(\alpha+\beta+1)}}, \qquad n = 0, 1, 2, \ldots .$$

Example 3.10 (product of homogeneous distributions). Let θ_1 and θ_2 be multiplicative characters, $\alpha, \beta \in \mathbb{C}$, $\theta_1\theta_2 \not\equiv 1$, $\alpha, \beta \neq -1$. In this nondegenerate case, the additional δ-term disappears:

$$R(\Delta_{\alpha,\theta_1}) \cdot R(\Delta_{\beta,\theta_2}) = R(\Delta_{\alpha+\beta,\theta_1\theta_2}).$$

Fourier transform and convolution. There are a Fourier transform $\mathcal{F}$ and a convolution $*$ defined on $\mathcal{S}$ that have usual properties. The Fourier transform of $u \in \mathcal{S}'$ is defined by duality, $\langle \mathcal{F}u, \varphi \rangle = \langle u, \mathcal{F}\varphi \rangle$, $\varphi \in \mathcal{S}$.

Definition. The Fourier transform $\mathcal{F}_M$ on $\mathcal{G}_M$ is defined componentwise,

$$\mathcal{F}_M \colon \mathcal{G}_M \to \mathcal{G}_M \colon (g_0, g_1, g_2, \ldots) \mapsto (\mathcal{F}g_0, \mathcal{F}g_1, \mathcal{F}g_2, \ldots),$$

$$\mathcal{F} \colon \mathcal{G} \to \mathcal{G} \colon [(f_n)] \mapsto [\mathcal{F}(f_n)].$$

Since $\mathcal{F}_M(\mathcal{N}) = \mathcal{N}$, we have the well-defined Fourier transform $\mathcal{F} \colon \mathcal{G} \to \mathcal{G}$.

Theorem 3.6. *The Fourier transform on $\mathcal{G}$ is in agreement with the Fourier transform on $\mathcal{S}'$, i.e.,*

$$\mathcal{F}(\tau(u)) = \tau(\mathcal{F}u) \qquad \forall u \in \mathcal{S}'.$$

This statement follows from the equality

$$\mathcal{F}\big((u * \delta_n) \cdot \widetilde{\delta}_n\big) = (\mathcal{F}(u * \delta_n)) * \big(\mathcal{F}\widetilde{\delta}_n\big) = \big((\mathcal{F}u) \cdot \widetilde{\delta}_n\big) * \delta_n = ((\mathcal{F}u) * \delta_n) \cdot \widetilde{\delta}_n.$$

The problem of defining a convolution on $\mathcal{S}'$ (i.e., a bilinear map $* \colon \mathcal{S}' \times \mathcal{S}' \to \mathcal{S}'$) that reasonably continues the convolution on $\mathcal{S}$ reduces, with the help of the Fourier transform, to the problem of defining a product of distributions, which has no solution inside $\mathcal{S}'$.

Definition. Let us define a convolution on $\mathcal{G}_M$ and $\mathcal{G}$ as follows:

$$* \colon \mathcal{G}_M \times \mathcal{G}_M \to \mathcal{G}_M \colon ((f_0, f_1, f_2, \ldots), (g_0, g_1, g_2, \ldots)) \mapsto (f_0 * g_0, f_1 * g_1, f_2 * g_2, \ldots),$$

$$* \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G} \colon ([(f_n)], [(g_n)]) \mapsto [(a_n) * (g_n)].$$

Theorem 3.7. *The convolution on $\mathcal{G}$ is in agreement with the convolution on $\mathcal{S}$; i.e.,*

$$\tau(a * b) = \tau(a) * \tau(b) \qquad \forall a, b \in \mathcal{S}.$$

$(\mathcal{G}, *, +)$ is a convolution algebra isomorphic to $(\mathcal{G}, \times, +)$, and the isomorphism is given by the Fourier transform.

4. MNEMOFUNCTIONS ON ADELES

In this section, we embed the linear space $\mathcal{S}'(\mathbb{A})$ of Schwartz–Bruhat distributions into the algebra of adelic mnemofunctions.

Separating Archimedean and non-Archimedean places. In the case of adeles, we can see that the Archimedean (real) place stands apart from the non-Archimedean places. It is natural to study these latter places together. A group of adeles splits as $\mathbb{A} = \mathbb{R} \times \mathbb{A}_0$, where $\mathbb{A}_0$ is a group of *discrete adeles* [4].

We can represent the Schwartz–Bruhat space as a tensor product (over $\mathbb{C}$):

$$\mathcal{S}(\mathbb{A}) = \mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{A}_0).$$

The space $\mathcal{S}(\mathbb{A}_0)$ of Schwartz–Bruhat functions on discrete adeles is a restricted tensor product of $\mathcal{S}(\mathbb{Q}_p)$ and can be represented as an inductive limit of finite-dimensional spaces in a way similar to the local case of $\mathcal{S}(\mathbb{Q}_p)$ [4]. As a consequence, any linear functional on $\mathcal{S}(\mathbb{A}_0)$ is continuous. On the contrary, the space $\mathcal{S}(\mathbb{R})$ cannot be represented as a limit of finite-dimensional spaces.

A similar factorization exists for the space $\mathcal{E} = \mathcal{E}(\mathbb{A})$ of *smooth functions* on adeles: $\mathcal{E}(\mathbb{A}) = \mathcal{E}(\mathbb{R}) \otimes \mathcal{E}(\mathbb{A}_0)$ (φ belongs to $\mathcal{E}(\mathbb{A}_0)$ if, for some neighborhood $U \subset \mathbb{A}_0$ of zero, $x - y \in U$ implies $\varphi(x) = \varphi(y)$) and for the space $\mathcal{D} = \mathcal{D}(\mathbb{A})$ of *smooth compactly supported functions*: $\mathcal{D}(\mathbb{A}) = \mathcal{D}(\mathbb{R}) \otimes \mathcal{D}(\mathbb{A}_0)$. Note that, because of the non-Archimedean structure of $\mathbb{A}_0$, one has $\mathcal{S}(\mathbb{A}_0) = \mathcal{D}(\mathbb{A}_0)$. As a consequence, we have factorizations for the spaces of distributions $\mathcal{D}'(\mathbb{A})$, tempered distributions $\mathcal{S}'(\mathbb{A})$, and compactly supported distributions $\mathcal{E}'(\mathbb{A})$:

$$\mathcal{D}'(\mathbb{A}) = \mathcal{D}'(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathcal{D}'(\mathbb{A}_0), \qquad \mathcal{S}'(\mathbb{A}) = \mathcal{S}'(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathcal{S}'(\mathbb{A}_0), \qquad \mathcal{E}'(\mathbb{A}) = \mathcal{E}(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathcal{E}(\mathbb{A}_0).$$

The first example of a mnemofunction algebra that contains distributions from $\mathcal{D}'(\mathbb{R})$ was introduced by Colombeau [8]. Since then, algebras of this type on a real domain have been extensively studied. A differential algebra containing tempered distributions from $\mathcal{S}'(\mathbb{R})$ with a Fourier transform and a convolution was constructed in [7]. Let us denote it by $\mathcal{G}^{(\infty)}$.

The algebra $\mathcal{G}^{(\infty)}$ is a quotient algebra $\mathcal{G}_M^{(\infty)}/\mathcal{N}^{(\infty)}$, where $\mathcal{G}_M^{(\infty)}$ consists of all $g_n \in \mathcal{S}(\mathbb{R})^{\mathbb{N}}$ such that, for any seminorm π on $\mathcal{S}$, $\pi(g_n)$ grows no faster than n^α for some α, and $\mathcal{N}^{(\infty)} \subset \mathcal{G}_M^{(\infty)}$ consists of all sequences such that $\pi(g_n)$ decreases faster than n^α for any $\alpha \in \mathbb{R}$.

A regularization operator $R_\varphi \colon \mathcal{S}'(\mathbb{R}) \to \mathcal{G}_M^{(\infty)}$ is constructed as follows. For $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}} \varphi(t)dt = 1$, we consider a δ-sequence $\varphi_n(t) = n\varphi(nt)$ and assume that $R_\varphi(u)_n = (u * \varphi_n) \cdot \widetilde{\varphi}_n$. Note that there is no canonical δ-sequence in the real case; therefore, we should consider the whole family of regularizations.

Let us introduce a mnemofunction algebra on discrete adeles. Then, we can glue together the Archimedean and non-Archimedean places to obtain mnemofunctions on adeles.

Egorov-type mnemofunctions on discrete adeles. Consider a subset $V_1 = \prod_p \mathbb{Z}_p \subset \mathbb{A}_0$. A family $V_n = nV_1$, $n = 1, 2, 3, \dots$, indexed by positive integers $\mathbb{N}$, forms a basis of neighborhoods of zero in $\mathbb{A}_0$. The embedding $V_m \subset V_n$ holds if and only if n divides m. In fact, V_1 is a ring of principal ideals, and its ideals are exactly V_n.

A family of sets $\{m \colon m \text{ is divisible by } n\} \subset \mathbb{N}$, $n \in \mathbb{N}$, is a filter different from the usual Frechet filter on $\mathbb{N}$. We will write $n \to \infty$ assuming this filter. Being a basis of neighborhoods of zero, V_n also forms a filter isomorphic to the one just considered.

Definition. As in the p-adic case, we define an *algebra of Egorov-type mnemofunctions on discrete adeles* $\mathcal{G}^{(0)}$ as the quotient algebra $\mathcal{G}_M^{(0)}/\mathcal{N}^{(0)}$, where $\mathcal{G}_M^{(0)}$ consists of all sequences of elements of $\mathcal{S}(\mathbb{A}_0)$ and $\mathcal{N}^{(0)}$ consists of finite sequences.

Definition. We define a *canonical δ-sequence* as

$$\delta_n(x) = nI_{V_n}(x) \in \mathcal{S}(\mathbb{A}_0).$$

Its Fourier transform $\widetilde{\delta_n} = I_{\frac{1}{n}V_1}$ forms a 1-sequence.

There is a simple relation between adelic and p-adic canonical δ-sequences. Let $n = \prod_p p^{\alpha_p}$ be a prime expansion of n, where all but finitely many α_p equal 0. Then,

$$\delta_n(x) = \bigotimes_p \delta_{\alpha_p}(x_p),$$

where $\delta_n(x) \in \mathcal{S}(\mathbb{A}_0)$, $\delta_{\alpha_p}(x_p) = p^{\alpha_p} I_{p^{\alpha_p}\mathbb{Z}_p}(x_p) \in \mathcal{S}(\mathbb{Q}_p)$, and $\delta_0(x_p) = I_{\mathbb{Z}_p}(x_p)$ is the characteristic function of p-adic integers.

Definition. A *regularization operator* $R\colon \mathcal{S}'(\mathbb{A}_0) \to \mathcal{G}_M$ and an *embedding* $\tau\colon \mathcal{S}'(\mathbb{A}_0) \to \mathcal{G}$ are defined as follows:

$$R(u) = \big((u * \delta_1)\cdot \widetilde{\delta}_1, (u*\delta_2)\cdot\widetilde{\delta}_2, \dots, (u*\delta_n)\cdot\widetilde{\delta}_n, \dots\big), \qquad \tau(u) = [R(u)] = R(u) + \mathcal{N}.$$

As in the local case, the regularization operator maps a distribution to a *specific* sequence of functions from $\mathcal{S}$ that weakly converges to this distribution.

Theorem 4.1. *The embedding τ is in agreement with the canonical embedding $\mathcal{S}(\mathbb{A}_0) \to \mathcal{G}$: $a \mapsto [(a,a,a,\dots)]$ and with the multiplication on $\mathcal{S}(\mathbb{A}_0)$; i.e.,*

$$\tau(a) = [(a,a,a,\dots)], \qquad \tau(ab) = \tau(a)\tau(b) \qquad \forall a,b \in \mathcal{S}(\mathbb{A}_0).$$

In the standard distribution theory, one defines the product of a distribution $u \in \mathcal{S}'$ and a locally constant function $a \in \mathcal{E}$ as follows: $\langle au, \varphi\rangle = \langle u, a\varphi\rangle$, $\varphi \in \mathcal{S}$.

Theorem 4.2. *The embedding τ is in agreement with the multiplication of distributions by locally constant functions; i.e., $\tau(au) = \tau(a)\tau(u)$ $\forall a \in \mathcal{E},\ u \in \mathcal{S}'$.*

Example 4.1. The Dirac δ is represented by $R(\delta) = (\delta_1, \delta_2, \dots, \delta_n, \dots)$. Its square is $R(\delta)^2 = (\delta_1, 2\delta_2, \dots, n\delta_n, \dots)$.

Definition. We define a *Fourier transform* and a *convolution* on $\mathcal{G}_M$ componentwise,

$$\mathcal{F}(g_1, g_2, \dots) = (\mathcal{F}g_1, \mathcal{F}g_2, \dots), \qquad (f_1, f_2, \dots) * (g_1, g_2, \dots) = (f_1 * g_1, \dots).$$

This yields well-defined Fourier transform $\mathcal{F}$ and convolution $*$ on $\mathcal{G}$.

Theorem 4.3. *The Fourier transform on $\mathcal{G}$ is in agreement with the Fourier transform on $\mathcal{S}'$, and the convolution on $\mathcal{G}$ is in agreement with the convolution on $\mathcal{S}$; i.e.,*

$$\mathcal{F}(\tau(u)) = \tau(\mathcal{F}u) \quad \forall u \in \mathcal{S}', \qquad \tau(a*b) = \tau(a)*\tau(b) \quad \forall a,b \in \mathcal{S}.$$

$(\mathcal{G}, *, +)$ is a convolution algebra isomorphic to $(\mathcal{G}, \times, +)$, and the isomorphism is given by the Fourier transform.

Gluing Archimedean and non-Archimedean places.

Definition. We define a *mnemofunction algebra on adeles* $\mathbb{A}$ as the tensor product (over $\mathbb{C}$) $\mathcal{G}^{(\infty)} \otimes \mathcal{G}^{(0)}$.

Theorem 4.4. *The algebra $\mathcal{G}^{(\infty)} \otimes \mathcal{G}^{(0)}$ with the Fourier transform and convolution contains the space $\mathcal{S}'(\mathbb{A})$ of Schwartz–Bruhat distributions as a linear subspace.*

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