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ON ADELIC ANALOGUE OF LAPLACIAN

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The purpose of this paper is de»ning an analogue of Laplacian (Vladimirov operator) on the group of »nite adeles and describing its spectra. We need generalized functions to expound it correctly. Necessary information from the theory of functions and distributions on the »eld of p-adic numbers $\mathbb{Q}_p$ as well as generalized functions on the group of adels are given at the beginning of the paper.

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1. p-adic numbers and integral on $\mathbb{Q}_p$

The aim of this paper is to develop some mathematical methods which will be useful for the development of quantum models on the group of »nite adeles. This paper is purely mathematical, so we do not plan to go into details of possible applications in mathematical physics, see, e.g., [1]- [25].

This part deals with main notions and facts of p-adic analysis we need further. For proof we refer to books [26] and [9].

Let us denote $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ sets of positive integer, integer, rational, real and complex numbers respectively, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.

Let p be prime, $x \in \mathbb{Q}$. Then $x = p^{\gamma(x)} \frac{m}{n}$, $m, n, \gamma \in \mathbb{Z}$, where m and n are not divisible by p.

Let $|x|_p = p^{-\gamma(x)}$, $|0|_p = 0$. Then $|x|_p$ is the valuation on $\mathbb{Q}_p$ satisfying strong triangle inequality $|x + y|_p \leq \max\{|x|_p, |y|_p\}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is denoted $\bar{\mathbb{Q}}_p$ or $\mathbb{Q}_p$ and called »eld of p-adic numbers. It is also assumed $\mathbb{Q}_\infty \equiv \mathbb{R}$.

Each element of $x \in \mathbb{Q}_p$ has a unique representation $x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...)$, $0 \leq x_k \leq p - 1$. The fractional part of x is number

$$\{x\}_p = \begin{cases} 0, & \text{if } \gamma(x) \geq 0 \text{ or } x = 0 \\ p^{\gamma}(x_0 + ... + x_{|\gamma|-1} p^{|\gamma|-1}), & \text{if } \gamma(x) < 0 \end{cases} \tag{1.1}$$

For any $x \in \mathbb{Q}_p$, the number $\{x\}_p$ is rational. The following equality holds for the fractional parts

$$\{x + y\}_p = \{x\}_p + \{y\}_p - N, \quad N = 0 \text{ or } 1; \quad x, y \in \mathbb{Q}_p. \tag{1.2}$$

Let us denote $B_p[a; p^\gamma] = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^\gamma\}$, $B_p(a; p^\gamma) = \{x \in \mathbb{Q}_p : |x - a|_p < p^\gamma\}$, $S_p(a; p^\gamma) = \{x \in \mathbb{Q}_p : |x - a|_p = p^\gamma\}$. We also assume $B_p[p^\gamma] = B_p[0; p^\gamma]$,

$B_p[p^\gamma] = B_p[0;p^\gamma]$, $S_p(p^\gamma) = S_p(0;p^\gamma)$. $\mathbb{Z}_p = B_p[1]$ is additive subgroup of $\mathbb{Q}_p$ called *p-adic integers*, $\mathbb{Q}_p^\times = \mathbb{Q}_p \setminus \{0\}$ is multiplicative group. Unit sphere $U_p = S_p(1)$ is a multiplicative subgroup of $\mathbb{Q}_p^\times$.

For any set K let us denote its characteristic function

$$\Delta_K(x) = \begin{cases} 1, & x \in K, \\ 0, & x \notin K. \end{cases}$$

Theorem 1.1. $\mathbb{Q}_p$ is locally compact totally disconnected Abelian topological group under addition and $\mathbb{Z}_p$ is its compact subgroup.

Recall that a topological space is called *totally disconnected* if a connected domain of each point consists of that point itself.

An additive character of group $\mathbb{Q}_p$ is any continuous complex-valued function χ_p de»ned on $\mathbb{Q}_p$ satisfying condition

$$\chi_p(x+y) = \chi_p(x)\chi_p(y), \quad |\chi_p(x)| = 1, \ x, y \in \mathbb{Q}_p.$$

Theorem 1.2. Additive characters of $\mathbb{Q}_p$ with pointwise multiplication form a group isomorphic to $\mathbb{Q}_p$. Isomorphism is given by mapping

$$\xi \mapsto \chi_p(\xi x) = e^{2\pi i \{\xi x\}_p}. \tag{1.3}$$

So far as $\mathbb{Q}_p$ is an Abelian locally compact group, there exists shift-invariant Haar measure dx_p on $\mathbb{Q}_p$. We normalize it with the condition

$$\int_{\mathbb{Z}_p} dx_p = 1.$$

Let Ω be measurable subset in $\mathbb{Q}_p$. We denote as usually $L_q(\Omega)$, $1 \le q < +\infty$ set of all measurable functions $f(x)$ on Ω such that

$$\int_\Omega |f(x)|^q \, dx_p < +\infty.$$

If Ω is open then the space $C_0(\Omega)$ of all continuous functions on Ω with compact support is dense in $L_q(\Omega)$, $1 \le q < +\infty$.

Let us also denote $L_q^{loc}(\Omega)$ the set of measurable functions $f(x)$ on Ω such that $f \in L_q(K)$ for any compact $K \subset \Omega \to$ not compact.

Here are some integrals we need further:

$$\int_{B[p^\gamma]} dx = p^\gamma, \quad \int_{S(p^\gamma)} dx = p^\gamma(1 - \frac{1}{p}), \quad \gamma \in \mathbb{Z}, \tag{1.4}$$

$$\int_{B[p^\gamma]} |x|_p^{\alpha-1} \, dx = p^{\alpha\gamma} \frac{1-p^{-1}}{1-p^{-\alpha}}, \quad \int_{B[1]} |x|_p^{\alpha-1} \, dx = \frac{1-p^{-1}}{1-p^{-\alpha}}, \quad \operatorname{Re}\alpha > 0, \tag{1.5}$$

$$\int_{B[p^\gamma]} x_p(\xi x) dx = \begin{cases} p^\gamma, & |\xi|_p \le p^{-\gamma} \\ 0, & |\xi|_p > p^{-\gamma+1} \end{cases}, \tag{1.6}$$

$$\int_{\mathbb{Q}_p} |x|_p^{\alpha-1} \chi_p(\xi x) dx = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}} |\xi|_p^{-\alpha}, \quad \xi \ne 0, \operatorname{Re}\alpha > 0. \tag{1.7}$$

2. Locally Constant Functions on Totally Disconnected Locally Compact Group

Here is an impotent de»nition of a locally constant function.

Let X be a Hausdor« space. A function $f : X \to \mathbb{R}$ is called *locally constant at point* $x \in X$ if there exists neighborhood U_x of point x such that $f|_{U_x}$, the restriction of f onto domain U_x, is a constant function. Function f is called *locally constant* if it is locally constant at every point $x \in X$.

Note that all locally constant functions are continuous. An example of such a function is a characteristic function of clopen (closed and open at the same time) $U \subset X$.

The following theorem describes all locally constant functions.

Theorem 2.1. A function $f : X \to \mathbb{R}$ is locally constant i« $X = \bigsqcup_{i \in I} U_i$, where U_i are clopen subsets in X and f is constant on U_i. Here and below we denote $\bigsqcup_{i \in I} U_i$ the union of non-intersecting sets U_i.

Proof. The suБciency of condition is obvious. Let us proove the necessity. Let $(a_i)_{i \in I}$ be the range of a function f. Let's consider a set $U_i = f^{-1}(a_i)$. It is a closed subset of X, because f is continuous and $\{a_i\}$ is closed in $\mathbb{R}$. Thus $X = \bigsqcup_{i \in I} U_i$. Let $x \in U_i$. There exists neighborhood U_x of x such that $f|_{U_x} \equiv f(x) = a_i$, that is $U_x \subset U_i$. It implies that U_i is open. ⊠

Corollary 1. Let X be a compact space. Each locally constant function $f : X \to \mathbb{R}$ takes only a »nite number of values.

Corollary 2. The existence of non-trivial locally constant function on X (i.e. not constant) implies that X is not connected.

Below at this part we assume that X is Hausdor« totally disconnected locally compact and σ-compact Abelian topological group. (2.1)

Recall that a topological space is called *σ-compact* if it is a countable union of compact sets.

Note that such spaces X are paracompact and there exists a continuous partition of unity. Let's show that the condition (2.1) implies the existence of the locally constant partition of unity on X.

Theorem 2.2. Any X satisfying (2.1) has a basis of neighborhoods of zero consisting of open compact subgroups $(V_i)_{i \in I}$.

Let us denote $C(X)$ the set of all continuous (real-valued) functions on X, $C_0(X)$ set of continuous functions with compact support, $\mathcal{E}(X)$ the set of locally constant functions on X and $S(X) = \mathcal{E}(X) \bigcap C_0(X)$.

Theorem 2.3. Let X satisfy (2.1). Then for any compact K and any open set $U \subset X$, $K \subset U$ there exists a »nite cover of compact K with non-intersecting open compact sets $(U_i)_{1 \le i \le n}$ such that $K \subset \bigsqcup_{i=1}^{n} U_i \subset U$.

Proof. Let's cover each point $x \in K$ with neighborhood $x + V_{i(x)}$, where V_i is open compact subgroup from basis of neighborhoods of zero (see theorem 2.2). We take $V_{i(x)}$ such that $x + V_{i(x)} \subset U$. The cover $K \subset \bigcup_{x \in K} [x + V_{i(x)}] \subset U$ has a »nite subcover

$$K \subset \bigcup_{k=1}^{m} \left[x_k + V_{i(x_k)}\right] \subset U. \tag{2.2}$$

Let us choose a open group V_0 from the basis of neighborhoods of zero such that $V_0 \subset \bigcap_{k=1}^{m} V_{i(x_k)}$. Thus X can be partitioned into factor-classes corresponding to V_0, i.e. $X = \bigsqcup_{y \in X/V_0} [y + V_0]$.

As far as $V_0 \subset \bigcap_{k=1}^{m} V_{i(x_k)}$, each compact set $x_k + V_{i(x_k)}$ can be partitioned into »nite number of sets $y + V_0$, $y \in V_{i(x_k)}/V_0$. Inclusion (2.2) implies that there exists n such that $K \subset \bigsqcup_{i=1}^{n} [y_i + V_0] \subset U$. The assertion $U_i = y_i + V_0$ »nishes the proof . ⊠

Corollary 1. For any compact K and any open set U such that $K \subset U$, there exists a function $\varphi \in \mathcal{E}(X)$ satisfying conditions: $0 \le \varphi \le 1$, $\operatorname{supp}\varphi \subset U$ and $\varphi(x) = 1$ on K.

Proof. Example of such φ is

$$\varphi(x) = \sum_{k=1}^{n} \Delta_{U_i}(x),$$

where U_i are clopen sets from theorem 2.3. ⊠

Corollary 2. *(Partition if unity).* Let X satisfy (2.1), Ω be an open set in X, $(U_i)_{i\in I}$ be cover of X with open sets. Then there exists a locally constant partition of unity subordinated to cover $(U_i)_{i\in I}$ i.e. there exists a family of locally constant functions $(\varphi_i)_{i\in I}$ such that

$$0 \le \varphi_i \le 1, \quad \operatorname{supp}\varphi_i \subset U_i,$$

the family $\{\operatorname{supp}\varphi_i\}_{i\in I}$ is locally »nite and

$$\sum_{i\in I} \varphi_i(x) = 1 \text{ for all } x \in \Omega.$$

Corollary 3. If X satis»es the condition (2.1), then $S(X)$ is a dense subset of $C_0(X)$.

Proof. Let $f \in C_0(X)$ and $K = \operatorname{supp} f$. Then f is uniformly continuous. That is why for any $\epsilon > 0$ there exists an open and compact neighborhood V of zero such that $x - y \in V$, $x, y \in K$ implies $|f(X) - f(y)| \le \epsilon$. So there exists a sequence $a_1, ..., a_n \in K$ such that $K \subset \bigsqcup_{i=1}^{n} (a_i + V) \equiv \bigsqcup_{i=1}^{n} U_i$. Let us consider

$$\varphi(x) = \sum_{i=1}^{n} f(a_i)\Delta_{U_i}(x) \in \mathcal{E}(X).$$

For any $x \in K$ we have

$$|f(x) - \varphi(x)| = \left| \sum_{i=1}^{n} [f(x) - f(a_i)]\Delta_{U_i}(x) \right| \le \epsilon. \quad ⊠$$

3. The Generalized Functions on $\mathbb{Q}_p$

Functions from $S(\mathbb{Q}_p) = \mathcal{E}(\mathbb{Q}_p) \cap C_0(\mathbb{Q}_p)$ are called *Schwartz-Bruhat functions.*

The group of p-adic numbers $\mathbb{Q}_p$ satis»es the condition (2.1). Thus the Corollary 3 of Theorem 2.3 gives $\overline{S(\mathbb{Q}_p)} = C_0(\mathbb{Q}_p)$. But always $\overline{C_0(\mathbb{Q}_p)} = L_q(\mathbb{Q}_p)$, $1 \le q < +\infty$. So we obtain that $\overline{S(\mathbb{Q}_p)} = L_q(\mathbb{Q}_p)$, $1 \le q < +\infty$.

For each $\varphi \in S(\mathbb{Q}_p)$ we can »nd $l \in \mathbb{Z}$ such that

$$\varphi(x + y) = \varphi(x), \quad |y|_p \le p^l.$$

The greatest of such integers $l = l(\varphi)$ is called the *parameter of constancy* of the function φ. Let us denote $S_N^l(\mathbb{Q}_p)$ the set of functions from $S(\mathbb{Q}_p)$ with support being contained in the ball $B_p[p^N]$ and the parameter of constancy not less than l. There is an inclusion

$$S_N^l \subset S_{N'}^{l'} \text{ if } N < N',\ l < l'. \tag{3.1}$$

The Space S_N^l is »nite dimensional and $\dim S_N^l = p^{N-l}$. Thus any of equivalent norm of a »nite dimensional space can be introduced naturally.

In view of inclusion (3.1), the natural topology of $S(\mathbb{Q}_p)$ is the topology of inductive limit

$$S(\mathbb{Q}_p) = \operatorname*{limind}_{N\to+\infty} S_N, \quad S_N = \operatorname*{limind}_{l\to-\infty} S_N^l(\mathbb{Q}_p).$$

The properties of $S(\mathbb{Q}_p)$ can be deduced from the general properties of inductive limits and the theory of topological vector spaces. They are gathered at the following theorem.

Theorem 3.1. 1) The space $S(\mathbb{Q}_p)$ is a Hausdor« complete nuclear-convex Montel space (thus it is a locally convex space ч l.c.s.).

2) For any l.c.s. X all linear operators $A : S(\mathbb{Q}_p) \to X$ are continuous.

3) Sequence $\varphi_k \in S(\mathbb{Q}_p)$ tends to zero in $S(\mathbb{Q}_p)$ means

a) $\varphi_k \in S^l_N(\mathbb{Q}_p)$, where N and l do not depend on k.

b) $\varphi_k \to 0$ uniformly as $k \to \infty$.

In particular, there is no unbounded functionals on $S(\mathbb{Q}_p)$ and the adjoint space $S'(\mathbb{Q}_p)$ of *p-adic distributions* (generalized functions) is complete with respect to the strong topology and sequentially complete with respect to a weak one.

Besides, the following statement holds. Any linear operator $A : S(\mathbb{Q}_p) \to S'(\mathbb{Q}_p)$ can be given via its kernel-function, i.e. function $K \in S'(\mathbb{Q}_p \times \mathbb{Q}_p)$ such that

$$< A\varphi, \psi > = < K(x,y), \varphi(x)\psi(y) >, \quad \varphi, \psi \in S(\mathbb{Q}_p).$$

If $\varphi \in S(\mathbb{Q}_p)$ then its *Fourier transform* is de»ned by formula

$$(\mathcal{F}\varphi)(x) = \int_{\mathbb{Q}_p} \varphi(x)\chi_p(\xi x)dx, \tag{3.2}$$

where χ_p is p-adic character given by (1.3).

Main properties of Fourier transform on $S(\mathbb{Q}_p)$ are collected at the following theorem [26], [9]:

Theorem 3.2. A Fourier transform $\mathcal{F}$ is a linear isomorphism of $S(\mathbb{Q}_p)$ into $S(\mathbb{Q}_p)$. The following equalities hold: an inversion formula

$$\varphi(x) = \int_{\mathbb{Q}_p} \widehat{\varphi}(\xi)\chi_p(-\xi x)d\xi, \tag{3.3}$$

Steklov-Parseval equality

$$\int_{\mathbb{Q}_p} \varphi(x)\overline{\psi}(x)dx = \int_{\mathbb{Q}_p} \hat{\varphi}(x)\overline{\hat{\psi}}(x)dx, \tag{3.4}$$

and one equivalent to (3.4)

$$\int_{\mathbb{Q}_p} \varphi(x)\widehat{\psi}(x)dx = \int_{\mathbb{Q}_p} \widehat{\varphi}(x)\psi(x)dx. \tag{3.5}$$

According to the general ideology of topological vector spaces, duality theory and equality (3.5), the Fourier transform of a generalized function $u \in S'(\mathbb{Q}_p)$ is de»ned as follows

$$< \widehat{u}, \varphi > = < u, \widehat{\varphi} >, \quad \varphi \in S(\mathbb{Q}_p).$$

Theorem 3.2 implies that a Fourier transform $\mathcal{F}$ performs an isomorphism of $S'(\mathbb{Q}_p)$ into $S'(\mathbb{Q}_p)$.

We have that $S(\mathbb{Q}_p) \subset L_2(\mathbb{Q}_p)$, $\overline{S(\mathbb{Q}_p)} = L_2(\mathbb{Q}_p)$. Thus $L_2(\mathbb{Q}_p) \subset S'(\mathbb{Q}_p)$. So we may consider the restriction of Fourier transform $\mathcal{F} : S'(\mathbb{Q}_p) \to S'(\mathbb{Q}_p)$ onto the domain $L_2(\mathbb{Q}_p)$ and denote this restriction again as $\mathcal{F}$.

Theorem 3.3. ([26], [9]). The operator $\mathcal{F}$ of a Fourier transform is unitary on $L_2(\mathbb{Q}_p)$.

Let us consider some examples of the Fourier transform of functions >from $S(\mathbb{Q}_p)$ and $S'(\mathbb{Q}_p)$ that we need below [26], [9]:

$$\widehat{\Delta}_{B[p^\gamma]}(\xi) = p^\gamma \widehat{\Delta}_{B[p^{-\gamma}]}(\xi). \tag{3.6}$$

In particular,

$$\widehat{\Delta}_{B[1]}(\xi) = \widehat{\Delta}_{B[1]}(\xi). \tag{3.7}$$

$$\widehat{\Delta}_{S(p^\gamma)}(\xi) = p^\gamma \widehat{\Delta}_{B[p^{-\gamma}]}(\xi) - p^{\gamma-1}\widehat{\Delta}_{B[p^{-\gamma+1}]}(\xi). \tag{3.7}$$

$$\widehat{|\xi|}^{\alpha-1} = \frac{1-p^{\alpha-1}}{1-p^\alpha}|\xi|_p^\alpha, \quad \mathrm{Re}\,\alpha > 0. \tag{3.9}$$

Case $\alpha = 1/2$ gives us eigenfunction $|\xi|_p^{-1/2}$ of the Fourier transform with eigenvalue $\lambda = -p^{-1/2}$.

4. Adeles and Ideles.

Let us consider the set $\mathbb{A}$ of all sequences of kind $a = (a_\infty, a_2, a_3, a_5, ...)$, where a_∞ is a real number, a_p is p-adic numbers. In addition, all a_p are p-adic integers beginning from some p (this p may vary for the di«erent a). The set of all such sequences form a ring with respect to a pointwise addition and multiplication. This ring is called *the ring of adeles*, the additive group of the ring is called *the group of adeles* and is denoted $\mathbb{A}$, the multiplicative group $\mathbb{A}^\times$ of the ring $\mathbb{A}$ is called *the group of ideles*. Thus the elements of group of ideles are sequances $\lambda = (\lambda_\infty, \lambda_2, \lambda_3, ..., \lambda_p, ...)$, where $\lambda_p \neq 0$ and $|\lambda_p|_p = 1$ for all p except a »nite number of indexes.

To de»ne topology on $\mathbb{A}^\times$ and $\mathbb{A}$ we consider the following procedure.

Let us consider a countable set of indexes $P = \{\infty, 2, 3, 5, ..., p, ...\}$ consisting of symbol ∞ and primes. Let us denote $\mathcal{P}$ the set of all »nite subsets of P containing ∞. The set $\mathcal{P}$ is ordered by inclusion, i.e. for $\pi_1, \pi_2 \in \mathcal{P}$ we say $\pi_1 \prec \pi_2$ if $\pi_1 \subset \pi_2$. The set $(\mathcal{P}, \prec)$ with order is directed. For any $\pi \in \mathcal{P}$ let us denote

$$\mathbb{A}(\pi) = Q_\infty \times \prod_{p\in\pi} \mathbb{Q}_p \times \prod_{p\notin\pi} \mathbb{Z}_p \text{ additive group of } \pi\text{-}adeles, \tag{4.1}$$

$$\mathbb{A}^\times(\pi) = Q_\infty^\times \times \prod_{p\in\pi} \mathbb{Q}_p^\times \times \prod_{p\notin\pi} U_p \text{ multiplicative group of } \pi\text{-}ideles. \tag{4.2}$$

The inequality $\pi_1 \prec \pi_2$, obviously, implies that the group $\mathbb{A}(\pi_1)$ is a subgroup of $\mathbb{A}(\pi_2)$ ($\mathbb{A}^\times(\pi_1)$ is a subgroup of $\mathbb{A}^\times(\pi_2)$ respectively). Thus

$$\mathbb{A} = \bigcup_{p\in\mathcal{P}} \mathbb{A}(\pi), \quad \mathbb{A}^\times = \bigcup_{p\in\mathcal{P}} \mathbb{A}^\times(\pi). \tag{4.3}$$

For any $\pi \in \mathcal{P}$ groups $\mathbb{A}(\pi)$ and $\mathbb{A}^\times(\pi)$ are endowed with a natural Tikhonov topology of Cartesian product. Sets $\mathbb{A}$ and $\mathbb{A}^\times$ are endowed with topologies of inductive limits, also known as a »nal topology :

$$\mathbb{A} = \operatorname*{limind}_{\pi\in\mathcal{P}} \mathbb{A}(\pi), \quad \mathbb{A}^\times = \operatorname*{limind}_{\pi\in\mathcal{P}} \mathbb{A}^\times(\pi).$$

These topologies are called *adelic* and *idelic* respectively.

Thus the basis of the topology in $\mathbb{A}$ ($\mathbb{A}^\times$ resp.) consists of sets of the following kind

$$\prod_{p\in S} V_p \times \prod_{p\notin S} \mathbb{Z}_p \quad (\prod_{p\in S} W_p \times \prod_{p\notin S} U_p \text{ resp.}), \tag{4.4}$$

where V_p (W_p resp.) are open sets in $\mathbb{Q}_p$ ($\mathbb{Q}_p^\times$ resp.) for any $S \in \mathcal{P}$.

Note. There is an inclusion $\mathbb{A}^\times \subset \mathbb{A} \subset \prod_p \mathbb{Q}_p$ and topology of $\mathbb{A}^\times$ is stronger than that of $\mathbb{A}$, and the latter is stronger than Tikhonov topology of Cartesian product $\prod_p \mathbb{Q}_p$.

Main properties of groups $\mathbb{A}$ and $\mathbb{A}^\times$ are collected at the following theorem.

Theorem 4.1. 1) Groups $\mathbb{A}$ and $\mathbb{A}^{\times}$ are locally compact and σ-compact. In particular, they are complete.

2) The group $\mathbb{A}^{\times}$ is dense in $\mathbb{A}$.

3) A set $K \subset \mathbb{A}$ ($K \subset \mathbb{A}^{\times}$ resp.) is compact in $\mathbb{A}$ ($\mathbb{A}^{\times}$ resp.) i« there exist $\pi \in \mathcal{P}$ such that $K \subset \mathbb{A}(\pi)$ ($K \subset \mathbb{A}^{\times}(\pi)$ resp.) and K is compact in $\mathbb{A}(\pi)$ ($\mathbb{A}^{\times}(\pi)$ resp.).

4) a sequence of adeles (ideles resp.) $a^{(n)} \in \mathbb{A}$ ($\lambda^{(n)} \in \mathbb{A}^{\times}$ resp.) tends to adele $a \in \mathbb{A}$ ($\lambda \in \mathbb{A}^{\times}$ resp.) if

a) $a^{(n)}$, $a \in \mathbb{A}(\pi)$ ($\lambda^{(n)}, \lambda \in \mathbb{A}^{\times}(\pi)$ resp.) for some $\pi \in \mathcal{P}$;

б) $a^{(n)} \to a$ ($\lambda^{(n)} \to \lambda$ resp.) pointwise.

Proof. 1) The locally compactness of $\mathbb{A}$ and $\mathbb{A}^{\times}$ is a direct consequence of the way we introduced topologies in $\mathbb{A}$ and $\mathbb{A}^{\times}$.

Let us check that $\mathbb{A}$ is σ-compact. Let us consider a sequence of expanding compacts in $\mathbb{A}$ numbered with 1 and primes:

$$K_1 = [-1,1] \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times ...$$

$$K_2 = [-2,2] \times 2^{-2}\mathbb{Z}_2 \times \mathbb{Z}_3 \times ...$$

$$...$$

$$K_7 = [-7,7] \times 2^{-7}\mathbb{Z}_2 \times 3^{-5}\mathbb{Z}_3 \times 5^{-3}\mathbb{Z}_5 \times 7^{-2}\mathbb{Z}_7 \times \mathbb{Z}_{11} \times ...$$

$$...$$

Obviously, $\mathbb{A} = \bigcup_p K_p$. By analogy we can check that $\mathbb{A}^{\times}$ is σ-compact.

2) Follows from the de»nitions of topologies in $\mathbb{A}^{\times}$ and $\mathbb{A}$.

3) Let $K \subset \mathbb{A}$ be a compact set. Then K is predcompact. So for a neighborhood of zero $V = [-1,1] \times \prod_p \mathbb{Z}_p$, there exist a »nite number of points $(x_i)_{1 \le i \le n(V)} \in K$ such that $K \subset \bigcup_{i=1}^{n}(x_i + V)$. In view of (4.3), a »nite set $x_1, x_2, ..., x_{n(V)}$ is contained in some $\mathbb{A}(\pi)$ and thus $\bigcup_{i=1}^{n(V)}(x_i + V) \subset \mathbb{A}(\pi)$. Besides, $\bigcup_{i=1}^{n}(x_i + V)$ is compact in $\mathbb{A}(\pi)$. The case of ideles is analogous.

5) This statement follows immediately from the previous one because any convergent sequence $a^{(n)} \in \mathbb{A}$ forms a compact set. ⊠

Example 4.1. Sequence $x^{(p)}$ (numbered with primes)

$$x^{(p)} = (0, ..., 0, p^{-1}, 0, ...) \in \mathbb{A} \quad \text{(number } p^{-1} \text{ has index } p)$$

converges to $x = (0, 0, ..., 0, ...)$ pointwise. Nevertheless, it doesn't converge in the topology of $\mathbb{A}$.

Example 4.2. An adele $a = (0, 0, ..., 0, ...) \in \mathbb{A} \setminus \mathbb{A}^{\times}$ is the limit of the sequence of ideles in $\mathbb{A}$: $x^{(p)} = (1/p, 2^p, 3^p, ..., p^p, 1, 1, ...) \in \mathbb{A}^{\times}$. But $x^{(p)}$ doesn't converge in the topology of $\mathbb{A}^{\times}$.

Below we consider only adeles and only the *discrete part* of them.

Let us denote $P_0 = \{2, 3, ..., p, ...\}$ the set of primes and $\mathcal{P}_0$ the set of all »nite subsets of P_0 ordered by inclusion. For any $\pi \in \mathcal{P}_0$, we denote

$$\mathbb{A}_0(\pi) = \prod_{p\in\pi} \mathbb{Q}_p \times \prod_{p\notin\pi} \mathbb{Z}_p \text{ the additive group of } \textit{»nite } \pi\textit{-adeles} \tag{4.5}$$

endowed with the natural Tikhonov topology. A group

$$\mathbb{A}_0 = \bigcup_{p\in\mathcal{P}_0} \mathbb{A}_0(\pi) \tag{4.6}$$

endowed with the topology of the inductive limit

$$\mathbb{A}_0 = \limind_{\pi \in \mathcal{P}_0} \mathbb{A}_0(\pi) \tag{4.7}$$

is called *the group of »nite adeles.*

Let us formulate all properties of the group of »nite adeles we need further.

Theorem 4.2. 1) Group $\mathbb{A}_0$ is locally compact and σ-compact. The following topological equality holds

$$\mathbb{A} = \mathbb{Q}_\infty \times \mathbb{A}_0. \tag{4.8}$$

2) A subset $K \subset \mathbb{A}_0$ is compact in $\mathbb{A}_0$ i« there exists$\pi \in \mathcal{P}_0$ such that $K \subset \mathbb{A}_0(\pi)$ and K is compact in $\mathbb{A}_0(\pi)$.

3) A sequence $a^{(n)} \in \mathbb{A}_0$ converges to $a \in \mathbb{A}_0$ i«

a) $a^{(n)}$, $a \in \mathbb{A}_0(\pi)$ for some $\pi \in \mathcal{P}_0$.

b) $a^{(n)} \to a$ pointwisely.

4) $\mathbb{A}_0$ is a totally disconnected group.

Proof. Statements 1)ч3) are prooved in the same way as in theorem 4.1. The equality (4.8) follows from the de»nition of the topology in $\mathbb{A}$ and $\mathbb{A}_0$.

4) $\mathbb{Q}_p$ is totally disconnected. So is $\prod_p \mathbb{Q}_p$. We have $\mathbb{A}_0 \subset \prod_p \mathbb{Q}_p$ and the topology in $\mathbb{A}_0$ is stronger than one induced from $\prod_p \mathbb{Q}_p$. Thus the identity mapping $I : \mathbb{A}_0 \to \prod_p \mathbb{Q}_p$ is continuous. If there existed a connected domain in $\mathbb{A}_0$ consisting of more than one point, then its image would be a connected domain in $\prod_p \mathbb{Q}_p$ but there isn't such domain. ⊠

5. Schwartz-Bruhat Functions on the Group of Adeles

The groups $\mathbb{A}$ and $\mathbb{A}_0$ are both locally compact Abelian. Let us denote dx and dx_0 Haar measures on $\mathbb{A}$ and $\mathbb{A}_0$ respectively. Here $x = (x_\infty, x_2, x_3, \ldots) \in \mathbb{A}$, $x_0 = (x_2, x_3, \ldots) \in \mathbb{A}_0$. These measures can be expressed via Haar measures dx_p on $\mathbb{Q}_p$:

$$dx = dx_\infty \cdot dx_2 \cdot \ldots \cdot dx_p \cdot \ldots \text{ and } dx_0 = dx_2 \cdot \ldots \cdot dx_p \cdot \ldots \qquad (5.1).$$

We assume that measures are normalized with the following conditions:

$$\int_0^1 dx_\infty = 1, \quad \int_{\mathbb{Z}_p} dx_p = 1. \tag{5.2}$$

The formula (5.1) should be interpreted as follows: for any integrable function of an adelic variable of the kind $f(x) = f_\infty(x) \cdot f_2(x) \cdot \ldots$ the equality holds

$$\int_{\mathbb{A}} f(x)dx = \int_{\mathbb{Q}_\infty} f_\infty(x_\infty)dx_\infty \cdot \int_{\mathbb{Q}_2} f_2(x_2)dx_2 \cdot \ldots \tag{5.3}$$

As a consequence of the formulas (5.1), (5.3) and (4.8), we obtain

$$dx = dx_\infty \cdot dx_0.$$

Thus the problem of constructing an analysis (of continuous, integrable and generalized functions) on the group of adeles reduced to separate problems 1) constructing an analysis on $\mathbb{R} = \mathbb{Q}_\infty$ and on the group of »nite adeles $\mathbb{A}_0$; 2) constructing function theory on Cartesian product $\mathbb{A} = \mathbb{Q}_\infty \times \mathbb{A}$.

Taking this into account, it is natural to give a special attention to »nite adeles $\mathbb{A}_0$. The purpose of this paper is to learn about the Vladimirov operator on the group of »nite adeles. So from now on saying 'adele' we mean *only* »nite adele. We write $x = (x_2, \ldots, x_p, \ldots) \in \mathbb{A}$, etc.

For any measurable set $T \subset \mathbb{A}$, we de»ne in a standard way a Lebesgue spaces $L_q(T)$, $1 \le q < +\infty$. According to the construction of Haar measure all compact subsets $K \subset \mathbb{A}$ are measurable. The space $C_0(\mathbb{A})$ is dense in $L_q(\mathbb{A})$, $1 \le q < +\infty$. Group $\mathbb{A}$ satis»es the condition (2.1), so all the statements of §2 is true for $\mathbb{A}$. In particular, the space $S(\mathbb{A})$ of locally constant functions with compact support is dense in $C_0(\mathbb{A})$. That is why

$$\overline{S(\mathbb{A})} = L_q(\mathbb{A}), \quad 1 \le q < +\infty. \tag{5.4}$$

Let us denote

$$L_q^{loc}(\mathbb{A}) = \{f : \mathbb{A} \to \mathbb{C} : f(x) \cdot \Delta_K(x) \in L_q(\mathbb{A}), \forall \text{ compact } K\}. \tag{5.5}$$

Let us describe the topology of $S(\mathbb{A})$.

The basis of neighborhoods of zero $\mathcal{V}$ in $\mathbb{A}$ consists of open compact subgroups

$$V_{K,N_p} = \prod_{p \in K} p^{N_p}\mathbb{Z}_p \times \prod_{p \notin K} \mathbb{Z}_p, \quad K \in \mathcal{V},\ N_p \in \mathbb{Z}. \tag{5.6}$$

For any given $V, W \in \mathcal{V}$ there exist (Theorem 2.3) $n(V,W) \in \mathcal{N}$ and $(x_i)_{1 \le i \le n} \in V$ such that

$$V \subset \coprod_{i=1}^{n} (x_i + W). \tag{5.7}$$

Let us denote $S_V^W(\mathbb{A})$ the set of a function $f \in S(\mathbb{A})$ such that

$$\operatorname{supp} f \subset V \text{ and } f(x+h) = f(x) \quad \text{for any } x \in V, h \in W \tag{5.8}$$

A neighborhood W is called *a domain of constancy* of the function f.

Theorem 5.1. The space $S_V^W(\mathbb{A})$ is »nite dimensional and it has $n(V,W)$ dimensions.

Proof. By Theorem 2.3 functions $(\Delta_{V \cap (x_i + W)})_{1 \le i \le n}$ form a basis of this space. ⊠

Let us endow $S_V^W(\mathbb{A})$ with the topology of $C_0(\mathbb{A})$ given by a norm. Set $\mathcal{V} \times \mathcal{V}$ can be ordered as follows:

$$(V,W) \prec (V',W') \text{ if } V \subset V' \text{ and } W' \subset W. \tag{5.9}$$

Set $(\mathcal{V} \times \mathcal{V}, \prec)$ is, obviously, directed and

$$S_V^W(\mathbb{A}) \subset S_{V'}^{W'}(\mathbb{A}) \text{ if } (V,W) \prec (V',W'). \tag{5.10}$$

In addition,

$$S(\mathbb{A}) = \bigcup_{(V,W)} S_V^W(\mathbb{A}). \tag{5.11}$$

Thus $S(\mathbb{A})$ is endowed with the topology of the inductive limit

$$S(\mathbb{A}) = \operatorname*{limind}_{(V,W)} S_V^W(\mathbb{A}). \tag{5.12}$$

This de»nition and general theory of locally convex spaces enable us to make the following statements.

Theorem 5.2. 1) Space $S(\mathbb{A})$ is Hausdor« nuclear-complete Montel l.c.s.

2) For any l.c.s X all linear operators $\mathbb{A} : S(\mathbb{A}) \to X$ are continuous.

3) Sequence $\varphi_k \in S(\mathbb{A})$ tends to zero in $S(\mathbb{A})$ means

a) $\varphi_k \in S_V^W(\mathbb{A})$ where V and W do not depend on k.

б) $\varphi_k \to 0$ uniformly.

Adjoint space $S'(\mathbb{A})$ of *distributions* (generalized functions) on $\mathbb{A}$ is complete with respect to strong topology and sequentially complete with respect to weak one.

Let us clear up the structure of functions from $S(\mathbb{A})$.

Theorem 5.3. Each function $f \in S_V^W(\mathbb{A})$ is "cylindric", i.e. has a form

$$f(x) = f_0(x_2, x_3, ..., x_p) \cdot \prod_{q>p} \Delta_{\mathbb{Z}_q}(x_q), \tag{5.13}$$

where f_0 is locally constant on $\mathbb{Q}_2 \times \mathbb{Q}_3 \times ... \times \mathbb{Q}_p$ for some p.

Proof. Without loss of generality we may assume that the domain of constancy W of a function f has a form

$$W = \prod_{2 \le q \le p} q^{N_q}\mathbb{Z}_q \times \prod_{q>p} \mathbb{Z}_q.$$

A function f is constant on the subgroup $W_0 = \{0\} \times \prod_{q>p} \mathbb{Z}_q \subset W$. Consequently, $f(x) = f_0(x_2, x_3, ..., x_p) \cdot \prod_{q>p} \Delta_{\mathbb{Z}_q}(x_q)$, where f_0 is a locally constant function with compact support on $\mathbb{Q}_2 \times ... \times \mathbb{Q}_p$ (the topology of $\mathbb{Q}_2 \times ... \times \mathbb{Q}_p$ induced from $\mathbb{A}$ coincides with the Tikhonov topology of Cartesian product). ⊠

Let us consider functions $\varphi(x)$ on $\mathbb{A}$ which can be represented as an infinite product

$$\varphi(x) = \prod_p \varphi_p(x_p), \quad x = (x_2, x_3, ..., x_p, ...) \in \mathbb{A}, \tag{5.14}$$

of multipliers satisfying conditions:

1) $\varphi_p \in S(\mathbb{Q}_p)$, $p \in P$;

2) $\varphi_p(x_p) = \Delta_{\mathbb{Z}_p}(x_p)$ for all except a finite number of p.

Any finite linear combination of such functions is called *Schwartz-Bruhat function* on $\mathbb{A}$. Let us denote $S_0(\mathbb{A})$ the space of all Schwartz-Bruhat functions.

Theorem 5.4. The space $S_0(\mathbb{A})$ of Schwartz-Bruhat functions coincides with the space of locally constant functions $S(\mathbb{A})$.

Proof. The inclusion $S_0(\mathbb{A}) \subset S(\mathbb{A})$ is obvious. Let us prove the backward inclusion. If $f \in S(\mathbb{A})$ then $f \in S_V^W(\mathbb{A})$ for some open compact neighborhoods (subgroups) of zero V and W in $\mathbb{A}$. According to the previous theorem 5.3

$$f(x) = f_0(x_2, x_3, ..., x_p) \cdot \prod_{q>p} \Delta_{\mathbb{Z}_q}(x_q),$$

where f_0 is locally constant on $\mathbb{Q}_2 \times \mathbb{Q}_3 \times ... \times \mathbb{Q}_p$.

To prove the statement it is sufficient to show that a locally constant function $f_0 : \mathbb{Q}_2 \times \mathbb{Q}_3 \times ... \times \mathbb{Q}_p \to \mathbb{R}$ is a linear combination of functions of the form

$$\varphi_2(x)...\varphi_p(x), \text{ where } \varphi_q(x_q) \in S(\mathbb{Q}_p), \quad 2 \le q \le p. \tag{5.15}$$

Let us denote $G = \mathbb{Q}_2 \times \mathbb{Q}_3 \times ... \times \mathbb{Q}_p$, $\tilde{x} = (x_2, x_3, ..., x_p) \in G$, with the norm $\|\tilde{x}\| = \max\{|x_2|_2, ..., |x_p|_p\}$. Assuming

$$\rho(\tilde{x}, \tilde{y}) = ||\tilde{x} - \tilde{y}|| \tag{5.16}$$

we turn (G, ρ) into an ultrametric space. The topology of G defined by metric ρ coincides with the Tikhonov topology. Besides, an opened ball $B_G(\tilde{a}, r)$ in G with the center $\tilde{a} = (a_2, ..., a_p)$ and radius $r > 0$ has the following form

$$B_G(\tilde{a}, r) = B_{\mathbb{Q}_2}(a_2, r) \times ... \times B_{\mathbb{Q}_p}(a_p, r). \tag{5.17}$$

So let $f_0 : G \to \mathbb{R}$ be a locally constant function with compact support $\operatorname{supp} f_0$ and let W be an open compact domain of constancy of function f_0. Theorem 2.3 yields

$$\operatorname{supp} f_0 = \bigsqcup_{i=1}^{n(f_0)} \{x_i + W\} \equiv K. \tag{5.18}$$

According to the general theory, an open set W in the ultrametric space G is nothing else than the disjoint union of open balls. The compactness of W yields the »niteness of the union. Let us choose the ball of the smallest radius r and divide the other balls into the balls of radius r. Thus we obtain the partition of the compact K:

$$K = \bigsqcup_{j=1}^{m} B_G(\tilde{a}^{(j)}; r).$$

There are a »nite number of balls $B_G(\tilde{a}^{(j)}; r)_{1 \le j \le m}$ of the same radius r. The function f_0 is constant on each of them and takes the corresponding values c_j (some c_j may be equal). Thus we obtain a representation of function $f_0(\tilde{x})$ as linear combination

$$f_0(\tilde{x}) = \sum_{j=1}^{m} c_j \Delta_{B_G(\tilde{a}^{(j)};\, r)}(\tilde{x}). \tag{5.19}$$

Remembering (5.17) we »nish the proof

$$\Delta_{B(\tilde{a}^{(j)};\, r)}(\tilde{x}) = \prod_{2 \le q \le p} \Delta_{B_{\mathbb{Q}_p}(a_q^{(j)};\, r)}(x_q). \qquad \boxtimes$$

This theorem and Corollary 2 of Theorem 2.3 imply that there exists a partition of the unity on $\mathbb{A}$ with the Schwartz-Bruhat functions. That is why we can de»ne support of any distribution $u \in S'(\mathbb{A})$ correctly.

Theorem 5.5 (Lemma of du Bua-Raimond). Let $f \in L_1^{loc}(\mathbb{A})$ and

$$\int_{\mathbb{A}} \varphi(x) f(x) dx = 0 \text{ for all } \varphi \in S(\mathbb{A}). \tag{5.20}$$

Then $f(x) = 0$ almost everywhere.

Proof. In view of inequality

$$\left| \int_{\mathbb{A}} \psi(x) f(x) dx \right| \le \sup |\psi(x)| \cdot \int_{\operatorname{supp} \psi} |f(x)|\, dx, \quad \psi \in C_0(\mathbb{A}),$$

we de»ne a linear continuous functional on $C_0(\mathbb{A})$ with the formula

$$\mu_f(x) = \int_{\mathbb{A}} \psi(x) f(x) dx$$

So μ_f represents the Radon measure on $\mathbb{A}$. The equality (5.20) means that the measure $\mu_f = f dx$ equals zero on $S(\mathbb{A})$. Since $S(\mathbb{A})$ is dense in $C_0(\mathbb{A})$, it gives that $\mu_f \equiv 0$, i.e. $u(x) = 0$ almost everywhere. $\boxtimes$

Corollary. The space $L_1^{loc}(\mathbb{A})$ can be canonically embedded into $S'(\mathbb{A})$.

6. Fourier Transform on the Group of Adeles

Let us de»ne a function χ_0 of $x = (x_2, x_3, ..., x_p, ...) \in \mathbb{A}$ with the formula

$$\chi_0(x) = \exp 2\pi i(x_2 + x_3 + ...). \tag{6.1}$$

The expression $x_2 + x_3 + ... + x_p + ...$ should be considered modulo 1, i.e. as

$$\{x_2\}_2 + \{x_3\}_3 + ... + \{x_p\}_p + ... \text{ (mod 1)}, \text{ (see. (1.1))}. \tag{6.2}$$

In other words,

$$\chi_0(x) = \prod_p \chi_p(x_p), \tag{6.3}$$

where χ_p are the additive characters on groups $\mathbb{Q}_p$ (see §1).

Theorem 6.1. The group of additive characters on the group of adeles $\mathbb{A}$ is isomorphic to $\mathbb{A}$ itself. This isomorphism is given by the formula

$$\xi \to \chi_0(\xi x),$$

where χ_0 is de»ned by (6.1). The expression

$$\xi x = \xi_2 x_2 + \ldots + \xi_p x_p + \ldots \tag{6.4}$$

is understood in the sense of (6.2).

For any $\varphi \in L_1(\mathbb{A})$, its Fourier transform is de»ned as follows

$$(\mathfrak{F}\varphi)(\xi) \equiv \widehat{\varphi}(\xi) = \int_{\mathbb{A}} \varphi(x)\chi_0(\xi x)dx. \tag{6.5}$$

Theorem 6.2. A Fourier transform $\mathfrak{F}$ performs a linear isomorphism of $S(\mathbb{A})$ into $S(\mathbb{A})$. The following equalities hold: the conversion formula

$$\varphi(x) = \int_{\mathbb{A}} \widehat{\varphi}(\xi)\chi_0(-\xi x)dx, \quad \varphi \in S(\mathbb{A}), \tag{6.6}$$

Steklov-Parseval equality

$$\int_{\mathbb{A}} \varphi(x)\overline{\psi(x)}dx = \int_{\mathbb{A}} \widehat{\varphi}(\xi)\overline{\widehat{\psi}}(\xi)d\xi, \quad \varphi, \psi \in S(\mathbb{A}) \tag{6.7}$$

and

$$\int_{\mathbb{A}} \varphi(x)\widehat{\psi}(x)dx = \int_{\mathbb{A}} \widehat{\varphi}(\xi)\psi(\xi)d\xi, \quad \varphi, \psi \in S(\mathbb{A}). \tag{6.8}$$

Proof. Let $\varphi(x) = \varphi_2(x_2)\ldots\varphi_p(x_p) \cdot \prod_{q>p} \Delta_{\mathbb{Z}_q}(x_q) \in S(\mathbb{A})$. The equality (6.3) yields that

$$\widehat{\varphi}(\xi) = \widehat{\varphi}_2(\xi_2)\ldots\widehat{\varphi}_p(\xi_p) \cdot \prod_{q>p} \widehat{\Delta}_{\mathbb{Z}_q}(\xi_q).$$

The inclusion $\widehat{\varphi}_p \in S(\mathbb{Q}_p)$ (Theorem 3.2) and equality $\widehat{\Delta}_{\mathbb{Z}_p} = \Delta_{\mathbb{Z}_p}$ (Formula 3.7) imply that $\widehat{\varphi} \in S(\mathbb{A})$.

The Conversion Formula (6.6) holds for any $\varphi \in L_1(\mathbb{A})$ such that $\widehat{\varphi} \in L_1(\mathbb{A})$. Since $S(\mathbb{A}) \subset L_1(\mathbb{A})$ and $\mathfrak{F}S(\mathbb{A}) \subset S(\mathbb{A})$, we have that the equality (6.6) holds and $\mathfrak{F}$ is an isomorphism.

The inclusion $S(\mathbb{A}) \subset L_1(\mathbb{A}) \cap L_2(\mathbb{A})$ and the Pontryagin duality theorem yield (6.7) and (6.8). ⊠

Since the Fourier transform $\mathfrak{F} : S(\mathbb{A}) \to S(\mathbb{A})$ is an isomorphism (Theorem 5.2) and the Equality (6.8) holds, we can use the duality arguments of l.c.s theory and de»ne a Fourier transform of a generalized function $u \in S'(\mathbb{A})$ with the formula

$$< \widehat{u}, \varphi > = < u, \widehat{\varphi} >, \quad \varphi \in S(\mathbb{A}). \tag{6.9}$$

That is $\mathfrak{F} : S'(\mathbb{A}) \to S'(\mathbb{A})$ is de»ned as an adjoint operator for an operator $\mathfrak{F} : S(\mathbb{A}) \to S(\mathbb{A})$. That is why $\mathfrak{F} : S'(\mathbb{A}) \to S'(\mathbb{A})$ is a continuous isomorphism in both cases of strong and weak topologies in $S'(\mathbb{A})$.

The equality $\overline{S(\mathbb{A})} = L_2(\mathbb{A})$ implies $L_2(\mathbb{A}) \subset S'(\mathbb{A})$. So we may consider the restriction of $\mathfrak{F} : S'(\mathbb{A}) \to S'(\mathbb{A})$ onto the domain $L_2(\mathbb{A})$ and denote it again as $\mathfrak{F}$.

Theorem 6.3. The Operator of Fourier transform $\mathfrak{F}$ is unitary in $L_2(\mathbb{A})$.

At last, let us de»ne a multiplication of distribution $u \in S'(\mathbb{A})$ and a locally constant function $a \in S(\mathbb{A})$ with the formula

$$< au, \varphi > = < u, a\varphi >, \quad \varphi \in S(\mathbb{A}). \tag{6.10}$$

7. The Operator M_α and its properties

Let $\xi = (\xi_2, \xi_3, ..., \xi_p, ...) \in \mathbb{A}$ and $\alpha = (\alpha_2, \alpha_3, ..., \alpha_p, ...) \in \mathbb{R}^\infty$ be an in»nite multi-index. Let us de»ne a formal expression

$$|\xi|^\alpha = \prod_p |\xi_p|_p^{\alpha_p}. \tag{7.1}$$

Theorem 7.1. The Expression (7.1) supplied with the condition

$$\lim_{p\to\infty} \alpha_p \ln p = 0 \tag{7.2}$$

de»nes a function $|\xi|^\alpha : \mathbb{A} \to \mathbb{R}$, »nitely almost everywhere on $\mathbb{A}$. It belongs to $L_q^{loc}(\mathbb{A})$, $1 \le q < +\infty$ if $q\alpha_p + 1 > 0$ but not to $L_q(\mathbb{A})$ for any $1 \le q < +\infty$.

Proof. The group $\mathbb{A}$ is locally compact and σ-compact (Theorem 4.2). Let us consider the following expanding sequence of compacts

$$K_p = 2^{-p}\mathbb{Z}_2 \times ... \times p^{-2}\mathbb{Z}_p \times \prod_{q>p} \mathbb{Z}_q, \quad K_1 = \prod_p \mathbb{Z}_p. \tag{7.3}$$

It covers $\mathbb{A}$, i.e.

$$\mathbb{A} = \bigcup_p K_p. \tag{7.4}$$

To prove the inclusion $|\xi|^\alpha \in L_q^{loc}(\mathbb{A})$, $1 \le q < +\infty$ it is suБcient to show the »niteness of integral

$$\int_{K_1} |\xi|^{\alpha q}\, d\xi. \tag{7.5}$$

Indeed, we have (see (1.5))

$$\int_{\mathbb{Z}_p} |\xi|_p^{\alpha_p}\, d\xi = \frac{1-p^{-1}}{1-p^{-(q\alpha_p+1)}}$$

as $\alpha_p + 1 > 0$. Thus

$$\int_{K_1} |\xi|^{\alpha q}\, d\xi = \prod_p \int_{\mathbb{Z}_p} |\xi|_p^{q\alpha_p}\, d\xi_p = \prod_p \frac{1-p^{-1}}{1-p^{-(q\alpha_p+1)}}, \quad (\alpha_p > -\frac{1}{q}). \tag{7.6}$$

The latter product converges i« the following sequence converges

$$\sum_p \left[\ln\left(1-\frac{1}{p}\right) - \ln\left(1-\frac{1}{p^{q\alpha_p+1}}\right)\right]. \tag{7.7}$$

It is equivalent to the convergence of the sequence

$$\sum_p \left[-\frac{1}{p} + \frac{1}{p^{q\alpha_p+1}}\right] = \sum_p \frac{1}{p}\left[\frac{1}{p^{q\alpha_p}} - 1\right] = \sum_p \frac{1}{p}\left[e^{q\alpha_p \ln p} - 1\right]. \tag{7.8}$$

The Condition (7.2) yields

$$e^{q\alpha_p \ln p} - 1 = -q\alpha_p \ln p + o(\alpha_p \ln p). \tag{7.9}$$

Putting (7.9) into (7.8) we obtain that the convergence of sequence (7.8) is equivalent to the convergence of

$$\sum_p \frac{\alpha_p \ln p}{p}. \tag{7.10}$$

But it is convergent due to the condition (7.2).

Thus $\alpha_p \ln p \to 0$ and $\alpha_p > -\frac{1}{q}$ imply $|\xi|^\alpha \in L_q^{loc}(\mathbb{A})$, $1 \le q < +\infty$.

If $|\xi|^{\alpha}$ belonged to $L_q(\mathbb{A})$ for some $1 \le q < +\infty$ (of cause, we assume $\alpha_p > -\frac{1}{q}$), then (7.4) would imply

$$\int_{\mathbb{A}} |\xi|^{q\alpha}\, d\xi = \lim_{p\to\infty} \int_{K_p} |\xi|^{q\alpha}\, d\xi. \tag{7.11}$$

According to (7.3) and (1.5) we have

$$\int_{K_p} |\xi|^{q\alpha}\, d\xi = \int_{2^{-p}Z_2} |\xi|_2^{q\alpha_2}\, d\xi_2 \cdot \ldots \cdot \int_{p^{-2}\mathbb{Z}_p} |\xi|_p^{q\alpha_p}\, d\xi_p \cdot \prod_{s>p} \int_{Z_s} |\xi|_s^{q\alpha_s}\, d\xi_s =$$

$$= 2^{(q\alpha_2+1)p} \frac{1-2^{-1}}{1-2^{-(\alpha_2+1)}} \cdot \ldots \cdot p^{(q\alpha_p+1)2} \frac{1-p^{-1}}{1-p^{-(\alpha_p+1)}} \cdot \prod_{s>p} \frac{1-s^{-1}}{1-s^{-(\alpha_s+1)}} =$$

$$= 2^{(q\alpha_2+1)p} \cdot \ldots \cdot p^{(q\alpha_p+1)2} \cdot \prod_p \frac{1-p^{-1}}{1-p^{-(\alpha_p+1)}} =$$

$$= 2^{(q\alpha_2+1)p} \cdot \ldots \cdot p^{(q\alpha_p+1)2} \cdot \int_{K_1} |\xi|^{\alpha q} d\xi >$$

$$> p^{(q\alpha_p+1)2} \cdot \int_{K_1} |\xi|^{\alpha q} d\xi \to \infty \text{ as } p \to \infty$$

independently on the choice of q, $1 \le q < +\infty$.

Thus $|\xi|^{\alpha} \notin L_q(\mathbb{A})$ for all $1 \le q < +\infty$. ⊠

Corollary 1. The function $|\xi|^{\alpha}$ de»ned with (7.1) supplied with the conditions (7.2) and $\alpha_p > -1$ is »nite almost everywhere.

Proof. The conditions mentioned above yield $|\xi|^{\alpha} \in L_q^{loc}(\mathbb{A})$. Consequently, $|\xi|^{\alpha}$ is »nite almost everywhere on every compact in $\mathbb{A}$. The function $|\xi|^{\alpha}$ is »nite almost everywhere on $\mathbb{A}$ due to σ-compactness of $\mathbb{A}$. ⊠

Corollary 2. The conditions (7.2) and $\alpha_p > -1$ imply $|\xi|^{\alpha} \in S'(\mathbb{A})$.

Note. From now on we assume the condition (7.2) to hold without special prescription.

Putting $q = 2$ into the equality (7.6) we obtain the »niteness of integral

$$\int_{\mathbb{A}} |\xi|^{2\alpha} \cdot |\psi(\xi)|\, d\xi,$$

where $\psi \in S(\mathbb{A})$ has the form

$$\psi(\xi) = \prod_p \Delta_{\mathbb{Z}_p}(\xi_p).$$

In fact, we de»ned operator

$$A_\alpha : L_2(\mathbb{A}) \to L_2(\mathbb{A}) : \psi(\xi) \longmapsto |\xi|^{\alpha} \psi(\xi) \tag{7.12}$$

with the domain $D(A_\alpha) = S(\mathbb{A})$.

On the other hand, in view of Theorem 7.1, we can de»ne an operator of a multiplication on almost everywhere »nite measurable function $|\xi|^{\alpha}$:

$$M_\alpha : L_2(\mathbb{A}) \to L_2(\mathbb{A}) : \varphi(\xi) \longmapsto |\xi|^{\alpha} \varphi(\xi). \tag{7.13}$$

This operator has the domain

$$D(M_\alpha) = \{\varphi \in L_2(\mathbb{A}) : |\xi|^{\alpha} \varphi(\xi) \in L_2(\mathbb{A})\}. \tag{7.14}$$

It is well-known that the operator M_α is self-adjoint.

Obviously, the operator A_α is symmetric and $(A_\alpha\psi, \psi) = \left\|A_{\alpha/2}\psi\right\|^2 \ge 0$. In view of Friedrichs theorem the operator A_α can be extended up to a self-adjoint operator.

On the other hand the following theorem is true.

Theorem 7.2. The operator A_α ($\alpha > -1/2$) is essentially self-adjoint. Its closure coincides with the operator M_α.

The proof uses lemma.

Lemma. Let A be a closable operator with the domain $D(A)$ dense in Hilbert space H and let M be its closed extension such that their adjoint operators coincide, i.e. $M^* = A^*$. Then $\overline{A} = M$.

Proof. By the condition $A \subset M$, M is closed and $M^* = A^*$. It implies $M^{**} = A^{**}$. The equalities $\overline{A} = A^{**}$ and $M = \overline{M} = M^{**}$ yield $\overline{A} = M$. ⊠

Proof of theorem 7.2. The operator A_α is symmetric and, consequently, closable; $A_\alpha \subset M_\alpha$, M_α is self-adjoint and, consequently, closed. In view of lemma, it is suﬃcient to show that $A_\alpha^* = M_\alpha^* = M_\alpha$. Let us take a pair of elements $v, w \in L_2(\mathbb{A})$ satisfying

$$(A_\alpha\varphi, v)_{L_2(\mathbb{A})} = (\varphi, w)_{L_2(\mathbb{A})}, \quad \varphi \in S(\mathbb{A}), \tag{7.15}$$

i.e. such that $v \in D(A_\alpha^*)$ and $w = A_\alpha^* v$. In other words,

$$\int_{\mathbb{A}} |\xi|^\alpha \varphi(\xi)\overline{v}(\xi)d\xi = \int_{\mathbb{A}} \varphi(\xi)\overline{w}(\xi)d\xi, \quad \forall\varphi \in S(\mathbb{A}), \tag{7.16}$$

or

$$\int_{\mathbb{A}} \varphi(\xi)[|\xi|^\alpha \overline{v}(\xi) - \overline{w}(\xi)]d\xi, \quad \forall\varphi \in S(\mathbb{A}). \tag{7.17}$$

We have $v, w \in L_2(\mathbb{A})$ and $|\xi|^\alpha \in L_2^{loc}(\mathbb{A})$ ($\alpha > -1/2$). Thus $h(\xi) = |\xi|^\alpha \overline{v}(\xi) - \overline{w}(\xi) \in L_1^{loc}(\mathbb{A})$.

According to the lemma of du Bua-Raimond (Theorem 5.5) the formula (7.17) yields that $h(\xi) = 0$ almost everywhere, i.e. $w(\xi) = |\xi|^\alpha v(\xi)$. Bearing this in mind, we obtain from (7.16) that

$$D(A_\alpha^*) = \{v \in L_2(\mathbb{A}) : |\xi|^\alpha v(\xi) \in L_2(\mathbb{A})\}$$

and

$$A_\alpha^* v(\xi) = w(\xi) = |\xi|^\alpha v(\xi),$$

i.e. $A_\alpha^* = M_\alpha$. ⊠

Let us ﬁnd spectrum of the operator M_α.

Theorem 7.3. Functions of the kind

$$\psi(\xi) = \prod_p c_p \Delta_{S(p^{N_p})}(\xi_p) \tag{7.18}$$

are an eigenfunction of the operator M_α corresponding to the eigenvalues

$$\lambda = \prod_p p^{\alpha_p N_p}. \tag{7.19}$$

Here $c_p \in \mathbb{R}$, $N_p \in \mathbb{Z}$ are some constants.

Proof. We are to show that the sequences c_p and N_p do exist.

The function $\psi(\xi)$ of kind (7.18) belongs to $D(M_\alpha)$ iﬀ $\psi \in L_2(\mathbb{A})$ and $|\xi|^\alpha \psi(\xi) \in L_2(\mathbb{A})$, i.e.

$$\int_{\mathbb{A}} |\psi(\xi)|^2 d\xi < +\infty \text{ и } \int_{\mathbb{A}} |\xi|^{2\alpha} |\psi(\xi)|^2 d\xi < +\infty. \tag{7.20}$$

Formula (1.4) yields

$$\int_{\mathbb{A}} |\psi(\xi)|^2 d\xi = \prod_p c_p^2 \int_{S(p^{N_p})} d\xi_p = \prod_p c_p^2 p^{N_p}(1 - \frac{1}{p}); \tag{7.21}$$

$$\int_{\mathbb{A}} |\xi|^{2\alpha} |\psi(\xi)|^2 d\xi = \prod_p c_p^2 \int_{S(p^{N_p})} |\xi|^{2\alpha_p} d\xi_p = \prod_p c_p^2 p^{2\alpha_p N_p} \int_{S(p^{N_p})} d\xi_p =$$

$$= \left(\prod_p c_p p^{\alpha_p N_p}\right)^2 \int_{\mathbb{A}} |\psi(\xi)|^2 d\xi_p. \tag{7.22}$$

Thus the function $\psi(\xi)$ of the kind (7.18) belongs to $D(M_\alpha)$ i« the following conditions hold

$$\sum_p N_p \alpha_p \ln p < +\infty \tag{7.23}$$

and

$$\prod_p c_p^2 p^{N_p}(1 - 1/p) \text{ converges (not to zero).} \tag{7.24}$$

For any given $\alpha_p > -1/2$ such that $\alpha_p \ln p \to +\infty$ and selected $N_p \in \mathbb{Z}$ satisfying (7.23) we can choose c_p to ful»l condition (7.24). For example, the following one will do

$$c_p = p^{-N_p/2}(1 - \frac{1}{p})^{-1/2}. \tag{7.25}$$

Note, there are many sequences $N = (N_p)_{p \in P}$, $N_p \in \mathbb{Z}$ satisfying (7.23). For example, all »nite sequences $(N_p)_{p \in P}$ (i.e. vanishing for all but a »nite number of N_p) will do.

To summarize, we have the functions $\psi(\xi)$ of kind (7.18) satisfying the conditions (7.23) and (7.25), λ de»ned by (7.19) and the equality $M_\alpha \varphi(\xi) = \varphi(\xi)$. Indeed,

$$M_\alpha \psi(\xi) = \prod_p |\xi_p|_p^{\alpha_p} \prod_p c_p \Delta_{S(p^{N_p})}(\xi_p) = \prod_p |\xi_p|_p^{\alpha_p} c_p \Delta_{S(p^{N_p})}(\xi_p) =$$

$$= \prod_p c_p p^{\alpha_p N_p} \Delta_{S(p^{N_p})}(\xi_p) = \left(\prod_p p^{\alpha_p N_p}\right)\left(\prod_p c_p \Delta_{S(p^{N_p})}(\xi_p)\right) = \lambda\psi(\xi). \boxtimes$$

Theorem 7.4. The spectrum $\sigma(M_\alpha)$ of the operator M_α (as $\alpha_p > -1$) coincides with the set of all non-negative real numbers $\sigma(M_\alpha) = \mathbb{R}_+$.

Proof. The equality $(A_\alpha \varphi, \varphi)_{L_2(\mathbb{A})} = \|A_{\alpha/2}\varphi\|^2 \geq 0$ yields $\sigma(M_\alpha) \subset [0, +\infty)$.

According to the theorem теорема 7.3, set

$$\Lambda = \left\{\lambda : \lambda = \prod_p p^{\alpha_p N_p},\ (N_p) \text{ is »nite sequence of integers}\right\} \subset \mathbb{R}_+ \tag{7.26}$$

is contained by point spectrum of the operator M_α.

As spectrum $\sigma(M_\alpha)$ is closed set, it is enough now to prove that Λ is dense in $\mathbb{R}_+$.

As a logarithmic function performs homomorphism of $\mathbb{R}_+$ into $\mathbb{R}$, it is enough to prove the density of

$$V = \left\{\mu \in \mathbb{R} : \mu = \sum_p N_p \alpha_p \ln p, N_p \in \mathbb{Z},\ \text{(sum is »nite)}\right\} \tag{7.27}$$

in $\mathbb{R}$.

Let us denote $h_p = \alpha_p \ln p$. We need to show the density of the set of the numbers of the form

$$\sum_p N_p h_p \text{ (»nite sum)} \tag{7.28}$$

in $\mathbb{R}$, supplied with the conditions $N_p \in \mathbb{Z}$ and $h_p \to 0$ as $p \to \infty$. It is simple. For any $\epsilon > 0$ let us take $h_p < \epsilon$. As $\mathbb{R} = \bigsqcup_{N\in\mathbb{Z}} (Nh_p, (N+1)h_p]$, then for any $x \in \mathbb{R}$ there exists $N_p(x)$ such that $x \in (N_p h_p, (N_p+1)h_p]$, i.e. $|x - N_p h_p| \le h_p < \epsilon$. ⊠

8. Vladimirov Operator on the Group of Finite Adeles

A pseudo-di«erential operator on the group of »nite adeles $\mathbb{A}$ is the operator A of the following form

$$(A\psi)(x) = \int_A a(\xi, x)\psi(\xi)\chi_0(-\xi x)d\xi \tag{8.1}$$

acting upon functions $\psi(x) \in S(\mathbb{A})$. The function $a(\xi, x)$, $(\xi, x \in \mathbb{A})$ is called *symbol of operator A*.

Vladimirov operator V_α (α is multi-index) is a pseudo-di«erential operator with a symbol $|\xi|^\alpha$ (we assume the conditions (7.2) and $\alpha_p > -1/2$ to hold).

Thus

$$V_\alpha\varphi(x) = \int_A |\xi|^\alpha \varphi(\xi)\chi_0(-\xi x)d\xi, \quad \varphi(x) \in S(A). \tag{8.2}$$

The operator V_α is unitary equivalent to the operator A_α, so we can state the following.

Theorem 8.1. the operator V_α with the domain $S(\mathbb{A})$ is essentially self-adjoint. Its closure (let us denote it V_α again) with the domain

$$D(V_\alpha) = \{\varphi \in L_2(A) : |\xi|^\alpha \varphi(\xi) \in L_2(A)\}$$

is a self-adjoint operator with spectrum $\sigma(V_\alpha)$ consisting of all non-negative real numbers $\mathbb{R}_+$.

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