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Algorithms of Solving Large Sparse Underdetermined Linear Systems with Embedded Network Structure

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Abstract

We consider algorithms for solving linear systems with embedded network structure. We investigate problems of network optimization based on research theoretic-graph specificity for the structure of the support and for properties of the base of a solution space.

1 Introduction

Let $G = \{I, U\}$ be a finite oriented graph without multiple arcs and loops. Consider the linear underdetermined system

$$\sum_{j \in I_i^+(U)} x_{ij} - \sum_{j \in I_i^-(U)} x_{ji} = \begin{cases} a_i, & i \in I \setminus I^* \\ x_i \cdot \mathrm{sign}[i], & i \in I^* \end{cases} \tag{1}$$

$$\sum_{(i,j) \in U} \lambda_{ij}^p x_{ij} = \alpha_p, \quad \text{for} \quad p = \overline{1, q} \tag{2}$$

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$$\text{sign}[i] = \begin{cases} 1, & i \in I^n \\ -1, & i \in I^* \setminus I^n \end{cases}, \qquad I^n \subseteq I^*$$

$$I_i^+(U) = \{j : (i,j) \in U\}, \qquad I_i^-(U) = \{j : (j,i) \in U\}$$

Restrictions (1) and (2) can be written as a block matrix in the following form:

$$A = \begin{bmatrix} M & R \\ Q & 0 \end{bmatrix}$$

Here submatrix M has size $|I| \times |U|$ and consists of elements 1 and -1 for every arc (i,j) with other elements being zeros. Submatrix R has size $|I| \times |I^*|$ consisting of signums of nodes. It has one nonzero element per column equal to sign$[i]$ to with the others equal to zero. Q is the submatrix of size $q \times |U|$ with additional restrictions on the variables x_{ij}, for $(i,j) \in U$.

Theorem 1. *The rank of the matrix of System (1) for a connectivity graph $G = \{I, U\}$ is equal to $|I|$.*

Theorem 2. *The value of any minor of the matrix of System (1) is equal to 0, 1, or -1.*

Definition 1. *Consider any cycle $L = (I_L, U_L)$ of the graph. Then construct a vector according to the following rules:*

- *Choose an arbitrary arc from the cycle. Let it be an arc $(\tau, \rho) \in U_L$. This sets the cycle detour direction and $\delta_{\tau\rho} = 1$.*
- *For cycle's forward arcs, let $\delta_{ij} = 1$.*
- *For cycle's backward arcs, let $\delta_{ij} = -1$.*
- *For nodes form the set I^* that form part of the cycle, let $\delta_i = 0$, for $i \in I_L \cap I^*$.*
- *For arcs that have not formed any part of the cycle, let $\delta_{ij} = 0$, for $(i,j) \in U \setminus U_L$.*
- *For nodes from the set I^* not included into the cycle, let $\delta_i = 0$, for $i \in I^* \setminus I_L$.*

The vector constructed according to the described rules is called the ***characteristic vector of the cycle****.*

Obviously, the characteristic vector depends on the chosen direction.

Definition 2. *Consider any chain $C = (I_C, U_C)$ of the graph connecting nodes $u, v \in I^*$. The vector constructed according to the following rules is the* ***characteristic vector of the chain with the direction according to a node:***

- *Let node u be the beginning of the chain and v be the end. Thus, we define the direction of the chain.*
- *For the node from which the chain begins, let $\delta_u = 1$.*
- *For the last node, let*

$$\delta_v = \begin{cases} \text{sign}[v] \cdot \delta_{vj}, & (v,j) \in U_C \\ -\text{sign}[v] \cdot \delta_{jv}, & (j,v) \in U_C \end{cases}$$

- *For forward arcs of the chain that correspond to the direction from u to v taken, let $\delta_{ij} = \text{sign}[u]$.*
- *For backward arcs of the chain, let $\delta_{ij} = -\text{sign}[u]$.*
- *For nodes $i \in I_C \cap I^* \setminus \{u,v\}$, let $\delta_i = 0$.*
- *For arcs that do not belong to the chain, let $\delta_{ij} = 0$, for $(i,j) \in U \setminus U_C$.*
- *For nodes $i \in I^* \setminus I_C$, let $\delta_i = 0$.*

Definition 3. *Consider any chain $C = (I_C, U_C)$ of the graph connecting two nodes $u, v \in I^*$. The vector constructed according to the following rules is the* ***characteristic vector of the chain with the direction according to an arc****:*

- *Choose any arc $(\tau, \rho) \in U_C$ that defines the direction of the chain.*
- *For the chain's forward arcs, let $\delta_{ij} = 1$.*
- *For the chain's backward arcs, let $\delta_{ij} = -1$.*
- *For nodes u and v, let,*

$$\delta_u = \begin{cases} \text{sign}[u] \cdot \delta_{uj}, & (u,j) \in U_C \\ -\text{sign}[u] \cdot \delta_{ju}, & (j,u) \in U_C \end{cases}$$

$$\delta_v = \begin{cases} \text{sign}[v] \cdot \delta_{vj}, & (v,j) \in U_C \\ -\text{sign}[v] \cdot \delta_{jv}, & (j,v) \in U_C \end{cases}$$

- *For nodes $i \in I_C \bigcap I^* \setminus \{u,v\}$, let $\delta_i = 0$.*
- *For the arcs that do not belong to the chain, let $\delta_{ij} = 0$, for $(i,j) \in U \setminus U_C$.*
- *For nodes $i \in I^* \setminus I_C$, let $\delta_i = 0$.*

Lemma 1. *The characteristic vector of a cycle, the characteristic vector of a chain with the direction according to a node, and the characteristic vector of a chain with the direction according to an arc satisfy the system*

$$\sum_{j\in I_i^+(U)} x_{ij} - \sum_{j\in I_i^-(U)} x_{ji} = \begin{cases} 0, & i \in I \setminus I^* \\ x_i \cdot \text{sign}[i], & i \in I^* \end{cases} \tag{3}$$

Theorem 3. *Any solution of System (3) is a linear combination of characteristic vectors.*

Proof Let $x = (x_{ij}, (i,j) \in U; \quad x_i, i \in I^*)$ be a solution of System (3). We show that vector x can be represented as the sum of the cyclic vector multiplied by some coefficient and some other solution $y = (y_{ij}, (i,j) \in U; \quad y_i, i \in I^*)$ of System (3), which has a smaller number of nonzero components.

Consider graph $R = \{I, T, S\}$ where T is the set of the arcs that correspond to nonzero components of the vector x and S is the set of the nodes that correspond to nonzero components of the vector x, $S = \{i : i \in I^*, x_i \neq 0\}$.

We find the subsets of the graph R with the following structural elements: a cycle or a chain between two nodes from the set S.

- At least one cycle exists in the graph R. Then we choose any one of them $L = \{I_L, U_L\}$ and compare it to the characteristic vector that corresponds to the cycle $\delta = (\delta_{ij}, (i,j) \in U; \quad \delta_i, i \in I^*)$. Let $(i_0, j_0) \in U_L$ be any of the cycle's arcs. Without commonness limitations, it can be considered a forward arc of the cycle. We represent components of the vector x as

$$\begin{cases} x_{ij} = x_{i_0 j_0}\delta_{ij} + x'_{ij}, (i,j) \in U \\ x_i = x_{i_0 j_0}\delta_i + x'_i, \qquad i \in I^* \end{cases} \tag{4}$$

  where the vector $x' = \left(x'_{ij}, (i,j) \in U; \quad x'_i, i \in I^*\right)$, where $x'_{ij} = x_{ij} - x_{i_0 j_0}\delta_{ij}$, for $(i,j) \in U$, and $x'_i = x_i - x_{i_0 j_0}\delta_i$, for $i \in I^*$. It also appears to be a solution of System (3) and contains at least one nonzero component less. Thus, we have reduced the number of elements of the set T.

- A chain $C = \{I_C, U_C\}$ that connects nodes $i_1, i_2 \in S$ exists in the graph R. We compare it to the characteristic vector with the direction according to an arc. Let $(i_0, j_0) \in U_C$ be any of the chain's arcs. Without commonness limitations, it can be considered forward. We represent vector x, the solution of System (3), as (4). Obviously, vector x' also appears to be a solution of System (3) and it has at least one nonzero component less. Thus, we have reduced the set T and broken the chain by dividing one of its coherence components into two: i_1 belongs to one of them and i_2 to the other. We then apply this process to the vector x' and all successive vectors constructed according to the rules in (4) while some cycles or chains with arcs from the set S exist in the graph R. During each step, one component is removed from the graph R.

We prove that if no chains or no cycles exist in the graph R then the system

$$\sum_{j\in I_i^+(T)} x_{ij} - \sum_{j\in I_i^-(T)} x_{ji} = \begin{cases} 0, & i \in I \setminus S \\ x_i \cdot \text{sign}[i], & i \in S \end{cases} \tag{5}$$

has only the trivial solution.

Let graph R be consisting of s coherence components. Then System (5) splits into s independent systems.

Consider any coherence component $R^k = \{I(T^k), T^k\}$ and $S^k = S \bigcap I(T^k)$. It has no cycles, so the set T^k has a tree structure and therefore $|T^k| = |I(T^k)| - 1$. Since graph R^k has no chain of a considered type, we have $|S^k| \leq 1$.

If $|S^k| = 0$ then there is one equation more than the number of variables. Since A^k is the block of the matrix of System (5) for the mentioned coherence component that corresponds to the incidence matrix of the tree T^k, for which we have $\text{rank}(A^k) = |T^k|$, the corresponding subsystem has only a trivial solution. If $|S^k| = 1$, then A^k is the incidence matrix of the tree T^k plus one column with one nonzero element (the signum of the node i_k, is $S^k = \{i_k\}$). In this case, $\text{rank}(A^k) = |T^k| + 1$, which coincides with the dimensions of A^k, and therefore the corresponding subsystem has only a trivial solution.

Thus, we have split the solution x of System (5) until we obtain the next vector $x' = 0$, and, therefore, we have a decomposition of the vector x as a linear combination of characteristic vectors. A constructive method of representation of the vector x as a linear combination of characteristic vectors is completely described. □

Definition 4. *We call an* ***aggregate of sets*** $R = \{U_R, I_R^*\}$, $U_R \subseteq U$, *and* $I_R^* \subseteq I^*$ *the support of the graph* G *for the System (1) if for* $\widetilde{R} = \{\widetilde{U}, \widetilde{I}^*\}, \widetilde{U} = U_R, \widetilde{I}^* = I_R^*$ *the system*

$$\sum_{j\in I_i^+(\widetilde{U})} x_{ij} - \sum_{j\in I_i^-(\widetilde{U})} x_{ji} = \begin{cases} 0, & i \in I \setminus \widetilde{I}^* \\ x_i \cdot \text{sign}[i], & i \in \widetilde{I}^* \end{cases} \tag{6}$$

has only a trivial solution, but has a nontrivial solution for any of the following set aggregations:

$$\begin{cases} \widetilde{R} = \{\widetilde{U}, \widetilde{I}^*\}, \quad \widetilde{U} = U_R \bigcup (i_0, j_0), & \textit{for } (i_0, j_0) \in U \setminus U_R \textit{ and }, \widetilde{I}^* = I_R^* \\ \widetilde{R} = \{\widetilde{U}, \quad \widetilde{I}^*\}, \quad \widetilde{U} = U_R, \quad \widetilde{I}^* = I_R^* \bigcup \{i_0\}, & \textit{for } i_0 \in I^* \setminus I_R^* \end{cases}$$

For some subset of arches $U_1 \subseteq U$, we introduce the set of incidental nodes $I(U_1) = \{i \in I : (i,j) \in U_1 \vee (j,i) \in U_1\}$.

We construct a forest from s the trees $T^k = \{I(U_T^k), U_T^k\}$, $s \leq |I^*|$, so that every tree has exactly one node $u_k \in I^*$, for $k = \overline{1,s}$, and $\bigcup_{k1}^{s} I(U_d^k) = I$. We form the sets

$$U_R = \bigcup_{k1}^{s} U_T^k, \qquad I_R^* = \bigcup_{k1}^{s} \{u_k\}$$

Theorem 4 (Network Criterion of Support). *An aggregate of sets $R = \{U_R, I_R^*\}$, $U_R \subseteq U$, and $I_R^* \subseteq I^*$ is the support of the graph G for System (1) if and only if the following conditions can be carried out:*

- *Each coherence component $T^k = \{I(U_T^k), U_T^k\}$, for $k = \overline{1,s}$, is a tree.*
- $I(\bigcup_{k=1}^{s} U_T^k) = \bigcup_{k=1}^{s} I(U_T^k) = I$
- $|I_k^*| = 1$, *where* $I_k^* = I_R^* \bigcap I(U_T^k)$, *for* $k = \overline{1,s}$.

After the support $R = \{U_R, I_R^*\}$ of System (1) is chosen, we determine what structures can be obtained after adding one nonsupporting element to the support.

Definition 5. The **characteristic vector entailed by an arc** $(\tau, \rho) \in U \backslash U_R$ is the vector constructed according to the following rules:

- If the set $U_R \bigcup \{(\tau, \rho)\}$ has a cycle $L = \{I_L, U_L\}$, then the entailed characteristic vector is the characteristic vector of that cycle, and the arc (τ, ρ) is chosen to define the detour direction of the cycle.
- If the set $U_R \bigcup (\tau, \rho)$ has a chain $C = \{I_C, U_C\}$ that connects nodes $u, v \in I_R^*$, then the entailed characteristic vector is the characteristic vector of that chain, and the arc that defines the detour direction is chosen to be (τ, ρ).

Definition 6. The **characteristic vector entailed by a node** $\gamma \in I^* \setminus I_R^*$ is the characteristic vector of the chain that connects nodes γ and $v \in I_R^*$ with node γ being chosen as the beginning of the chain.

Theorem 5. *Any solution of the homogeneous System (1) may be uniquely represented as a linear combination of characteristic vectors entailed by the nonsupporting for System (1) components of the graph $G = \{I, U\}$.*

Proof We have to prove that the aggregate of entailed characteristic vectors make up the basis of the space of solutions of the System (1).

The fact that each characteristic vector satisfies System (1) comes from Lemma 1.

Let a support of the graph $R = \{U_R, I_R^*\}$ of the graph G for the System (1) be consisting of s coherence components, then the number of nonsupporting arches equals $|U \setminus U_R| = |U| - (|I| - s)$ and the number of nonsupporting nodes equals $|I^* \setminus I_R^*| = |I^*| - s$. We have

$$|U \setminus U_R| + |I^* \setminus I_R^*| = |U| - (|I| - s) + |I^*| - s = |U| - |I| + |I^*|$$

Each entailed characteristic vector always has one and only one nonsupporting component that equals one. It corresponds to the arc or the node that has entailed this vector. All the other components of the characteristic vector are equal to zero. This means that any two different entailed characteristic vectors are linearly independent.

Thus, the aggregate of entailed characteristic vectors is a basis of the space of solutions. Therefore, any of the solutions may be uniquely represented as their linear combination. □

We choose a support of the network $R = \{U_R, I_R^*\}$ of the network G for System (1). It consists of the trees $T^k = \{I(U_T^k), U_T^k\}$, for $k = \overline{1, s}$, where each tree has the only one node $u_k \in I(U_T^k) \cap I_R^*$. We find characteristic vectors-columns $\delta(\tau, \rho) = \left(\delta_{ij}^{\tau\rho}, (i,j) \in U; \quad \delta_i^{\tau\rho}, i \in I^*\right)$, entailed by nonsupporting arcs $(\tau, \rho) \in U \setminus U_R$, and $\delta(\gamma) = \left(\delta_{ij}^{\gamma}, (i,j) \in U; \quad \delta_i^{\gamma}, i \in I^*\right)$, entailed by nonsupporting nodes $\gamma \in I^* \setminus I_R^*$.

Theorem 6. *The general solution of the System (1) may be uniquely represented using the following look:*

$$x_{ij} = \sum_{(\tau,\rho) \in U \setminus U_R} x_{\tau\rho} \delta_{ij}^{\tau\rho} + \sum_{\gamma \in I^* \setminus I_R^*} x_\gamma \delta_{ij}^{\gamma} + \widetilde{x}_{ij}, \qquad \text{for} \quad (i,j) \in U_R \tag{7}$$

$$x_i = \sum_{(\tau,\rho) \in U \setminus U_R} x_{\tau\rho} \delta_i^{\tau\rho} + \sum_{\gamma \in I^* \setminus I_R^*} x_\gamma \delta_i^{\gamma} + \widetilde{x}_i, \quad \text{for} \quad i \in I_R^* \cap I(U_T^k) \text{ and } k = \overline{1, s} \tag{8}$$

where $\widetilde{x} = (\widetilde{x}_{ij}, (i,j) \in U, \quad \widetilde{x}_i, i \in I^*)$ is a partial solution of the inhomogeneous system.

Proof We choose a supporting set $R = \{U_R, I_R^*\}$ for the graph $G = \{I, U\}$ for the System (1) and find the general solution of System (1). We consider it to be the sum of the general solution of the homogeneous system and a partial solution of the inhomogeneous system.

Let $y = (y_{ij}, (i,j) \in U; \quad y_k, k \in I^*)$ be any of the solutions of the homogeneous System (1). We consider vector

$$y' = y - \sum_{(\tau,\rho) \in U \setminus U_R} y_{\tau\rho} \delta(\tau, \rho) - \sum_{\gamma \in I^* \setminus I_R^*} y_\gamma \delta(\gamma)$$

According to Lemma 1, the characteristic vector of the chain and characteristic vector of the cycle are the solutions of the homogeneous System (1). Moreover, y is also a solution of the homogeneous system and therefore their linear combination also satisfies System (1). And, furthermore, vector y' is constructed in such a way that $y'_{ij} = 0$, for $(i,j) \in U \setminus U_R$, and $y'_k = 0, k \in I^* \setminus I^*_R$. In other words, all nonbasis components of y' are equal to zero and therefore y' satisfies the following system:

$$\sum_{j \in I_i^+(U_R)} x_{ij} - \sum_{j \in I_i^-(U_R)} x_{ji} = \begin{cases} 0, & i \in I \setminus I^*_R \\ x_i \cdot \mathrm{sign}[i], & i \in I^*_R \end{cases}$$

However, according to the support definition such a system has only a trivial solution. Consequently, $y' = 0$ and the general solution of the homogeneous System (1) has the following look:

$$y = \sum_{(\tau,\rho) \in U \setminus U_R} y_{\tau\rho} \delta(\tau,\rho) + \sum_{\gamma \in I^* \setminus I^*_R} y_\gamma \delta(\gamma)$$

We have found the general solution of the homogeneous System (1). We write down the general solution of System (1) in network form.

$$\begin{cases} y_{ij} = \displaystyle\sum_{(\tau,\rho) \in U \setminus U_R} y_{\tau\rho} \delta_{ij}^{\tau,\rho} + \sum_{\gamma \in I^* \setminus I^*_R} y_\gamma \delta_{ij}^{\gamma}, \text{for} \quad (i,j) \in U_R \\ y_i = \displaystyle\sum_{(\tau,\rho) \in U \setminus U_R} y_{\tau\rho} \delta_i^{\tau,\rho} + \sum_{\gamma \in I^* \setminus I^*_R} y_\gamma \delta_i^{\gamma}, \quad \text{for} \quad i \in I^*_R \end{cases} \tag{9}$$

The general solution of the inhomogeneous System (1) is the sum of the general solution of the homogeneous system and a partial solution of the inhomogeneous system. □

2 Support of the Graph

Let $R = \{U_R, I^*_R\}$ be a support of the graph $G = \{I, U\}$ for System (1). In arbitrary order, we choose sets $W = \{U_W, I^*_W\}, |W| = q, U_W \subseteq U \setminus U_R$, and $I^*_W \subseteq I^* \setminus I^*_R$.

By substituting the general solution of System (1), which has the form (7)–(8), into the system of linear equations (2), we obtain:

$$\sum_{(i,j) \in U_R} \lambda_{ij}^p \left[\sum_{(\tau,\rho) \in U \setminus U_R} x_{\tau\rho} \delta_{ij}^{\tau\rho} + \sum_{\gamma \in I^* \setminus I^*_R} x_\gamma \delta_{ij}^{\gamma} + \tilde{x}_{ij} \right] + \sum_{(i,j) \in U_W} \lambda_{ij}^p x_{ij} + \sum_{(i,j) \in U \setminus (U_W \cup U_R)} \lambda_{ij}^p x_{ij} = \alpha_p$$

We change the summing order:

$$\sum_{(\tau,\rho)\in U\setminus U_R} x_{\tau\rho} \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} x_\gamma \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^\gamma + \sum_{(i,j)\in U_R} \lambda_{ij}^p \widetilde{x}_{ij} + \sum_{(\tau,\rho)\in U_W} \lambda_{\tau\rho}^p x_{\tau\rho} + \sum_{(\tau,\rho)\in U\setminus(U_W\cup U_R)} \lambda_{\tau\rho}^p x_{\tau\rho} = \alpha_p,$$

$$\sum_{(\tau,\rho)\in U\setminus U_R} x_{\tau\rho} \left[\sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{\tau\rho} + \lambda_{\tau\rho}^p\right] + \sum_{\gamma\in I^*\setminus I_R^*} x_\gamma \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^\gamma + \sum_{(i,j)\in U_R} \lambda_{ij}^p \widetilde{x}_{ij} = \alpha_p$$

Definition 7. *The number*

$$\Lambda_{\tau\rho}^p ::= \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^{\tau\rho} + \lambda_{\tau\rho}^p \tag{10}$$

is the ***determinant of the structure entailed by the arc*** $(\tau,\rho) \in U \setminus U_R$ *relatively to restriction (2) with the number* p.

Definition 8. *The number*

$$\Lambda_\tau^p ::= \sum_{(i,j)\in U_R} \lambda_{ij}^p \delta_{ij}^\gamma \tag{11}$$

is the ***determinant of the structure entailed by the node*** $\gamma \in I^* \setminus I_R^*$ *relatively to restriction (2) with the number* p.

We introduce the designations:

$$A^p ::= \alpha_p - \sum_{(i,j)\in U_R} \lambda_{ij}^p \widetilde{x}_{ij} \tag{12}$$

Then System (2) takes the form

$$\sum_{(\tau,\rho)\in U\setminus U_R} \Lambda_{\tau\rho}^p x_{\tau\rho} + \sum_{\gamma\in I^*\setminus I_R^*} \Lambda_\gamma^p x_\gamma = A^p, \qquad \text{for} \quad p = \overline{1,q} \tag{13}$$

In System (13), we separate variables that correspond to W and then we obtain

$$\sum_{(\tau,\rho)\in U_W} \Lambda_{\tau\rho}^p x_{\tau\rho} + \sum_{\gamma\in I_W^*} \Lambda_\gamma^p x_\gamma = A^p - \sum_{(\tau,\rho)\in U\setminus(U_W\cup U_R)} \Lambda_{\tau\rho}^p x_{\tau\rho} - \sum_{\gamma\in I^*\setminus(I_W^*\cup I_R^*)} \Lambda_\gamma^p x_\gamma \tag{14}$$

for $p = \overline{1,q}$. In the matrix of System (14), it has the form

$$Dx_W = \beta \tag{15}$$

where matrix D consists of the determinants of the structures entailed by the components of the set W, $x_W = (x_{ij}, (i,j) \in U_W; \quad x_i, i \in I_W^*)$, for $\beta = (\beta_p, p = \overline{1,q})$

$$\beta_p = A^p - \sum_{(\tau,\rho)\in U\setminus(U_W\cup U_R)} \Lambda_{\tau\rho}^p x_{\tau\rho} - \sum_{\gamma\in I^*\setminus(I_W^*\cup I_R^*)} \Lambda_\gamma^p x_\gamma$$

for $p = \overline{1,q}$.

Definition 9. *We call the support of the graph G for Systems (1)–(2) such an aggregate of sets* $K = \{U_k, I_k^*\}$ *that given* $\widetilde{K} = \{\widetilde{U}, \widetilde{I}^*\}, \widetilde{U} = U_k$, *and* $\widetilde{I}^* = I_k^*$, *the system*

$$\begin{cases} \sum\limits_{j \in I_i^+(\widetilde{U})} x_{ij} - \sum\limits_{j \in I_i^-(\widetilde{U})} x_{ji} = \begin{cases} 0, & i \in I \setminus \widetilde{I}^* \\ x_i \cdot \mathrm{sign}[i], & i \in \widetilde{I}^* \end{cases} \\ \sum\limits_{(i,j) \in (\widetilde{U})} \lambda_{ij}^p x_{ij} = 0, \qquad for \quad p = \overline{1,q} \end{cases} \tag{16}$$

has only a trivial solution. Moreover, it has a nontrivial solution for any of the following aggregations of sets:

$$\begin{cases} \widetilde{K} = \{\widetilde{U}, \quad \widetilde{I}^*\}, \quad \widetilde{U} = U_K \bigcup (i_0, j_0), \qquad for\ (i_0, j_0) \in U \setminus U_K \ and\ \widetilde{I}^* = I_K^* \\ \widetilde{K} = \{\widetilde{U}, \widetilde{I}^*\}, \quad \widetilde{U} = U_K, \quad \widetilde{I}^* = I_K^* \bigcup \{i_0\}, \qquad for\ i_0 \in I^* \setminus I_K^* \end{cases}$$

Theorem 7 (Network Support Criterion). *The aggregation of sets* $K = \{U_K, I_K^*\}$ *is a support of the network* $G = \{I, U\}$ *for System (1)–(2) if and only if*

- *the aggregation of sets* $K = \{U_K, I_K^*\}$ *may be divided into two aggregations:* $R = \{U_R, I_R^*\}$ *and* $W = \{U_W, I_W^*\}$, *such as* $U_R \bigcup U_W = U_K$, $U_R \bigcap U_W = \emptyset$, $I_R^* \bigcup I_W^* = I_K^*$, $\delta_R = 0$, *and the set R is a support of the network* $G = \{I, U\}$ *for System (1);*
- $|W| = q$, *where q is the number of equations in System (2);*
- *matrix D, which consists of determinants of the structures entailed by the arcs and nodes of the aggregation W, is not degenerated.*

3 Theoretical-Graphical Properties

We now investigate theoretical-graphical properties of the structure of the support of the network $G = \{I, U\}$ for Systems (1)–(2). According to Theorem 7, the supporting set aggregate $K = \{U_K, I_K^*\}$ includes the support $R = \{U_R, I_R^*\}$ of the network G for System (1). Supporting elements that correspond to the aggregate R make up a forest of trees that covers all the nodes of the set I, and each tree of the forest has exactly one node from the set I_R^*. Adding each additional element from $W = \{U_W, I_W^*\}$ to the elements from $R = \{U_R, I_R^*\}$, we made a cycle or a chain in the set $K = \{U_K, I_K^*\}$.

We write the general solution of Systems (1)–(2) in matrix form. We designate $N = \{U_N, I_N^*\}$, $U_N = U \setminus U_K$, and $I_N^* = I^* \setminus I_K^*$ to be components of the nonsupport of the network $G = \{I, U\}$ for Systems (1)–(2).

For any set $Z = \{U_Z, I_Z^*\}, U_Z \subseteq U$, and $I_Z^* \subseteq I^*$, we introduce a vector $x_Z = (x_{ij}, (i,j) \in U_Z; \quad x_i, i \in I_Z^*)$. Relations (7) (8) in the matrix have the following form

$$x_R = S_W x_W + S_N x_N - \widetilde{b}$$

where in matrix $S_Z, Z \in \{W, N\}$ consists of columns of two types $S_Z = (\delta_R(\tau,\rho), (\tau,\rho) \in U_Z; \delta_R(\gamma), \gamma \in I_Z^*)$. Here each column can be written as:

$$\begin{aligned}
\delta_R(\tau,\rho) &= \left(\delta_{ij}^{\tau\rho}, (i,j) \in U_R; \quad \delta_i^{\tau\rho}, i \in I_R^*\right)' \\
\delta_R(\gamma) &= \left(\delta_{ij}^{\gamma}, (i,j) \in U_R; \quad \delta_i^{\gamma}, i \in I_R^*\right)' \\
\widetilde{b} &= \left(\widetilde{b}_{ij}, (i,j) \in U; \quad \widetilde{b}_i \in I_R^*\right)' \\
\widetilde{b}_{ij} &= -\mathrm{sign}(i,j) \cdot P_i \bigcup P_j \cdot \textstyle\sum_{k \in P_i \cap P_j} b_k, \quad \text{for } (i,j) \in U_R \\
\widetilde{b}_i &= \mathrm{sign}[i] \cdot \textstyle\sum_{j \in I(U_T^k) \setminus I^*} a_j, \quad \text{for } i \in I_R^* \cap I(U_T^k) \text{ and } k = \overline{1,s}
\end{aligned}$$

According to (15), $Dx_W = \beta$. Therefore, we have

$$x_W = D^{-1}\beta$$

Now we have obtained the general solution of Systems (1) (2).

$$\begin{aligned}
x_W &= D^{-1}\beta \\
x_R &= S_W D^{-1}\beta + S_N x_N + \widetilde{b}
\end{aligned}$$

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